THE SUM-OF-SQUARES PROBLEM AND DISSIPATIVE SYSTEMS

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ABSTRACT

In this presentation, we discuss the theory of dissipativeness of systems described by linear constant coefficient PDE's with respect to supply rates that are quadratic differential forms in the variables and their derivatives. The main issue considered is the equivalence of global and local dissipativeness. This leads to the construction of the storage function, the flux, and the dissipation rate. We show that mathematically this leads to Hilbert's 17-th problem on the factorization of a polynomial in n variables as a sum of squares.

1. INTRODUCTION

The notion of a dissipative system is among one of the more useful concepts in systems theory. It may be viewed as a natural generalization of a Lyapunov function to 'open' systems. Many results involving stability of systems and design of robust controllers make use of this notion. The theory of dissipative systems has been developed mainly for systems that have time as its only independent variable (1-D systems). However, models of physical systems often have several independent variables (i.e., they are n-D systems), for example, time and space variables. In this paper we develop the theory of dissipative systems for n-D systems.

The central problem in the theory of dissipative systems is the construction of a storage function. Examples of storage functions in the literature include Lyapunov functions in stability analysis and internal energy, entropy in thermodynamics, etc. The construction of such storage functions for 1-D systems is well understood for general nonlinear systems and for linear systems with quadratic supply rates. In this paper, we obtain analogous results for n-D systems that are described by constant coefficient partial differential equations with quadratic differential forms as supply rates. However, there are some important differences between the theory involved in 1-D systems and the one applicable to n-D systems. The most important one being the dependence of storage functions on the unobservable (or hidden) latent variables.

First, a few words about notation. We use the standard notation $\mathbb{R}^n$, $\mathbb{R}^{n_1 \times n_2}$ etc., for finite-dimensional vectors and matrices. When the dimension is not specified (but, of course, finite), we write $\mathbb{R}^\bullet$, $\mathbb{R}^{n\times\bullet}$, $\mathbb{R}^{\bullet\times\bullet}$, etc. In order to enhance readability, we typically use the notation $\mathbb{R}^{\mathtt{w}}$ when functions taking their values in that vector space are denoted by w. Real polynomials in the indeterminates $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ are denoted by $\mathbb{R}[\xi]$ and real rational functions by $\mathbb{R}(\xi)$, with obvious modifications for the matrix case. The space of infinitely differentiable functions with domain $\mathbb{R}^n$ and co-domain $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$, and its subspace consisting of elements with compact support by $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.

2. n-D SYSTEMS

We view a system as a triplet $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$. Here $\mathbb{T}$ is the set of 'independent' variables (for example time, space, time and space). $\mathbb{W}$ stands for the set of 'dependent' variables, i.e., where the variables take on their values – often called the signal space or the space of field variables. Finally the *behavior* $\mathscr{B}$ is viewed as a subset of the family of all trajectories that map the set of independent variables into the set of dependent variables. The behavior $\mathscr{B}$ consists of the set of admissible trajectories that satisfy the system laws (for example, the set of partial differential equations that constitute the system laws). In this paper, we consider systems with $\mathbb{T} = \mathbb{R}^n$ (n-D systems). We assume throughout that $\mathbb{W}$ is a finite dimensional real vector space, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$.

We look at behaviors that arise as a consequence of a system of PDE's. More precisely, if there exists a real polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ in n indeterminates, $\xi = (\xi_1, \ldots, \xi_n)$, then we consider $\mathscr{B}$ to be the $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$-solutions of

$$R(\frac{d}{d\mathbf{x}})w = 0. \tag{1}$$

This equation reflects the multi-index notation with $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \frac{\partial^{k_2}}{\partial x_2^{k_2}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$. The $\mathscr{C}^\infty$ assumption on the solutions is made for ease of exposition. The results of this paper also hold for other solution concepts like distributions, though the mathematics needed is more involved. Systems $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$

that are defined by a set of constant coefficient PDE's (or equivalently, behaviors that arise as a consequence of a set of constant coefficient PDE's) will be called differential systems and denoted as $\mathscr{L}_n^{\mathtt{w}}$. We often abuse the notation by stating $\mathscr{B} \in \mathscr{L}_n^{\mathtt{w}}$, as the indexing set and the signal space are then obvious from the notation.

