Stability and Quadratic Lyapunov Functions for nD Systems

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***Abstract*— We discuss some ideas and preliminary results on the stability of nD systems described by linear constant coefficient PDE's. The stability concept used is $\mathscr{L}_2$-stability and $\mathscr{L}_2$-asymptotic stability, with time as a distinguished variable. For scalar equations, stability conditions are derived, including methods to make these conditions into LMI's in the system parameters. These conditions are interpreted in terms of Lyapunov functions for systems involving many independent variables. Several open problems for multivariable nD systems are formulated.**

***Index Terms*— nD systems, PDE's, stability, Lyapunov functions, quadratic differential forms, LMI's.**

I. Introduction

We use standard symbols for $\mathbb{R}, \mathbb{C}, \mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc. When the number of rows or columns in vectors or matrices is immaterial (but finite), we use $^{\bullet}$, $^{\bullet\times\mathtt{w}}$, etc. Of course, when we then add or multiply vectors or matrices, we assume that the dimensions are compatible. $\succeq 0$ means that a symmetric matrix is nonnegative definite, with obvious changes to other domains and for positivity, etc. $\mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}), \mathscr{D}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}), \mathscr{L}_p(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ denote the set of infinitely differentiable, infinitely differentiable with compact support, and $\mathscr{L}_p$ functions from $\mathbb{R}^{\mathtt{n}}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$ denotes the set of polynomials, and $\mathbb{R}(\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}})$ the set of rational functions with real coefficients in the indeterminates $\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}$.

Let $R \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]^{\bullet\times\mathtt{w}}$, and consider the system of linear constant coefficient partial differential equations

$$R\left(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, \tfrac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0. \tag{1}$$

This PDE defines, through its solutions, the *behavior*

$$\begin{aligned}\mathscr{B} &= \text{kernel}\left(R\left(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, \tfrac{\partial}{\partial x_{\mathtt{n}}}\right)\right) \\ &= \{w \in \mathscr{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, \tfrac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0\}. \end{aligned} \tag{2}$$

The $\mathscr{C}^\infty$ assumption is made purely for ease of exposition. We refer to the domain variables $x_1, x_2, \ldots, x_{\mathtt{n}}$ of $w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}}$ as *independent* variables, and to the codomain variables $w_1, w_2, \ldots, w_{\mathtt{w}}$ as *dependent* variables. We denote this class of nD systems $\Sigma = (\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ and their behaviors by $\mathscr{L}^{\mathtt{w}}_{\mathtt{n}}$. When $\mathtt{n} = 1$, we write $\mathscr{L}^{\mathtt{w}}$. In many applications, one of the independent variables is time, and the others are spatial variables.

This class of systems has been studied very deeply in pure mathematics. In the system theory literature, it has received a great deal of attention in recent years, with the work of Oberst [4], [6], Pillai and Shankar [7], Rocha [9], [10], Zerz [5], [14], and many others. Very nice results concerning controllability, observability, elimination, i/o and state representations, etc. have been obtained. One of the attractive aspects of the theory put forward in these papers, as opposed to the semi-group approach [1] as applied to PDE's, is the fact that all the independent variables are treated on an equal footing. Also, these methods have put the algebraic structure to this subject, which had been dominated by functional analysis, into the foreground.

A nagging issue in this area is the question whether also in stability one should let all variables play a symmetric role, or if it makes more sense to consider time as a distinguished variable. In [11], [3] a stability theory is put forward in which all the independent variables play symmetric roles. In the present paper, however, we discuss stability and the construction of Lyapunov functions with time as a distinguished variable.

II. Stability of nD systems

For 1D systems the natural definition of stability is, following Lyapunov, as follows.

