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PARAMETRIZATION OF THE SET OF REGULAR AND SUPERREGULAR STABILIZING CONTROLLERS

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Abstract— **The characterization of stabilizing controllers is discussed from the behavioral point of view. The main results provide parametrizations of the set of regular and superregular stabilizing controllers in terms of rational kernel representations of the plant.**

Index Terms— **Behaviors, control in a behavioral setting, stabilizing controllers, regular controllers, superregular controllers, dead-beat controllers.**

I. INTRODUCTION

In the behavioral approach, a dynamical system is viewed as a family of time trajectories, called the *behavior* of the system. The behavior can be specified in many ways, for example, as the kernel of a differential operator, as an input/state/output model, or, if the system is controllable, as the image of a differential operator or a transfer function. Recently, these representations have been extended to rational symbols [8].

The behavioral vantage point allows to define a system in a completely representation free manner. But it imposes the challenge of defining system properties in an intrinsic, representation free manner as well. This has led, for example, to new definitions of classical system theoretic concepts as controllability and stabilizability, and observability and detectability. Of course, one of the main tasks remains to obtain tests that allow to verify various properties in terms of a specific system representation.

We view control simply as restricting the plant behavior. A controller is a dynamical system, with its own behavior, and the intersection of the plant behavior and the controller behavior is the subset of the plant behavior that is implemented by this controller. The aim of the present article is to study the parametrization of the set of regular and superregular stabilizing controllers. The results obtained are very akin to the Kučera-Youla parametrization [3], [9], see also [4]. However, we start from a different definition of stability, and hence of stabilizability, and since in the classical Kučera-Youla parametrization all systems with the same transfer function are identified, our results are sharper since they respect the uncontrollable part of the system involved.

The notation for polynomial matrices and matrices of rational functions is discussed in the appendix, along with certain subrings (proper, stable) of $\mathbb{R}(\xi)$. These subrings play a central role in the sequel. In this paper, we state the results. Proofs will be given elsewhere.

II. REVIEW: RATIONAL REPRESENTATIONS OF LINEAR TIME-INVARIANT SYSTEMS

A *dynamical system* is a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ the *time-set*, $\mathbb{W}$ the *signal space*, and $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior*. We consider behaviors $\mathscr{B} \subseteq (\mathbb{R}^\bullet)^{\mathbb{R}}$ that are the solution set of a system of linear constant coefficient differential equations. In other words, there exists a polynomial matrix $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $\mathscr{B}$ is the solution set of

$$\boxed{R\left(\tfrac{d}{dt}\right) w = 0} \qquad (\mathscr{R})$$

We need to make precise when we want to call $w : \mathbb{R} \to \mathbb{R}^\bullet$ a solution of $(\mathscr{R})$. For simplicity of exposition, we deal with infinitely differentiable solutions only. Hence $(\mathscr{R})$ defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^\bullet, \mathscr{B})$ with

$$\mathscr{B} = \left\{ w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \;\middle|\; R\left(\tfrac{d}{dt}\right) w = 0 \right\}.$$

Note that we may as well write $\mathscr{B} = \ker\left(R\left(\frac{d}{dt}\right)\right)$, since $\mathscr{B}$ is actually the kernel of the differential operator $R\left(\frac{d}{dt}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{coldim}(R)}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{rowdim}(R)})$.

We denote the set of linear time-invariant differential systems or their behaviors by $\mathscr{L}^\bullet$, and by $\mathscr{L}^{\mathtt{w}}$ when the number of variables is $\mathtt{w}$.

We also use rational representations of $\mathscr{L}^\bullet$. These are defined as follows. Let $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and consider the system of 'differential equations'

$$\boxed{G\left(\tfrac{d}{dt}\right) w = 0} \qquad (\mathscr{G})$$

The matrix of rational functions G is called the *symbol* of $(\mathscr{G})$. Since G is a matrix of rational functions, it is not clear when $w : \mathbb{R} \to \mathbb{R}^\bullet$ is a solution of equation $(\mathscr{G})$. We define solutions as follows. Let (P, Q) be a left coprime matrix factorization over $\mathbb{R}[\xi]$ of $G = P^{-1}Q$. Define

$$[\![w : \mathbb{R} \to \mathbb{R}^\bullet \text{ is a solution of } (\mathscr{G})]\!] :\Leftrightarrow [\![Q\left(\tfrac{d}{dt}\right) w = 0]\!].$$

Whence $(\mathscr{G})$ defines the system $\Sigma = \left(\mathbb{R}, \mathbb{R}^\bullet, \ker\left(Q\left(\frac{d}{dt}\right)\right)\right) \in \mathscr{L}^\bullet$. We refer to [8] for motivation and details on system representation with rational symbols.