Whereas we have *defined* the behavior of a system in $\mathscr{L}_n^{\mathtt{w}}$ as the set of solutions of a system of PDE's in the system variables, often, in applications, the specification of the behavior involves other, auxiliary variables, which we call *latent variables*. Specifically, consider the system of PDE's

$$R(\frac{d}{dx})w = M(\frac{d}{dx})\ell \tag{2}$$

with $w \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and $\ell \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell})$ and with $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\bullet\times\ell}[\xi]$ polynomial matrices with the same number of rows. The set

$$\mathscr{B}_f = \{(w,\ell) \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\ell}) \mid (2) \text{ holds}\} \tag{3}$$

obviously belongs to $\mathscr{L}_n^{\mathtt{w}+\ell}$. It follows from a classical result in the theory of PDE's, *the fundamental principle*, that the set

$$\{w \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell}) : (w,\ell) \in \mathscr{B}_f\} \tag{4}$$

also belongs to $\mathscr{L}_n^{\mathtt{w}}$. We call (2) a latent variable representation with manifest variables w and latent variables ℓ, of the system with *full* behavior (3) and *manifest* behavior (4). Correspondingly, we call (1) a kernel representation of the system with the behavior $\ker(R(\frac{d}{d\mathbf{x}}))$. We shall soon meet another sort of representation, the image representations, in the context of controllability.

3. CONTROLLABILITY AND OBSERVABILITY

Two very influential classical properties of dynamical systems are those of controllability and observability. Their generalization to behavioral systems leads to even more appealing concepts (see [5] and [3] for generalizations to n-D systems). We discuss these concepts here exclusively in the context of systems described by linear constant coefficient PDE's.

Definition 1: A system $\mathscr{B} \in \mathscr{L}_n^{\mathtt{w}}$ is said to be *controllable* if for all $w_1, w_2 \in \mathscr{B}$ and for all sets $U_1, U_2 \subset \mathbb{R}^n$ with disjoint closure, there exists a $w \in \mathscr{B}$ such that $w\,|_{U_1} = w_1\,|_{U_1}$ and $w\,|_{U_2} = w_2\,|_{U_2}$
Thus controllable PDE's are those in which the solutions can be 'patched up' from solutions on subsets.

Though there are several characterizations and tests of controllability, the characterization that is important for the purposes of this paper is the equivalence of controllability with the existence of an image representation. Consider the following special latent variable representation

$$w = M(\frac{d}{d\mathbf{x}})\ell \tag{5}$$

with $M \in \mathbb{R}^{\mathtt{w}\times\ell}[\xi]$. Obviously, by the elimination theorem, its manifest behavior $\mathscr{B} \in \mathscr{L}_n^{\mathtt{w}}$. Such special latent variable representations often appear in physics, where the latent variables involved in such a representation are called *potentials*. Obviously $\mathscr{B} = \mathrm{im}(M(\frac{d}{d\mathbf{x}}))$ with $M(\frac{d}{d\mathbf{x}})$ viewed as a map from $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\ell})$ to $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. For this reason, we call (5) an *image representation* of its manifest behavior. Whereas every $\mathscr{B} \in \mathscr{L}_n^{\mathtt{w}}$ allows (by definition) a kernel representation and hence trivially a latent variable representation, not every $\mathscr{B} \in \mathscr{L}_n^{\mathtt{w}}$ allows an image representation. In fact:

Theorem 2: $\mathscr{B} \in \mathscr{L}_n^{\mathtt{w}}$ admits an image representation if and only if it is controllable.
We denote the set of controllable systems in $\mathscr{L}_n^{\mathtt{w}}$ by $\mathscr{L}_{n,\mathrm{cont}}^{\mathtt{w}}$.

Observability is the property of systems that have two kinds of variables – the first set of variables are the 'observed' variables, and the second set of variables are the ones that are 'to-be-deduced' from the observed variables.