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}} \text{ is } \textit{stable}]\!] :\Leftrightarrow [\![(w \in \mathscr{B}) \Rightarrow (w \text{ is bounded on } [0,\infty))]\!],$$

and

$$[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}} \textit{is asymptotically stable}]\!] :\Leftrightarrow [\![(w \in \mathscr{B}) \Rightarrow (w(t) \to 0 \text{ for } t \to \infty)]\!].$$

We now generalize these notions to PDE's as (1). But, since we wish to have time as a special variable, we first incorporate this in the notation. For stability, we consider (n+1)D systems

$$R\left(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, \tfrac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0, \tag{3}$$

with $R \in \mathbb{R}[\xi_0, \xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]^{\bullet\times\mathtt{w}}$. Denote elements of its behavior as $w : (t,x) \mapsto w(t,x)$. The stability concepts that we will use for PDE's is as follows. Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}+1}$ be given in kernel representation by (3).

$$\begin{aligned}&[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}+1} \text{ is } \mathscr{L}_2\text{-}\textit{stable}]\!] :\Leftrightarrow \\ &\quad [\![(w \in \mathscr{B} \text{ and } w(t,\cdot) \in \mathscr{L}_2(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}}) \text{ for } t \geq 0) \Rightarrow \\ &\qquad (\|w(t,\cdot)\|_{\mathscr{L}_2(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}})} \text{ is bounded on } [0,\infty))]\!],\end{aligned}$$

and

$$\begin{aligned}&[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{n}+1} \text{ is } \mathscr{L}_2\text{-}\textit{asymptotically stable}]\!] :\Leftrightarrow \\ &\quad [\![(w \in \mathscr{B} \text{ and } w(t,\cdot) \in \mathscr{L}_2(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}}) \text{ for } t \geq 0) \Rightarrow \\ &\qquad (\|w(t,\cdot)\|_{\mathscr{L}_2(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{w}})} \to 0 \text{ as } t \to \infty]\!].\end{aligned}$$

In order to motivate this definition, consider the diffusion equation

$$\frac{\partial}{\partial t} w = \frac{\partial^2}{\partial x^2} w \tag{4}$$

This system, perhaps the most studied one of applied mathematics, defines a system in $\mathscr{L}^1_2$. We wish it to define an $\mathscr{L}_2$-asymptotically stable system. Perhaps the fact that for this system

$$\begin{aligned}&[\![(w \in \mathscr{B} \text{ and } w(t,\cdot) \in \mathscr{L}_1(\mathbb{R},\mathbb{R}) \text{ for } t \geq 0)]\!] \Rightarrow \\ &\qquad [\![\int_{-\infty}^{+\infty} w(t,x)\,dx = \int_{-\infty}^{+\infty} w(0,x)\,dx \text{ for } t \geq 0]\!],\end{aligned}$$

may be felt to contradict asymptotic stability. But this shows only that the system is not $\mathscr{L}_1$-asymptotically stable. As we shall see later, it is indeed $\mathscr{L}_2$-asymptotically stable. In fact, it is $\mathscr{L}_p$-stable for $1 \leq p$, and $\mathscr{L}_p$-asymptotically stable for $1 < p$. If this is bothersome, replace (4) by

$$\frac{\partial}{\partial t} w = -w + \frac{\partial^2}{\partial x^2} w,$$

which is $\mathscr{L}_p$-asymptotically stable for $1 \leq p$.

(4) is, by all intuitive reasonings, asymptotically stable. But it nevertheless has solutions that increase exponentially in time. Indeed, $w : (t,x) \mapsto e^{\alpha t + \beta x}$ is a solution iff $\alpha = \beta^2$. Hence $0 \neq \beta$ leads to a solution that increases exponentially in time. Note that if we consider spatially oscillatory solutions, ($0 \neq \beta$ imaginary), then we do have $\alpha < 0$. In any case, we cannot simply identify stability with the fact that all solutions are somehow bounded (or go to zero) for $t > 0$. We have to limit the class of solutions considered.