Since the representations ($\mathscr{R}$) are merely a subset of the representations ($\mathscr{G}$), matrices of rational functions form a representation class of $\mathscr{L}^\bullet$ that is more redundant, and hence richer, than the polynomial matrices. This redundancy can be used to obtain rational representations with properties that cannot be obtained using polynomial representations. In this paper, we use this richness in order to parametrize the set of stabilizing controllers.

III. Integer invariants

There are a number of integer invariants $\mathtt{w},\mathtt{m},\mathtt{p},\mathtt{n}$ (maps from $\mathscr{L}^\bullet$ to $\mathbb{Z}_+$) that play an important role in the present paper.

(i) $\mathtt{w}(\mathscr{B})$ equals the number of variables in $\mathscr{B}$,
(ii) $\mathtt{m}(\mathscr{B})$ equals the number of free variables in $\mathscr{B}$, i.e. the number of input variables in $\mathscr{B}$,
(iii) $\mathtt{p}(\mathscr{B})$ equals the number of bound variables in $\mathscr{B}$, i.e. the number of output variables in $\mathscr{B}$, and
(iv) $\mathtt{n}(\mathscr{B})$ equals the number of state variables in $\mathscr{B}$.

There are numerous ways to define these integer invariants formally, see [5].

IV. Review: Controllability, stability, and stabilizability

The time-invariant system $\Sigma = (\mathbb{R},\mathbb{R}^\bullet,\mathscr{B})$ is said to be

(i) *controllable* if $\forall\ w_1,w_2 \in \mathscr{B}$, $\exists\ T \geq 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geq T$;
(ii) *stabilizable* if $\forall\ w \in \mathscr{B}$, $\exists\ w' \in \mathscr{B}$, such that $w'(t) = w(t)$ for $t < 0$ and $w'(t) \to 0$ for $t \to \infty$;
(iii) *autonomous* if $w_1,w_2 \in \mathscr{B}$ and $w_1(t) = w_2(t)$ for $t < 0$ implies $w_1 = w_2$;
(iv) *stable* if $w \in \mathscr{B}$ implies $w(t) \to 0$ for $t \to \infty$;
(v) *memoryless* if $\mathtt{n}(\mathscr{B}) = 0$;
(vi) *dead-beat* if $\mathscr{B} = \{0\}$.

The latter two notions are added only for the sake of completeness.

The system $\mathscr{B}_{\text{controllable}} \in \mathscr{L}^{\mathtt{w}}$ is called the *controllable part* of $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ if (i) $\mathscr{B}_{\text{controllable}} \subseteq \mathscr{B}$, (ii) $\mathscr{B}_{\text{controllable}}$ is controllable, and (iii) $[\![\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}, \mathscr{B}'$ controllable , and $\mathscr{B}' \subseteq \mathscr{B}]\!] \Rightarrow [\![\mathscr{B}_{\text{controllable}} \subseteq \mathscr{B}']\!]$. In words, $\mathscr{B}_{\text{controllable}}$ is the largest controllable system contained in $\mathscr{B}$. It is well known that $\mathscr{B}_{\text{controllable}}$ exists.

The controllable part induces an equivalence relation, called *controllability equivalence*, on $\mathscr{L}^{\mathtt{w}}$ by setting $[\![\mathscr{B}' \sim_{\text{controllability}} \mathscr{B}'']\!] :\Leftrightarrow [\![\ \mathscr{B}'_{\text{controllable}} = \mathscr{B}''_{\text{controllable}}]\!]$. It is easy to prove that $\mathscr{B}'_{\text{controllable}} = \mathscr{B}''_{\text{controllable}}$ if and only if $\mathscr{B}'$ and $\mathscr{B}''$ have the same compact support trajectories.

It can be shown that the system $G\left(\frac{d}{dt}\right)w = 0$, where $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and $F\left(\frac{d}{dt}\right)G\left(\frac{d}{dt}\right)w = 0$ are controllability equivalent if $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ is square and nonsingular. Each equivalence class modulo controllability contains exactly one controllable behavior, and its behavior contains all the other behaviors that belong that the equivalence class. In particular, two input/output systems have the same transfer function if and only if they are controllability equivalent.

The following result links controllability and stabilizability of systems in $\mathscr{L}^\bullet$ to the existence of left prime representations over the various subrings of $\mathbb{R}(\xi)$ introduced in the appendix.

Proposition 1: Let $\mathscr{B} \in \mathscr{L}^\bullet$.