Definition 3: Let $w = (w_1, w_2)$ be a partition of the variables in $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1+\mathtt{w}_2}, \mathscr{B})$. Then w_2 is said to be *observable* from w_1 in $\mathscr{B}$ if given any two trajectories $(w_1', w_2'), (w_1'', w_2'') \in \mathscr{B}$ such that $w_1' = w_1''$, then $w_2' = w_2''$.

A natural situation to use observability is when one looks at the latent variable representation of a behavior. Then one may ask whether the latent variables are observable from the manifest variables. If this is the case, then we call the latent variable representation *observable*.

As we have already mentioned, every controllable behavior has an image representation. Whereas every controllable behavior has an observable image representation in 1-D systems, this is no longer true for n-D systems.

4. QUADRATIC DIFFERENTIAL FORMS

In [6] the theory for QDF's has been developed for systems described by *one-variable* polynomial matrices. The appropriate tool to express quadratic functionals in the variables and their derivatives are *two-variable* polynomial matrices. In this paper we will use polynomial matrices in $2n$ variables to express quadratic functionals for functions of n variables.

For convenience, let ζ denote $(\zeta_1, \ldots, \zeta_n)$ and η denote $(\eta_1, \ldots, \eta_n)$. Let $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ denote the set of real polynomial matrices in the $2n$ indeterminates ζ and η. We will consider quadratic forms of the type $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$. Explicitly,

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}}$$

The sum above ranges over all non-negative multi-indices $\mathbf{k} = (k_1, k_2, \ldots, k_n)$, and $\mathbf{l} = (l_1, l_2, \ldots, l_n) \in \mathbb{N}^n$ and the sum is assumed to be finite, and $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$. The polynomial matrix Φ induces a *bilinear differential form* (BLDF), that is, the map

$$L_\Phi : \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1}) \times \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_2}) \to \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

defined by

$$L_\Phi(v,w)(\mathbf{x}) := \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}} v}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) \right)^T \Phi_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{l}} w}{d\mathbf{x}^{\mathbf{l}}}(\mathbf{x}) \right).$$

Note that ζ corresponds to differentiation of terms to the left and η refers to differentiation of the terms to the right.

If $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$, then Φ induces the *quadratic differential form* (QDF)

$$Q_\Phi : \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \to \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

defined by

$$Q_\Phi(w) := L_\Phi(w,w)$$

Define the * operator

$$^* : \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \to \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$$

by

$$\Phi^*(\zeta,\eta) := \Phi^T(\eta,\zeta)$$

where ^{T} denotes transposition. If $\Phi = \Phi^*$, then Φ is called *symmetric*. For the purposes of QDF's induced by polynomial matrices, it suffices to consider the symmetric QDF's, since $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi+\Phi^*)}$.

We also consider vectors $\Psi \in (\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta])^n$, i.e. $\Psi = (\Psi_1,\ldots,\Psi_n)$. Analogous to the quadratic differential form Φ, Ψ induces a *vector of QDF's* (VQDF)

$$Q_\Psi(w) : \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \to (\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}))^n$$

defined by $Q_\Psi = (Q_{\Psi_1},\ldots,Q_{\Psi_n})$. We define the 'div' (divergence) operator that associates with the VQDF induced by Ψ, the scalar QDF

$$(\operatorname{div} Q_\Psi)(w) := \frac{\partial}{\partial x_1} Q_{\Psi_1}(w) + \cdots + \frac{\partial}{\partial x_n} Q_{\Psi_n}(w)$$

5. LOSSLESS AND DISSIPATIVE SYSTEMS

Quadratic functionals play an important role in control theory. Quite often, the rate of supply of some physical quantity (for example, the rate of energy, i.e., the power) delivered to a system is given by a quadratic functional in the variables and their derivatives. We make use of QDF's to define such supply rates for controllable systems $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\mathrm{cont}}$.

Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. We consider the QDF $Q_\Phi(w)$ as a supply rate for trajectories $w \in \mathscr{B}$. More precisely, we consider $Q_\Phi(w)(\mathbf{x})$ (with $\mathbf{x} \in \mathbb{R}^n$) as the rate of supply of some physical quantity delivered to the system at the point $\mathbf{x}$. Thus, $Q_\Phi(w)(\mathbf{x})$ being positive implies that the system absorbs the physical quantity that is being supplied.

Definition 4: The system $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\mathrm{cont}}$ is said to be *lossless* with respect to the supply rate Q_Φ induced by $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ if

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} = 0$$

for all $w \in \mathscr{B} \cap \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ (i.e., trajectories in the behavior $\mathscr{B}$ with compact support).

The system $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\mathrm{cont}}$ is said to be *dissipative with respect to* Q_Φ (briefly Φ-dissipative) if

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0$$

for all $w \in \mathscr{B} \cap \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.

We now explain the physical interpretation of the definition above. $\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$ denotes the net amount of supply that the system absorbs integrated over 'time' and 'space'. So the system is lossless with respect to the QDF if this integral is zero, since any supply absorbed at some time or place is temporarily stored but eventually recovered (perhaps at some other time or space). On the other hand, if the integral is always non-negative, then the net amount of supply is absorbed, expressing is dissipativeness.

6. STORAGE FUNCTIONS FOR LOSSLESS SYSTEMS

We shall first look at lossless systems. The following theorem gives some equivalent conditions for a system to be lossless.

Theorem 5: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\mathrm{cont}}$. Let $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi]$ induce respectively a kernel and image representation of $\mathscr{B}$; i.e. $\mathscr{B} = \ker(R(\frac{d}{d\mathbf{x}})) = \operatorname{im}(M(\frac{d}{d\mathbf{x}}))$. Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induce a QDF on $\mathscr{B}$. Then the following conditions are equivalent :

1) $\mathscr{B}$ is lossless with respect to the QDF Q_Φ;
2) $\Phi'(-\xi,\xi) = 0$ where

$$\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta);$$

3) there exists a VQDF Q_Ψ, with $\Psi \in (\mathbb{R}^{m\times m}[\zeta,\eta])^n$, where m is the number of columns of M, such that

$$\operatorname{div} Q_\Psi(\ell) = Q_{\Phi'}(\ell) = Q_\Phi(w) \qquad (6)$$

for all $\ell \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

Note that the condition 1 in the above theorem states that $\mathscr{B}$ is lossless with respect to Q_Φ, i.e. that

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} = 0 \qquad (7)$$

for all $w \in \mathscr{B} \cap \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. This is a global statement about the concerned trajectory $w \in \mathscr{B}$. On the other hand, condition 3 of the above theorem states that $\mathscr{B}$ admits an image representation $w = M(\frac{d}{d\mathbf{x}})\ell$ and there exists some VQDF Ψ such that

$$\operatorname{div} Q_\Psi(\ell) = Q_\Phi(w) \qquad (8)$$

for all $w \in \mathscr{B}$ and ℓ such that $w = M(\frac{d}{d\mathbf{x}})\ell$. This statement gives a local characterization of losslessness. This equivalence of the global version of losslessness (7) with the local version (8) is a recurrent theme in the theory of dissipative systems.

The local version states that there is a function, $Q_\Psi(\ell)(\mathbf{x})$ that plays the role of amount of supply stored at $\mathbf{x} \in \mathbb{R}^n$. Thus (8) says that for lossless systems, it is *always* possible to define a *storage function* Q_Ψ such that the *conservation equation*

$$\text{div } Q_\Psi(\ell) = Q_\Phi(w) \tag{9}$$

is satisfied for all w, ℓ such that $w = M(\frac{d}{d\mathbf{x}})\ell$.