Note that assuming $w(0,\cdot) \in \mathscr{L}_2(\mathbb{R},\mathbb{R})$ does not suffice either, for (4) is known [2, pp. 50-51] to have solutions with $w(0,\cdot) = 0$ and $w(t,\cdot) \notin \mathscr{L}_2(\mathbb{R},\mathbb{R})$ for $t > 0$. So imposing $\mathscr{L}_2$-boundedness of the initial condition does not suffice either. We therefore consider in our definition of $\mathscr{L}_2$-stability only solutions of which we assume a priori that $w(t,\cdot) \in \mathscr{L}_2(\mathbb{R},\mathbb{R})$ for $t \in [0,\infty)$. In this sense, stability as we consider it here, is very much like what is done in the semi-group approach [1].

III. STABILITY OF SCALAR SYSTEMS

In this section, we consider the stability of systems described by

$$\frac{\partial}{\partial t} w = a\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w, \tag{5}$$

with $a \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$. This defines a system in $\mathscr{L}^1_{\mathtt{n}}$. The problem is to determine conditions on a for $\mathscr{L}_2$-stability and asymptotic stability. Such conditions are easy to obtain. Define $\omega := (\omega_1, \omega_2, \ldots, \omega_{\mathtt{n}}) \in \mathbb{R}^{\mathtt{n}}$.

$$\begin{aligned}&[\![(5) \text{ is } \mathscr{L}_2 - \text{stable}]\!] \Leftrightarrow \\ &\qquad [\![a(i\omega) + a(-i\omega) \leq 0 \quad \forall\, \omega \in \mathbb{R}^{\mathtt{n}}]\!],\end{aligned} \tag{6}$$

and

$$\begin{aligned}&[\![(5) \text{ is } \mathscr{L}_2 - \text{asymptotically stable}]\!] \Leftrightarrow \\ &\qquad [\![a(i\omega) + a(-i\omega) \leq 0 \text{ for almost all } \omega \in \mathbb{R}^{\mathtt{n}}]\!].\end{aligned} \tag{7}$$

These conditions can be proven using the $\mathscr{L}_2$-Fourier transform $\hat{w}(t,i\cdot) \in \mathscr{L}_2(\mathbb{R}^{\mathtt{n}},\mathbb{R})$ of $w(t,\cdot) \in \mathscr{L}_2(\mathbb{R}^{\mathtt{n}},\mathbb{R})$. This Fourier transform is governed by the ordinary differential equation

$$\frac{d}{dt}\hat{w}(\cdot,i\omega) = a(i\omega)\hat{w}(\cdot,i\omega),$$

parametrized by $\omega \in \mathbb{R}^{\mathtt{n}}$. This yields

$$\hat{w}(t,i\omega) = e^{a(i\omega)t}\hat{w}(0,i\omega),$$

and the stability results follow from some simple estimates.

Generally (6) and (7) are not considered satisfactory answers from a computational point of view. However, it is sometimes possible to turn these conditions into an LMI. We discuss this in the next sections.

Identical stability results hold for scalar systems

$$\frac{\partial}{\partial t} b\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = a\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w, \tag{8}$$

with $a, b \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$. Simply replace in (6) and (7) the polynomial a by the rational function $g = b^{-1}a$, or by $b(-\xi)a(\xi)$, and proceed in exactly the same way.

IV. QUADRATIC FORMS

In this section, we first assume that $\mathtt{n} = 1$. We introduce one-variable quadratic differential forms. These have been studied in depth in [12].

Let $F \in \mathbb{R}[\xi]$. Define $F^* \in \mathbb{R}[\xi]$ by $F^*(\xi) := F(-\xi)$. Call $F \in \mathbb{R}[\xi]$

$$[\![\textit{symmetric}]\!] :\Leftrightarrow [\![F = F^*]\!].$$

Denote the symmetric elements of $\mathbb{R}[\xi]$ by $\mathbb{R}[\xi]_S$. Define the quadratic form $Q^{\mathbb{R}}_F : \mathscr{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ induced by $F \in \mathbb{R}[\xi]_S$ by

$$Q^{\mathbb{R}}_F(v) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{v}(-i\omega)^\top F(i\omega)\hat{v}(i\omega)\,d\omega$$