1) $\mathscr{B}$ admits a representation ($\mathscr{R}$) with R of full row rank, and a representation ($\mathscr{G}$) with G of full row rank and $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{P}\mathscr{S}}$, that is, with all its elements proper and stable, meaning that they have no poles in $\overline{\mathbb{C}}_+$.
2) $\mathscr{B}$ admits a representation ($\mathscr{G}$) with G left prime over $\mathbb{R}(\xi)$, that is, with G of full row rank.
3) $\mathscr{B}$ is controllable if and only if it admits a representation ($\mathscr{R}$) with $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ left prime over $\mathbb{R}[\xi]$, that is, with $R(\lambda)$ of full row rank for all $\lambda \in \mathbb{C}$.
4) $\mathscr{B}$ admits a representation ($\mathscr{G}$) with G left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$, that is, with all its elements proper and G^∞ of full row rank.
5) $\mathscr{B}$ is stabilizable if and only if it admits a representation ($\mathscr{G}$) with $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{S}}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$, that is, with G of full row rank and no poles and no zeros in $\overline{\mathbb{C}}_+$.
6) $\mathscr{B}$ is stabilizable if and only if it admits a representation ($\mathscr{G}$) with $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{P}\mathscr{S}}$ left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$, that is, with G^∞ of full row rank and no poles and no zeros in $\overline{\mathbb{C}}_+$.
7) $\mathscr{B}$ is memoryless if and only if it admits a representation ($\mathscr{R}$) with $Rin\mathbb{R}^{\bullet\times\bullet}$, in which case R can be taken to be left prime over $\mathbb{R}$, that is with R a constant matrix of full row rank.

Proposition 1 illustrates how system properties can be translated into rational symbols. Roughly speaking, every $\mathscr{B} \in \mathscr{L}^\bullet$ has a full row rank polynomial and a full row rank proper and/or stable representation. As long as we allow a non-empty region where to put the poles, we can obtain a representation with the poles in that region. It is only when we allow no finite and no infinite poles that we are restricted to memoryless systems. The zeros of the representation are more significant, however. No zeros corresponds to controllability. Stable zeros corresponds to stabilizability.

Proposition 1 spells out exactly what the condition is for the existence of a kernel representation that is left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$: *stabilizability*. It is of interest to compare this with the classical results obtained by Vidyasagar in his book [6] (this builds on a series of earlier results, for example [3], [9], [2]). In these publications, the aim is to obtain a representation of a system that is given as a transfer function to start with,

$$y = F\left(\tfrac{d}{dt}\right)u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

where $F \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}$. This is a special case of ($\mathscr{G}$), and, since $\begin{bmatrix} I_{\mathtt{p}} & -F \end{bmatrix}$ has no zeros, $y = F\left(\frac{d}{dt}\right)u$ is controllable, and therefore stabilizable. Thus, by proposition 1, it also admits a representation $G_1\left(\frac{d}{dt}\right)y = G_2\left(\frac{d}{dt}\right)u$ with $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{P}\mathscr{S}}$, and left coprime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$. This is an important, classical, result. However, in the controllable case, we obtain a representation that is left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$, and such that $\begin{bmatrix} G_1 & G_2 \end{bmatrix}$ has no zeros at all.

The main difference of our result from the classical left coprime factorization results over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$ is that we faithfully preserve controllability, or, more generally, the exact behavior, and not only the controllable part of a behavior, whereas in the classical approach all stabilizable systems with the same transfer function are identified. Loosely speaking, left coprime factorizations over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$ of a transfer function avoid unstable pole-zero cancellations. Our approach avoids altogether introducing common poles and zeros as well as pole-zero cancellations. Since the whole issue of coprime factorizations over the ring of stable rational functions started from a need to deal carefully with pole-zero cancellations [9], we feel that our trajectory based mode of thinking offers a useful point of view.

V. Interconnection

Let $\Sigma_1 = (\mathbb{T}, \mathbb{W}, \mathscr{B}_1)$ and $\Sigma_2 = (\mathbb{T}, \mathbb{W}, \mathscr{B}_2)$ be two dynamical systems. The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B}_1 \cap \mathscr{B}_2) =: \Sigma_1 \wedge \Sigma_2$ is called the *interconnection* of Σ_1 and Σ_2. This definition signifies that in the interconnected system, w has to obey both the laws of Σ_1 and Σ_2.

Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$. Their interconnection has behavior $\mathscr{B}_1 \cap \mathscr{B}_2$, and obviously $\mathscr{B}_1 \cap \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$. This interconnection is said to be *regular* if

$$\boxed{\mathsf{p}(\mathscr{B}_1 \cap \mathscr{B}_2) = \mathsf{p}(\mathscr{B}_1) + \mathsf{p}(\mathscr{B}_2)}$$

Since $\mathsf{m}(\mathscr{B}_1 + \mathscr{B}_2) = \mathsf{m}(\mathscr{B}_1) + \mathsf{m}(\mathscr{B}_2) - \mathsf{m}(\mathscr{B}_1 \cap \mathscr{B}_2)$, regularity is equivalent to $\mathsf{m}(\mathscr{B}_1 + \mathscr{B}_2) = \mathtt{w}$, and therefore to

$$\boxed{\mathscr{B}_1 + \mathscr{B}_2 = \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})}$$

It is said to be *superregular* if *in addition*

$$\boxed{\mathsf{n}(\mathscr{B}_1 + \mathscr{B}_2) = \mathsf{n}(\mathscr{B}_1) + \mathsf{n}(\mathscr{B}_2)}$$