At this point it is worth emphasizing some basic differences between 1-D and n-D systems. Since every controllable 1-D behavior has an observable image representation, it can be shown that the conservation equation can be always rewritten as

$$\frac{d}{dt} Q_{\Psi'}(w) = Q_\Phi(w)$$

with some QDF $Q_{\Psi'}$ that acts directly on the manifest variables. Here t is assumed to be the independent variable for the 1-D behavior. On the other hand, since every controllable n-D behavior need not necessarily have an observable image representation, there may not exist any storage function of the form $Q_{\Psi'}(w)$, that depend only on the manifest variables. Thus, the storage function in the conservation equation (9) may involve 'hidden' (i.e., non-observable) variables. Another important difference between 1-D and n-D behaviors is the non-uniqueness of the vector of QDF's Q_Ψ involved in the conservation equation (9) for the n-D case. As a result of this non-uniqueness, there will be several possible storage functions in the n-D case that satisfy the conservation equation.

7. STORAGE FUNCTIONS FOR DISSIPATIVE SYSTEMS

As we have already seen in the context of lossless systems, the storage function is in general a function of the unobservable latent variables that appear in an image representation of the behavior $\mathscr{B}$. We now incorporate this in the definition and show later that the function Q_Ψ defined in the conservation equation (9) is indeed a storage function.

Definition 6: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\text{cont}}$, $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $w = M(\frac{d}{d\mathbf{x}})\ell$ be an image representation of $\mathscr{B}$ with $M \in \mathbb{R}^{\mathtt{w}\times\ell}[\xi]$. Let $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with $\Psi_k = \Psi_k^* \in \mathbb{R}^{\ell\times\ell}[\zeta,\eta]$ for $k = 1, 2, \ldots, n$. The VQDF Q_Ψ is said to be a *storage function* for $\mathscr{B}$ with respect to Q_Φ if

$$\text{div } Q_\Psi(\ell) \leq Q_\Phi(w) \tag{10}$$

for all $\ell \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^\ell)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.
$\Delta = \Delta^* \in \mathbb{R}^{\ell\times\ell}[\zeta,\eta]$ is said to be a *dissipation rate* for $\mathscr{B}$ with respect to Q_Φ if

$$Q_\Delta \geq 0 \text{ and } \int_{\mathbb{R}^n} Q_\Delta(\ell) d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$$

for all $\ell \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^\ell)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

By $Q_\Delta \geq 0$, we mean that $Q_\Delta(w(\mathbf{x})) \geq 0$ for all $w \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^w)$ and all $\mathbf{x} \in \mathbb{R}^n$. This defines a pointwise positivity condition. Thus $\int_\Omega Q_\Delta(w) d\mathbf{x} \geq 0$ for every $\Omega \subset \mathbb{R}^n$ if $Q_\Delta \geq 0$.

In the case of lossless systems, we had obtained the conservation equation

$$\text{div } Q_\Psi(\ell) = Q_\Phi(w)$$

Clearly, this Q_Ψ qualifies to be a storage function as it satisfies the inequality stated in the definition above.

From the above definitions, it is also easy to see that there is a relation between a storage function for $\mathscr{B}$ with respect to Q_Φ and a dissipation rate for $\mathscr{B}$ with respect to Q_Φ, given by

$$\text{div } Q_\Psi(\ell) = Q_\Phi(M(\frac{d}{d\mathbf{x}})\ell) - Q_\Delta(\ell) \tag{11}$$

The above definitions of the storage function and the dissipation rate, combined with (11), yield intuitive interpretations. The dissipation rate can be thought of as the rate of supply that is dissipated in the system and the storage function as the rate of supply stored in the system. Intuitively, we can think of the QDF Q_Φ as measuring the power going into the system. Φ-dissipativity then implies that the net power flowing into a system is non-negative which in turn implies that the system dissipates energy. Of course, locally the flow of energy could be positive or negative, leading to variations in $Q_\Psi(\ell)$ (in many practical situations $Q_\Psi(\ell)$ play the role of energy density and fluxes). If the system is dissipative, then the rate of change of energy density and fluxes cannot exceed the power delivered into the system. This is captured by inequality (10) in definition 6. The excess is precisely what is lost (or dissipated). This interaction between supply, storage and dissipation is formalized by the equation (11).