where ^ denotes Fourier transform. Call $Q^{\mathbb{R}}_F$

$$\begin{aligned}&[\![\textit{non-negative}, \text{ denoted } Q^{\mathbb{R}}_F \succcurlyeq 0]\!] :\Leftrightarrow \\ &\qquad [\![Q^{\mathbb{R}}_F(v) \geq 0 \ \forall v \in \mathscr{D}(\mathbb{R},\mathbb{R})]\!],\end{aligned}$$

and

$$\begin{aligned}&[\![\textit{positive}, \text{ denoted } Q^{\mathbb{R}}_F \succ 0]\!] :\Leftrightarrow \\ &\qquad [\![(Q^{\mathbb{R}}_F \succcurlyeq 0) \text{ and } ((Q^{\mathbb{R}}_F(v) = 0) \Leftrightarrow (v = 0))]\!].\end{aligned}$$

It is easy to prove that $[\![Q_F^{\mathbb{R}} \succcurlyeq 0]\!] \Leftrightarrow [\![F(i\omega) \geq 0\ \forall \omega \in \mathbb{R}]\!]$, and $[\![Q_F^{\mathbb{R}} \succ 0]\!] \Leftrightarrow [\![F(i\omega) > 0$ for almost all $\omega \in \mathbb{R}]\!]$. Moreover, $[\![Q_F^{\mathbb{R}} \succcurlyeq 0(\succ 0)]\!] \Leftrightarrow [\![\exists (0 \neq) D \in \mathbb{R}[\xi]$ such that $F = D^* D]\!]$.

Also two-variable polynomials lead to quadratic forms. Let $\Phi \in \mathbb{R}[\zeta,\eta]$. Written out in terms of the coefficients,

$$\Phi(\zeta,\eta) = \Sigma_{\mathtt{k},\ell} \Phi_{\mathtt{k},\ell} \zeta^{\mathtt{k}} \eta^{\ell},$$

with the sum assumed to be finite. Let $\mathtt{d}$ denote the highest power that appears in Φ. The matrix

$$\mathrm{Mat}(\Phi) := \begin{bmatrix} \Phi_{0,0} & \Phi_{1,0} & \cdots & \Phi_{\mathtt{d},0} \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{\mathtt{d},1} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{\mathtt{d},0} & \Phi_{\mathtt{d},1} & \cdots & \Phi_{\mathtt{d},\mathtt{d}} \end{bmatrix}$$

is called the matrix associated with Φ. Define $\Phi^\star \in \mathbb{R}[\zeta,\eta]$ by $\Phi^\star(\zeta,\eta) := \Phi(\eta,\zeta)$. Call $\Phi \in \mathbb{R}[\zeta,\eta]$

$$[\![\textit{symmetric}]\!] :\Leftrightarrow [\![\Phi = \Phi^\star]\!] \Leftrightarrow [\![\mathrm{Mat}(\Phi) = \mathrm{Mat}(\Phi)^\top]\!].$$

Denote the symmetric elements of $\mathbb{R}[\zeta,\eta]$ by $\mathbb{R}[\zeta,\eta]_S$. Symmetric two-variable polynomials are in one-to-one relation with quadratic forms in $v \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ and its derivatives. The quadratic differential form (QDF) induced by $\Phi \in \mathbb{R}[\zeta,\eta]_S$ is defined as the map from $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ into $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ given by

$$Q_\Phi(v) := \Sigma_{\mathtt{k},\ell} \tfrac{d^{\mathtt{k}}}{dx^{\mathtt{k}}} v^\top \Phi_{\mathtt{k},\ell} \tfrac{d^\ell}{dx^\ell} v.$$

Observe that $\frac{d}{dx} Q_\Phi$ is also a QDF, corresponding to the two-variable polynomial $(\zeta+\eta)\Phi(\zeta,\eta)$.