The significance of these concepts may be explained as follows. Regularity of the interconnection means that the interconnection does not put conditions on the exogenous variables that may be present in the systems $\mathscr{B}_1$ and $\mathscr{B}_2$. It ensures that the exogenous behavior is left unchanged by the interconnection. See [1, section VII] for an explanation of this. Superregularity means that the interconnection can take place at any moment in time. Precisely, the interconnection of $\mathscr{B}_1 \in \mathscr{L}^{\mathtt{w}}$ and $w_2 \in \mathscr{L}^{\mathtt{w}}$ is superregular if and only if for all $w_1 \in \mathscr{B}_1, w_2 \in \mathscr{B}_2$, there exists a $w \in (\mathscr{B}_1 \cap \mathscr{B}_2)^{\text{closure}}$ such that $w_1', w_2' \in \mathscr{B}_1 \cap \mathscr{B}_2$, with w_1' and w_2' defined by

$$w_1'(t) = \begin{cases} w_1(t) & \text{for } t \leq 0 \\ w(t) & \text{for } t > 0 \end{cases},$$

and

$$w_2'(t) = \begin{cases} w_2(t) & \text{for } t \leq 0, \\ w(t) & \text{for } t > 0 \end{cases}.$$

Hence, for a superregular interconnection, any distinct past histories in $\mathscr{B}_1$ and $\mathscr{B}_2$ can harmoniously be continued as the same future trajectory in $\mathscr{B}_1 \cap \mathscr{B}_2$. In [7] it has been shown that superregular interconnection can also be viewed as feedback interconnection.

We now prove a proposition about (super)regularity in terms of left-prime kernel representations with rational symbols.

Proposition 2: Consider $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$. Let $G_{\mathtt{k}}\left(\frac{d}{dt}\right) w = 0, \mathtt{k} = 1, 2,\ G_{\mathtt{k}} \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, be a (matrix of rational functions based) kernel representation of $\mathscr{B}_{\mathtt{k}}$.

1. Assume that the $G_{\mathtt{k}}$'s are left prime over $\mathbb{R}(\xi)$ (by proposition 1 such representations exist). $\mathscr{B}_1 \cap \mathscr{B}_2$ is a regular interconnection if and only if

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \text{ is also left prime over } \mathbb{R}(\xi),$$

that is, if and only if

$$\operatorname{rank}\left(\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}\right) = \operatorname{rank}(G_1) + \operatorname{rank}(G_2).$$

2. Assume that the $G_{\mathtt{k}}$'s are left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$ (we have seen that such representations exist). $\mathscr{B}_1 \cap \mathscr{B}_2$ is a superregular interconnection if and only if

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \text{ is also left prime over } \mathbb{R}(\xi)_{\mathscr{P}},$$

that is, if and only if

$$\operatorname{rank}\left(\begin{bmatrix} G_1^\infty \\ G_2^\infty \end{bmatrix}\right) = \operatorname{rank}(G_1^\infty) + \operatorname{rank}(G_2^\infty).$$

In the next proposition, we establish the conditions for stability of (super)regular interconnections in terms of kernel representations.

Proposition 3: Consider $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}}$.

1. Assume that $\mathscr{B}_1$ and $\mathscr{B}_2$ are stabilizable. Let $G_{\mathtt{k}}\left(\frac{d}{dt}\right) w = 0, \mathtt{k} = 1, 2,\ G_{\mathtt{k}} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet \times \mathtt{w}}$, be a (matrix of rational functions based) kernel representation of $\mathscr{B}_{\mathtt{k}}$. Assume that the $G_{\mathtt{k}}$'s are left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$ (by proposition 1 such representations exist). Then $\mathscr{B}_1 \cap \mathscr{B}_2$ is a regular interconnection and stable if and only if $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ is square and unimodular over $\mathbb{R}(\xi)_{\mathscr{S}}$, that is, $\det(G)$ is miniphase.

2. Assume that $\mathscr{B}_1$ and $\mathscr{B}_2$ are stabilizable. Let $G_{\mathtt{k}}\left(\frac{d}{dt}\right) w = 0, \mathtt{k} = 1, 2,\ G_{\mathtt{k}} \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\bullet \times \mathtt{w}}$, be a (matrix of rational functions based) kernel representation of $\mathscr{B}_{\mathtt{k}}$. Assume that the $G_{\mathtt{k}}$'s are left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$ (by proposition 1 such representations exist). Then $\mathscr{B}_1 \cap \mathscr{B}_2$ is a superregular interconnection and stable if and only if $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ is square and unimodular over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$, that is, $\det(G)$ is biproper and miniphase.