When the independent variables are time ($\partial t \in \mathbb{R}$) and space ($\mathbf{x} \in \mathbb{R}^{\not\models}$), we can rewrite (11) as

$$\frac{\partial \mathbf{U}(\ell)}{\partial t} = Q_\Phi(M(\frac{d}{d\mathbf{x}})\ell) - \nabla \cdot \mathbf{S}(\ell) - Q_\Delta(\ell) \tag{12}$$

where we substitute $Q_\Psi = (\mathbf{U}, \mathbf{S})$, with $\mathbf{U} = \Psi_t$ the stored energy, and $\mathbf{S} = (\Psi_x, \Psi_y, \Psi_z)$ the flux. Moreover $w = M(\frac{d}{d\mathbf{x}})\ell$. The above equation is reminiscent of energy balance equations that appear in several fields like fluid mechanics, thermodynamics, etc. Thus (12) states that the change in the stored energy ($\frac{\partial \mathbf{U}(\ell)}{\partial t}$) in an infinitesimal volume is exactly equal to the difference between the energy supplied ($Q_\Phi(w)$) into the infinitesimal volume and the energy lost by the infinitesimal volume by means of energy flux flowing out of the volume ($\nabla \cdot \mathbf{S}(\ell)$) and the energy dissipated ($Q_\Delta(\ell)$) within the volume.

8. THE SUM OF SQUARES PROBLEM AND THE STORAGE FUNCTION

The problem we now address is the equivalence of (i) dissipativeness of $\mathscr{B}$ with respect to Q_Φ, (ii) the existence of a storage function and (iii) the existence of a dissipation rate. Note that this problem also involves the construction of an appropriate image representation. We first consider the case where $\mathscr{B} = \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. In this case, the definition of the dissipation rate requires that for all $\ell \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^\ell)$

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Delta(\ell) d\mathbf{x} \qquad (13)$$

with $w = M(\frac{d}{d\mathbf{x}})\ell$; $M(\frac{d}{d\mathbf{x}})$ a surjective partial differential operator and $Q_\Delta(\ell) \geq 0$ for all $\ell \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^\ell)$. By stacking the variables and their various derivatives to form a new vector of variables, this latter condition is easily seen to be equivalent to the existence of a polynomial matrix $D \in \mathbb{R}^{\bullet \times \ell}[\xi]$ such that $\Delta(\zeta, \eta) = D^T(\zeta) D(\eta)$. Using Theorem 5, it follows that (13) is equivalent to the factorization equation

$$M^T(-\xi)\Phi(-\xi,\xi)M(\xi) = D^T(-\xi)D(\xi) \qquad (14)$$

A very well known problem in 1-D systems is that of *spectral factorization* which involves the factorization of a matrix $\Gamma(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ into the form

$$\Gamma(\xi) = F^T(-\xi)F(\xi)$$

with $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ (the matrix F is often required to satisfy some additional conditions like being Hurwitz, but that does not concern us here). It is well known that a polynomial matrix $\Gamma(\xi)$ in one variable ξ admits a solution $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ if and only if $\Gamma^T(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. The above factorization problem for n-D systems (14) is very similar in flavor. We can reformulate the problem as follows : given $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, a polynomial matrix in n commuting variables $\xi = (\xi_1, \ldots, \xi_n)$, is it possible to factorize it as

$$\Gamma(\xi) = F^T(-\xi)F(\xi) \qquad (15)$$

with $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ itself a polynomial matrix? Quite clearly, $\Gamma^T(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$ are necessary conditions for the existence of a factor $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. The important question is whether these conditions are also sufficient (as in the 1-D case).

If we consider the case when $\mathtt{w} = 1$ (the scalar case), substituting $i\omega$ for ξ, (15) reduces to finding F such that

$$\Gamma(i\omega) = F^T(-i\omega)F(i\omega)$$

Separating the real and imaginary parts of the above equation, the problem further reduces to the case of finding a sum of two squares which add up to a given positive (or non-negative) polynomial. Generally, in fact, the factorizability problem (15) can be reduced to the more ordinary looking question of factoring a given n-variable polynomial matrix $Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ as

$$Y(\xi) = X^\top(\xi)X(\xi). \qquad (16)$$

In the special case $\mathtt{w} = 1$, this is the question of writing a n-variable polynomial Y as a sum of squares:

$$Y = x_1^2 + x_2^2 + \cdots + x_k^2.$$

Obviously $Y = Y^\top$ and $Y(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}^n$ is a necessary condition for factorizability. In fact for $n = 1$, these conditions are sufficient as well. But our problem is the case $n \geq 1$.