Call the QDF Q_Φ induced by $\Phi = \Phi^\star \in \mathbb{R}[\zeta,\eta]$

$$[\![\textit{non-negative}, \text{ denoted } Q_\Phi \succeq 0]\!] \\ :\Leftrightarrow [\![Q_\Phi(v) \geq 0\ \forall v \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})]\!] \Leftrightarrow [\![\mathrm{Mat}(\Phi) \succeq 0]\!],$$

and

$$[\![\textit{positive}, \text{ denoted } Q_\Phi \succ 0]\!] :\Leftrightarrow [\![(Q_\Phi \succcurlyeq 0) \text{ and} \\ ((Q_\Phi(v) = 0), (v \in \mathscr{L}_2(\mathbb{R},\mathbb{R}))]\!] \Rightarrow [\![(v = 0))]\!].$$

Non-negativity of Q_Φ is equivalent to the existence of $D \in \mathbb{R}[\xi]$ such that $\Phi(\zeta,\eta) = D^\top(\zeta) D(\eta)$, i.e. $Q_\Phi(v) = ||D(\frac{d}{dx} v||^2$. For positivity, add $D \neq 0$.

Define $\Delta : \mathbb{R}[\zeta,\eta] \to \mathbb{R}[\xi]$ by

$$\Delta(\Phi) := \Phi(-\xi,\xi).$$

Δ maps symmetric elements of $\mathbb{R}[\zeta,\eta]$ into symmetric elements of $\mathbb{R}[\xi]$. Note that

$$\ker(\Delta) = \{\Phi \in \mathbb{R}[\zeta,\eta] \ \mid\ \exists \Phi' \in \mathbb{R}[\zeta,\eta] \\ \text{such that } \Phi(\zeta,\eta) = (\zeta+\eta)\Phi'(\zeta,\eta).$$

Equivalently, $Q_\Phi = \frac{d}{dt} Q'_\Phi$.

In addition to the pointwise behavior of a QDF, we are also interested in the behavior of its integral over $\mathbb{R}$,

$$\int_{-\infty}^{+\infty} Q_\Phi(v)\, dx \quad \text{for } v \in \mathscr{D}(\mathbb{R},\mathbb{R}).$$

Observe that

$$\int_{-\infty}^{+\infty} Q_\Phi(v)(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{v}(-i\omega)^\top \Delta(\Phi)(i\omega) \hat{v}(i\omega)\, d\omega,$$

with $\hat{}$ the Fourier transform. Note that this integral equals

$$Q^{\mathbb{R}}_{\Delta(\Phi)} =: Q^{\mathbb{R}}_\Phi,$$

with a small abuse of notation.

This leads to an interplay between the two kinds of quadratic forms introduced. It follows that

$$[\![Q^{\mathbb{R}}_\Phi = 0]\!] \Leftrightarrow [\![\Delta(\Phi) = 0]\!] \\ \Leftrightarrow [\![\exists \Phi' \in \mathbb{R}[\zeta,\eta] \text{ such that } Q_\Phi = \tfrac{d}{dx} Q_{\Phi'}]\!].$$

Similarly

$$[\![Q^{\mathbb{R}}_\Phi \succcurlyeq 0]\!] \Leftrightarrow [\![\Delta(\Phi)(i\omega) \geq 0\ \forall \omega \in \mathbb{R}]\!] \\ \Leftrightarrow [\![\exists W \in \mathbb{R}[\zeta,\eta] \text{ such that the QDF induced by} \\ \Phi(\zeta,\eta) + (\zeta+\eta)W(\zeta,\eta) \text{ is } \succcurlyeq 0]\!],$$

and

$$[\![Q^{\mathbb{R}}_\Phi \succ_{\mathbb{R}} 0]\!] \Leftrightarrow [\![\Delta(\Phi)(i\omega) > 0 \text{ for almost all } \omega \in \mathbb{R}]\!] \\ \Leftrightarrow [\![\exists W \in \mathbb{R}[\zeta,\eta] \text{ such that the QDF induced by} \\ \Phi(\zeta,\eta) + (\zeta+\eta)W(\zeta,\eta) \text{ is } \succ 0]\!],$$

All this is readily generalized to $\mathtt{n} > 1$, by considering ξ,ζ,η and ω as multindices, and with obvious notational changes.