3. Assume that $\mathscr{B}_1$ and $\mathscr{B}_2$ are controllable Let $R_{\mathtt{k}}\left(\frac{d}{dt}\right) w = 0, \mathtt{k} = 1, 2,\ R_{\mathtt{k}} \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$, be a (polynomial matrix based) kernel representation of $\mathscr{B}_{\mathtt{k}}$ with $R_{\mathtt{k}}$ of full row rank (by proposition 1 such representations exist). Then $\mathscr{B}_1 \cap \mathscr{B}_2$ is a regular interconnection and dead-beat if and only if $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ is square and unimodular over $\mathbb{R}[\xi]$, that is, $\det(R)$ is a non-zero constant.

VI. Control

We refer to [7], [1] for an extensive treatment of control in a behavioral setting. In terms of the notions introduced in these references, we shall be only concerned with *full* interconnection, meaning that the controller has access to all the system variables. We refer to [1] for a nice discussion of the concepts involved.

Let $\mathscr{B}$ (henceforth $\in \mathscr{L}^{\mathtt{w}}$) be called the *plant*, $\mathscr{C}$ (henceforth $\in \mathscr{L}^{\mathtt{w}}$) the *controller*, and their interconnection $\mathscr{B} \cap \mathscr{C}$ (hence also $\in \mathscr{L}^{\mathtt{w}}$), the *controlled system*. Call the controller (super)regular if the interconnection of the plant and the controller is (super)regular: the controller restricts the behavior of the plant to a subset. In [7], [1], the relevance of (super)regularity of the controller has been discussed. The classical input/state/output based sensor-output-to-actuator-input controllers that dominate the field of control are superregular. Controllers that are regular, but not superregular, are relevant in control, much more so than is appreciated, for example as PID controllers, or as control devices that do not act as sensor-output-to-actuator-input controllers.

$\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is said to be a *stabilizing* controller for the plant $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ if $\mathscr{B} \cap \mathscr{C}$ is stable. $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is said to be a *dead-beat* controller for the plant $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ if $\mathscr{B} \cap \mathscr{C}$ is dead-beat. The following proposition establishes that stabilizability is equivalent to the existence of a superregular stabilizing controller, and that controllability is equivalent to a regular dead-beat controller.

Proposition 4:

1. $[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is stabilizable$]\!]$
$\Leftrightarrow [\![\exists$ a regular controller $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ stabilizing $\mathscr{B}]\!]$
$\Leftrightarrow [\![\exists$ a superregular controller $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ stabilizing $\mathscr{B}]\!]$.
2. $[\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is controllable$]\!]$
$\Leftrightarrow [\![\exists$ a regular dead-beat controller $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}]\!]$.

The following corollary is an immediate consequence of this proposition.

Corollary 5:

1. Assume that $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. Then there exists $F \in \mathbb{R}(\xi)_{\mathscr{S}}^{(\mathtt{n}_2-\mathtt{n}_1)\times \mathtt{n}_2}$ such that

$$\begin{bmatrix} G \\ F \end{bmatrix} \text{ is } \mathbb{R}(\xi)_{\mathscr{S}}\text{-unimodular.}$$

2. Assume that $G \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ is left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$. Then there exists $F \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{(\mathtt{n}_2-\mathtt{n}_1)\times \mathtt{n}_2}$ such that

$$\begin{bmatrix} G \\ F \end{bmatrix} \text{ is } \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}\text{-unimodular.}$$

3. Assume that $R \in \mathbb{R}[\xi]^{\mathtt{n}_1 \times \mathtt{n}_2}$ is left prime over $\mathbb{R}[\xi]$. Then there exists $F \in \mathbb{R}[\xi]^{(\mathtt{n}_2-\mathtt{n}_1)\times \mathtt{n}_2}$ such that

$$\begin{bmatrix} G \\ F \end{bmatrix} \text{ is } \mathbb{R}[\xi]\text{-unimodular.}$$

VII. Parametrization of the set of regular stabilizing, superregular stabilizing, and dead-beat controllers

A. Regular stabilizing controllers

In this section, we parametrize the set of regular controllers that stabilize a given stabilizable plant.

Step 1. The parametrization starts from a (matrix of rational functions based) kernel representation ($\mathscr{G}$) of the plant $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, assumed stabilizable. Assume that $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{p}(\mathscr{B})\times \mathtt{w}(\mathscr{B})}$ is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. By proposition 1, such a representation exists.

Step 2. Construct a $G' \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times \mathtt{w}(\mathscr{B})}$ such that $\begin{bmatrix} G \\ G' \end{bmatrix}$ is $\mathbb{R}(\xi)_{\mathscr{S}}$-unimodular. By corollary 5, such a G' exists.

Step 3. The set of regular stabilizing controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is given as the systems with (matrix of rational functions based) kernel representation $C(\frac{d}{dt})w = 0$, where

$$\boxed{C = F_1 G + F_2 G'}$$

with $F_1 \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times \mathtt{p}(\mathscr{B})}$ free, and $F_2 \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times \mathtt{m}(\mathscr{B})}$ $\mathbb{R}(\xi)_{\mathscr{S}}$-unimodular, that is, $\det(F_2)$ miniphase.