V. Quadratic Lyapunov functions

We aim at explaining how QDF's can be used as Lyapunov functions for (5). We consider again first the case $\mathtt{n} = 1$. Consider the QDF (in the x-variable) defined by $V = V^\star \in \mathbb{R}[\zeta,\eta]_S$, leading to $Q_V(w(t,\cdot))$. Its time-derivative along solutions of (5) is also a symmetric QDF, induced by

$$\dot{V}(\zeta,\eta) := a(\zeta)V(\zeta,\eta) + V(\eta,\zeta)a(\eta).$$

The basic quadratic Lyapunov function result now states that (5) is $\mathscr{L}_2$-stable iff there exists a $V = V^\star$ with $Q^{\mathbb{R}}_V \succ 0$ such that $Q^{\mathbb{R}}_{\dot{V}} \succcurlyeq 0$, and $\mathscr{L}_2$-asymptotically stable iff there exists a $V = V^\star \succ 0$ with $Q^{\mathbb{R}}_V \succ 0$ such that $Q^{\mathbb{R}}_{\dot{V}} \succ 0$. This result can be further simplified and leads to

$$[\![(5) \text{ is } \mathscr{L}_2\text{-stable}]\!] \Leftrightarrow [\![\exists V = V^\star, W = W^\star \in \mathbb{R}[\zeta,\eta]_S, \\ \text{such that (i) } V(\zeta,\eta) + (\zeta+\eta)W(\zeta,\eta) = \dot{V}(\zeta,\eta), \\ \text{(ii) } Q_V \succ 0, \text{and (iii) } Q_{\dot{V}} \preccurlyeq 0]\!],$$

and

$$[\![(5) \text{ is } \mathscr{L}_2\text{-asymptotically stable}]\!] \Leftrightarrow \\ [\![\exists V = V^\star, W = W^\star \in \mathbb{R}[\zeta,\eta]_S, \text{ such that} \\ \text{(i) } V(\zeta,\eta) + (\zeta+\eta)W(\zeta,\eta) = \dot{V}(\zeta,\eta), \\ \text{(ii) } Q_V \succ 0, \text{and (iii) } Q_{\dot{V}} \prec 0]\!].$$

Note that this condition is actually an LMI, viewed as a condition on $a \in \mathbb{R}[\xi]$.

(6) is equivalent to the factorizability of $a(\xi)+a(-\xi)$ as $a(\xi)+a(-\xi)=-H^\top(-\xi)H(\xi)$ for some $H\in\mathbb{R}[\xi]^\bullet$, while (7) requires in addition $H\neq 0$. This factorizability is an LMI. It requires the solvability for $X\in\mathbb{R}^{\mathtt{d}\times\mathtt{d}}$, with $\mathtt{d}=\text{degree}(a)$, of

$$a(\xi)+a(-\xi)=\begin{bmatrix}1\\-\xi\\\vdots\\(-\xi)^{\mathtt{d}}\end{bmatrix}^\top X\begin{bmatrix}1\\\xi\\\vdots\\\xi^{\mathtt{d}}\end{bmatrix},\quad X=X^\top\preccurlyeq 0. \quad (9)$$

This can be given a Lyapunov interpretation. Let $V=V^*\in\mathbb{R}[\xi]_S$. Consider the quadratic form $Q_V^{\mathbb{R}}$. Its derivative along solutions of (5) is the quadratic form $Q_{\dot V}^{\mathbb{R}}$, induced by

$$\dot V := a^*V+V^*a.$$

It follows that

$$[\![(5) \text{ is } \mathscr{L}_2\text{-stable}]\!]\Leftrightarrow[\![\exists V=V^*\in\mathbb{R}[\xi]_S,\\ \text{such that (i) } Q_V^{\mathbb{R}}\succ 0,\text{and (ii) } Q_{\dot V}^{\mathbb{R}}\preccurlyeq 0]\!],$$

and

$$[\![(5) \text{ is } \mathscr{L}_2\text{-asymptotically stable}]\!]\Leftrightarrow[\![\exists V=V^*\in\mathbb{R}[\xi]_S,\\ \text{such that (i) } Q_V^{\mathbb{R}}\succ 0,\text{and (ii) } Q_{\dot V}^{\mathbb{R}}\prec 0]\!].$$