Step 3'. This parametrization may be further simplified using controllability equivalence, by identifying controllers that have the same controllable part, that is, by considering controllers up to controllability equivalence. The set of controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ with kernel representation $C(\frac{d}{dt})w = 0$ and C of the form

$$\boxed{C = FG + G'}$$

with $F \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times \mathtt{p}(\mathscr{B})}$ is free, contains an element of the equivalence class modulo controllability of each regular stabilizing controller for $\mathscr{B}$.

Proof of the parametrization: Note that, since $\begin{bmatrix} G \\ G' \end{bmatrix}$, is $\mathbb{R}(\xi)$-unimodular, *any* $C \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ can be written as

$$C = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} G \\ G' \end{bmatrix},$$

for some $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet \times \mathtt{w}}$. Take C left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. By proposition 1, such a representation exists, since a stabilizing controller is obviously stabilizable. Then also F is left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. Taken as a controller (since we are only interested in stabilizing controllers, we only need to consider stabilizable controllers), $C\left(\frac{d}{dt}\right) w = 0$ leads to the controlled system

$$\begin{bmatrix} I & 0 \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} G \\ G' \end{bmatrix} \left(\tfrac{d}{dt}\right) w = 0.$$

By proposition 3, this controller is stabilizing and regular if and only if F_2 is $\mathbb{R}(\xi)_{\mathscr{S}}$-unimodular. This yields the parametrization. To obtain step 3', observe that the controller $\begin{bmatrix} F_2^{-1}F_1 & I \end{bmatrix} \begin{bmatrix} G \\ G' \end{bmatrix} \left(\frac{d}{dt}\right) w = 0$ is controllability equivalent to $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} G \\ G' \end{bmatrix} \left(\frac{d}{dt}\right) w = 0$. ∎

B. Superregular stabilizing controllers

In this section, we parametrize the set of regular controllers that stabilize a given stabilizable plant.

Step 1. The parametrization starts from a (matrix of rational functions based) kernel representation ($\mathscr{G}$) of the plant $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, assumed stabilizable. Assume that $G \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{p}(\mathscr{B})\times\mathtt{w}(\mathscr{B})}$ is left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$. By proposition 1, such a representation exists.

Step 2. Construct a $G' \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times\mathtt{w}(\mathscr{B})}$ such that $\begin{bmatrix} G \\ G' \end{bmatrix}$ is $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$-unimodular. By corollary 5, such a G' exists.

Step 3. The set of regular stabilizing controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is given as the systems with (matrix of rational functions based) kernel representation $C(\frac{d}{dt})w = 0$, where

$$\boxed{C = F_1 G + F_2 G'}$$

with $F_1 \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times\mathtt{p}(\mathscr{B})}$ free, and $F_2 \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times\mathtt{m}(\mathscr{B})}$ $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$-unimodular, that is, $\det(F_2)$ biproper and miniphase.

Step 3'. This parametrization may be further simplified using controllability equivalence, by identifying controllers that have the same controllable part, that is, by considering controllers up to controllability equivalence. The set of controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ with kernel representation $C(\frac{d}{dt})w = 0$ and C of the form

$$\boxed{C = FG + G'}$$

with $F \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{B})\times\mathtt{p}(\mathscr{B})}$ is free, contains an element of the equivalence class modulo controllability of each superregular stabilizing controller for $\mathscr{B}$.

This parametrization is proven in the same way as the regular case.

C. Regular dead-beat controllers

In this section, we parametrize the set of regular dead-beat controllers for a given plant.

Step 1. The parametrization starts from a (polynomial matrix based) kernel representation ($\mathscr{R}$) of the plant $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, assumed controllable. Assume that $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, that is, that R is left prime over $\mathbb{R}[\xi]$. By controllability of $\mathscr{B}$, such a kernel representation exists.

Step 2. Construct an $R' \in \mathbb{R}[\xi]^{\mathtt{m}(\mathscr{B})\times\mathtt{w}(\mathscr{B})}$ such that $\begin{bmatrix} R \\ R' \end{bmatrix}$ is $\mathbb{R}[\xi]$-unimodular. Since R is left prime over $\mathbb{R}[\xi]$, such an R' exists.

Step 3. The set of regular dead-beat controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is given as the systems with (polynomial matrix based) kernel representation $C(\frac{d}{dt})w = 0$, where

$$\boxed{C = FC + C'}$$

$F \in \mathbb{R}[\xi]^{\mathtt{m}(\mathscr{B})\times\mathtt{p}(\mathscr{B})}$ free.

Proof of the parametrization: Note that, since $\begin{bmatrix} R \\ R' \end{bmatrix}$, is $\mathbb{R}[\xi]$-unimodular, *any* $C \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$ can be written as

$$C = \begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} R \\ R' \end{bmatrix},$$

for some $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \in \mathbb{R}[\xi]^{\bullet\times\mathtt{w}}$. If $C\left(\frac{d}{dt}\right)w = 0$ defines a regular dead-beat controller, C can be taken to be left prime over $\mathbb{R}[\xi]$. Then F is also left prime over $\mathbb{R}[\xi]$. Taken as a controller $C\left(\frac{d}{dt}\right)w = 0$ leads to the controlled system

$$\begin{bmatrix} I & 0 \\ F_1 & F_2 \end{bmatrix} \begin{bmatrix} R \\ R' \end{bmatrix} \left(\tfrac{d}{dt}\right) w = 0.$$

By proposition 3, this controller is dead-beat and regular if and only if F_2 is $\mathbb{R}[\xi]$-unimodular. the controller $\begin{bmatrix} F_2^{-1}F_1 & I \end{bmatrix} \begin{bmatrix} R \\ R' \end{bmatrix} \left(\frac{d}{dt}\right) w = 0$ has the same behavior as $\begin{bmatrix} F_1 & F_2 \end{bmatrix} \begin{bmatrix} G \\ G' \end{bmatrix} \left(\frac{d}{dt}\right) w = 0$. This yields the parametrization. ∎

Example: To illustrate the parametrizations obtained above, consider the plant $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$, and the superregular stabilizing controller $w_2 + \alpha \frac{d}{dt} w_2 = 0$, with $\alpha \geq 0$.

Take $\begin{bmatrix} 0 & 1 \end{bmatrix}$ for G' in the parametrizations. The set of (super)regular stabilizing controllers is given by $C\left(\frac{d}{dt}\right)w_2 = 0$, with $C \in \mathbb{R}$ miniphase in the regular case, and miniphase and biproper in the superregular case. Taking $F_2(\xi) = (1+\alpha\xi)/(1+2\alpha\xi)$, for example, yields the controller $w_2 + \alpha\frac{d}{dt}w_2 = 0$, with $\alpha \geq 0$. The parametrization in step 3' yields only the controller $w_2 = 0$, which is indeed the controllable part of these controllers. This example illustrates that the parametrization in step 3' does not yield all the (super)regular stabilizing controllers, although it yields all the stabilizing controller transfer functions.

VIII. Concluding Remark

The parametrization of superregular stabilizing controllers is identical to what in the classical literature is called the Kučera-Youla parametrization of stabilizing controllers. But, there are two differences.

The first difference is that in the classical theory all systems with the same transfer function are identified, whereas we take carefully the uncontrollable modes into account. In a transfer-function based theory every system is considered to be controllable, and uncontrollable (stable) modes can be added and cancelled at will.

The second difference is the stability concept used. In the classical setting the interconnection of $\mathscr{B}$ and $\mathscr{C}$ is defined to be stable if the system obtained by injecting (artificial) arbitrary inputs at the interconnection terminals is bounded-input/bounded-output stable. Our stability definition requires that $w(t) \to 0$ for $t \to \infty$ in the interconnected behavior $\mathscr{B} \cap \mathscr{C}$. It turns out that bounded-input/bounded-output stability requires (i) our stability, combined with (ii) superregularity. Interconnections that are not superregular cannot be bounded-input/bounded-output stable. However, for physical systems these concepts (stability and superregularity) are quite unrelated. For example the harmonic oscillator $M\frac{d^2}{dt^2}w_1 + Kw_1 = w_2$, with $M, K > 0$ parameters, is obviously stabilized by the damper $w_2 = -D\frac{d}{dt}w_1$ if $D > 0$. In our opinion, it makes little sense to call the interconnection unstable, just because the interconnection is not superregular.

IX. Appendix: Notation and Nomenclature

We use the standard symbols $\mathbb{R}, \mathbb{N}, \mathbb{Z}$, etc. $\mathbb{C}$ denotes the complex numbers, $\mathbb{C}_- := \{s \in \mathbb{C} \mid \mathrm{Re}(s) < 0\}$ the open left half, and $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 0\}$ the closed right half of the complex plane. When the number of rows or columns is immaterial (but finite), we use $^\bullet$, $^{\bullet\times\bullet}$, etc. Of course, when we then add, multiply, etc., we assume that the dimensions are compatible. $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$. The notation rank, det, ker, degree, diag, etc. is self-explanatory.

$\mathbb{R}[\xi]$ denotes the set of polynomials with real coefficients and $\mathbb{R}(\xi)$ the set of real rational functions in the indeterminate ξ. $p_1, p_2 \in \mathbb{R}[\xi]$ are said to be *coprime* if they have no common roots. $p \in \mathbb{R}[\xi]$ is said to be *Hurwitz* if it has no roots in $\overline{\mathbb{C}}_+$. The *relative degree* of $f \in \mathbb{R}(\xi), f = n/d$, with $n, d \in \mathbb{R}[\xi]$, is the degree of the denominator d minus the degree of the numerator n; $f \in \mathbb{R}(\xi)$ is said to be *proper* if the relative degree is ≥ 0, *strictly proper* if it is > 0, and *biproper* if it is equal to 0. $f \in \mathbb{R}(\xi), f = n/d$, with $n, d \in \mathbb{R}[\xi]$ coprime, is said to be *stable* if d is Hurwitz, and *miniphase* if n and d are both Hurwitz. More general stability domains are of interest, but we stick with the 'Hurwitz' domain for the sake of concreteness.