The above is readily generalized to the case $\mathtt{n}>1$. QDF's acting on $\mathtt{n}$ variables are defined completely analogously to the 1D case (see [8]). It leads to the following. Let V_0 be scalar QDF and V a $\mathtt{n}-vector$ of QDF's, acting on $\mathtt{n}$ variables. Define ∇ by $\nabla:=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\ldots,\frac{\partial}{\partial x_{\mathtt{n}}}\right)$. Then (5) is $\mathscr{L}_2$-stable iff there exists QDF's (V_0,V) acting on $\mathtt{n}$ variables, such that

$$Q_{V_0}\succ 0 \text{ and } \tfrac{\partial}{\partial t}Q_V+\nabla\cdot Q_V\preccurlyeq 0$$

along solutions of (5). In terms of $2\mathtt{n}$-variable polynomials this leads to, in the obvious multiindex notation,

$$V_0(\zeta,\eta)\succ 0,\\ a(\zeta)V_0(\zeta,\eta)+V_0^*(\eta,\zeta)a(\eta)+(\zeta+\eta)^\top V(\zeta,\eta)\preccurlyeq 0.$$

This is easily generalized to $\mathscr{L}_2$-asymptotic stability, and to quadratic forms on $\mathbb{R}$.

VI. Examples

The diffusion equation

$$\tfrac{\partial}{\partial t}w=\tfrac{\partial^2}{\partial x^2}w \quad (10)$$

is $\mathscr{L}_2$-asymptotically stable. The test (7) yields

$$a(i\omega)+a(-i\omega)=-\omega^2,$$

which is negative almost everywhere. The Lyapunov function can be constructed as follows. Differentiating $||w||^2$ w.r.t. t along solutions leads to the QDF defined by the two-variable polynomial

$$\zeta^2+\eta^2.$$

The following are the quadratic Lyapunov functions obtained for the case at hand. Since

$$\zeta^2+\eta^2-(\zeta+\eta)^2=-2\zeta\eta,$$

we obtain

$$\tfrac{\partial}{\partial t}|w(t,x)|^2\tfrac{\partial}{\partial x}(-\tfrac{\partial}{\partial x}|w(t,x)|^2)=-2|\tfrac{\partial}{\partial x}w(t,x)|^2,$$

leading to

$$\tfrac{d}{dt}\int_{-\infty}^{+\infty}|w(t,x)|^2\,dx=-2\int_{-\infty}^{+\infty}|\tfrac{\partial}{\partial x}w(t,x)|^2\,dx.$$

The wave equation

$$\tfrac{\partial^2}{\partial t^2}w=\tfrac{\partial^2}{\partial x^2}w \quad (11)$$

is $\mathscr{L}_2$-stable but not $\mathscr{L}_2$-asymptotically stable. The behavior of the wave equation is the sum of the behaviors of

$$\tfrac{\partial}{\partial t}w=\tfrac{\partial}{\partial x}w \text{ and } \tfrac{\partial}{\partial t}w=-\tfrac{\partial}{\partial x}w.$$

The test given in the previous section yields

$$a(i\omega)+a(-i\omega)=0,$$

for both systems. The stability results follows.

VII. Extensions

The results obtained for (5) and (8) are very simple and preliminary in nature, but the methods of analysis open up a way of dealing with the stability of large classes of PDE's.