The definition of the behavior of $(\mathscr{G})$ involves left coprime polynomial matrix factorizations of elements of $\mathbb{R}(\xi)^{\bullet\times\bullet}$. These factorizations are reviewed in [8], along with the *Smith-McMillan form*, and *poles* and *zeros* of matrices of real rational functions.

Several subrings of $\mathbb{R}(\xi)$ play an important role in this article, namely,

1) $\mathbb{R}(\xi)$ itself, the rational functions;
2) $\mathbb{R}[\xi]$, the polynomials;
3) $\mathbb{R}(\xi)_{\mathscr{P}}$, the set elements of $\mathbb{R}(\xi)$ that are proper;
4) $\mathbb{R}(\xi)_{\mathscr{S}}$, the set elements of $\mathbb{R}(\xi)$ that are stable;
5) $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}} = \mathbb{R}(\xi)_{\mathscr{P}} \cap \mathbb{R}(\xi)_{\mathscr{S}}$, the proper stable rational functions;
6) $\mathbb{R}$, the reals.

The last ring is added for the sake of completeness. The notation is different from the one used in [8]. We can think of these subrings in terms of poles. Indeed, these subrings are characterized by, respectively, arbitrary poles, no finite poles, no poles at $\{\infty\}$, no poles in $\overline{\mathbb{C}}_+$, no poles in $\overline{\mathbb{C}}_+ \cup \{\infty\}$, and no poles in $\mathbb{C} \cup \{\infty\}$. It is easy to identify the unimodular elements (that is, the elements that have an inverse in the ring) of these rings. They consist of, respectively, the nonzero elements, the non-zero constants, the biproper elements, the miniphase elements, the biproper miniphase elements of $\mathbb{R}(\xi)$, and the non-zero constants.

We also consider matrices over these rings. Call an element of $\mathbb{R}(\xi)^{\bullet\times\bullet}$ *proper*, *stable*, or *proper stable* if each of its entries is. The square matrices over these rings are unimodular if and only if their determinant is unimodular. For $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\bullet\times\bullet}$, define $M^\infty := \lim_{x\in\mathbb{R}, x\to\infty} M(x)$. $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$ is called *biproper* if $\det(\mathscr{M}^\infty) \neq 0$. The unimodular elements of $\mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$ are the biproper ones.

Let $\mathscr{R}$ denote any of the rings $\mathbb{R}(\xi)$, $\mathbb{R}[\xi]$, $\mathbb{R}(\xi)_{\mathscr{P}}$, $\mathbb{R}(\xi)_{\mathscr{S}}$, $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}} = \mathbb{R}(\xi)_{\mathscr{P}} \cap \mathbb{R}(\xi)_{\mathscr{S}}$, or $\mathbb{R}$. $M \in \mathscr{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be *left prime* over $\mathscr{R}$ if for every factorization of M the form $M = FM'$ with $F \in \mathscr{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ and $M' \in \mathscr{R}^{\mathtt{n}_1\times\mathtt{n}_2}$, F is unimodular over $\mathscr{R}$. It is easy to characterize left-prime elements. $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ is left prime over $\mathscr{R}$ if and only if

1) M is of full row rank if $\mathscr{R} = \mathbb{R}(\xi)$,
2) $M \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$ and $M(\lambda)$ is of full row rank for all $\lambda \in \mathbb{C}$ if $\mathscr{R} = \mathbb{R}[\xi]$,
3) $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1\times\mathtt{n}_2}$ and M^∞ is of full row rank if $\mathscr{R} = \mathbb{R}(\xi)_{\mathscr{P}}$,
4) M is of full row rank and has no poles and no zeros in $\overline{\mathbb{C}}_+$ if $\mathscr{R} = \mathbb{R}(\xi)_{\mathscr{S}}$,
5) $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1\times\mathtt{n}_2}$, M^∞ is of full row rank, and M has no poles and no zeros in $\overline{\mathbb{C}}_+$, if $\mathscr{R} = \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$.
6) $M \in \mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ is of full row rank if $\mathscr{R} = \mathbb{R}$.

The matrices of rational functions $M_1, M_2, \ldots, M_{\mathtt{n}} \in \mathscr{R}$ are said to be *left coprime over* $\mathscr{R}$ if $M = \mathrm{row}(M_1, M_2, \ldots, M_{\mathtt{n}})$, is left prime over $\mathscr{R}$.

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