The ultimate goal is to obtain $\mathscr{L}_2$-stability results for general PDE's (3) with $R\in\mathbb{R}[\xi_0,\xi_1,\xi_2,\ldots,\xi_{\mathtt{n}}]^{\bullet\times\mathtt{w}}$. We emphasize that we view the stability of nD systems with time considered as a distinguished variable. Using Fourier transforms in the spatial variables, one can expect that such conditions can be brought back to the analysis of the 1D systems

$$R\left(\tfrac{d}{dt},i\omega\right)\hat w(\cdot,i\omega)=0, \quad (12)$$

parametrized by $\omega=(\omega_1,\omega_2,\ldots,\omega_{\mathtt{n}})\in\mathbb{R}^{\mathtt{n}}$. A necessary condition for the $\mathscr{L}_2$-asymptotic stability of (3) is that the 1D polynomial matrix $R(\xi,i\omega)\in\mathbb{C}[\xi]^{\mathtt{w}\times\mathtt{w}}$ should be a Hurwitz polynomial matrix for almost all $\omega\in\mathbb{R}^{\mathtt{n}}$. More precisely, that for almost all $\omega\in\mathbb{R}^{\mathtt{n}}$,

$$\text{rank}\,(R(\lambda,i\omega))=\mathtt{w}\quad\forall\lambda\in\mathbb{C}\text{ with Re}(\lambda)\geq 0.$$

Establishing the sufficiency of this condition appears to be hard. It may require more conditions on R.

The case that (3) is first order in $\frac{\partial}{\partial t}$ is of particular interest. In this case (12) can be brought back to the system

$$\tfrac{d}{dt}\hat w(\cdot,i\omega)=A\,(i\omega)\,\hat w(\cdot,i\omega), \quad (13)$$

parametrized by $\omega\in\mathbb{R}^{\mathtt{n}}$, with A a matrix of rational functions $A\,(i\omega)\in\mathbb{C}(\xi_0,\xi_1,\xi_2,\ldots,\xi_{\mathtt{n}})^{\mathtt{w}\times\mathtt{w}}$. Our conjecture is that the corresponding PDE (3) is $\mathscr{L}_2$-asymptotically stable iff $A\,(i\omega)$ is Hurwitz for almost all $\omega\in\mathbb{R}^{\mathtt{n}}$. It is possible to analyze this question using Lyapunov functions.

One of the most important results of linear system theory is the stability of the system

$$\tfrac{d}{dt}w=Aw\quad\text{with } A\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}.$$

This system is asymptotically stable iff A is Hurwitz. This is equivalent to the solvability of

$$\exists Q \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}} \text{ such that } Q = Q^\top \succ 0, \quad A^\top Q + QA = -I.$$

Since this condition is an LMI, it can be argued to be an effective result computationally. It can also be used to analyze the stability of (13). Indeed, if $A(i\omega)$ is Hurwitz for almost all $\omega \in \mathbb{R}^{\mathtt{n}}$, then there exists $Q \in \mathbb{R}(\xi_1, \xi_2, \dots, \xi_{\mathtt{n}})^{\mathtt{w}\times\mathtt{w}}$ with $Q(i\omega) = Q(-i\omega)^\top \succ 0$, such that

$$A(-i\omega)^\top Q(i\omega) + Q(-i\omega)^\top A(i\omega) = -I$$

for almost all $\omega \in \mathbb{R}^{\mathtt{n}}$. It follows that along solutions of (13) there holds

$$\frac{d}{dt}\int_{-\infty}^{+\infty} \hat{w}(-i\omega)^\top Q(i\omega)\hat{w}(i\omega)\, d\omega = -\int_{-\infty}^{+\infty} ||\hat{w}(i\omega)||^2\, d\omega.$$

From this it is possible to prove $\mathscr{L}_2$-asymptotic stability under certain conditions on $Q(i\omega)$, for example, if $\mathtt{n} = 1$, if $A(\xi)$ is bi-proper, and if $A(i\omega)$ is Hurwitz for all $\omega \in \mathbb{R}$ and $\omega = \infty$.

A second avenue of research suggested by the analysis of the stability of (5) is the construction of QDF's and quadratic forms for use of Lyapunov functions for (3) for the stability of multivariable high order nD PDE's. It calls for a generalization of the analysis in sections III and IV to multivariable systems (this is easy, see [12]), to high order differential equations (in the spirit of what is done in [12, section 4]), to nD systems (this involves vector QDF's, and the sum-of-squares problem for nD polynomials, see [8]), and to QDF's involving $\mathtt{n}$-variable rational functions ([13]).

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