This turns out to be a problem with a long history. It is Hilbert's 17^{th} problem, which deals with the representation of positive definite functions as sums of squares [2]. This investigation of positive definite functions began in the year 1888 with the following 'negative' result of Hilbert : If $f(\xi) \in \mathbb{R}[\xi]$ is a positive definite polynomial in n variables, then f need not be a sum of squares of polynomials in $\mathbb{R}[\xi]$, except in the case when $n = 1$. Several examples of such positive definite polynomials which cannot be expressed as sum of squares of polynomials are available in the literature, for example the polynomial

$$\xi_1^2\xi_2^2(\xi_1^2 + \xi_2^2 - 1) + 1$$

is not factorizable as a sum of squares of polynomials [1]. But (16) is factorizable when $Y = Y^\top$ and $Y(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}^n$ if we allow X to be a matrix of real rational functions.

Thus the two conditions that we mentioned earlier (namely $\Gamma^T(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$) are not sufficient to guarantee a polynomial factor $F \in \mathbb{R}^{\bullet\times\mathtt{w}}$ (even for the scalar case). However, we have the following result.

Theorem 7: Assume that $\Gamma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ satisfies $\Gamma^T(-\xi) = \Gamma(\xi)$ and $\Gamma(i\omega) \geq 0$ for all $\omega \in \mathbb{R}^n$. Then there exists an $F \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi)$ such that $\Gamma(\xi) = F^T(-\xi)F(\xi)$.

Note that even when Γ is a polynomial matrix, the entries of the matrix F are rational functions in n-indeterminates with real coefficients, whereas for the 1-D case one can obtain an F with polynomial entries. Combining the result of Theorem 7 along with the factorization problem (14), we obtain the following theorem. This result is a consequence of the SOS factorizability over the rational functions.

Theorem 8: Let $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Then the following conditions are equivalent:

1) $\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0$ for all $w \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$.
2) there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $M(\frac{d}{d\mathbf{x}})$ is surjective and $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ for $k = 1, 2, \ldots, n$ such that the VQDF Q_Ψ is a storage function, i.e.,

$$\text{div } Q_\Psi(\ell) \leq Q_\Phi(w)$$

for all $\ell \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and $w = M(\frac{d}{d\mathbf{x}})\ell$
3) there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $M(\frac{d}{d\mathbf{x}})$ is surjective and a $\Delta = \Delta^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that Q_Δ is a dissipation rate, i.e.,

$$Q_\Delta \geq 0 \text{ and } \int_{\mathbb{R}^n} Q_\Delta(\ell) d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x}$$

for all $\ell \in \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and $w = M(\frac{d}{d\mathbf{x}})\ell$
4) there exists a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $M(\frac{d}{d\mathbf{x}})$ is surjective, a $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with

$\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ for $k = 1,2,\ldots,n$ and a $\Delta = \Delta^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that

$$\begin{aligned} Q_\Delta &\geq 0 \\ &\text{and} \\ \operatorname{div} Q_\Psi(\ell) &= Q_\Phi(w) - Q_\Delta(\ell) \end{aligned} \tag{17}$$

for all $\ell \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and $w = M(\frac{d}{d\mathbf{x}})\ell$. Note that this states that the VQDF Q_Ψ is a storage function and that Q_Δ is a dissipation rate.

The above theorem considers the case when $\mathscr{B}$ is all of $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ and it shows the equivalence of dissipativeness of $\mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ with respect to Q_Φ, the existence of a storage function (Q_Ψ) and the existence of a dissipation rate (Q_Δ).

The important message of this theorem is the unavoidable emergence of latent variables in the dissipation equation (17) for n-D systems. Also note that the storage and dissipation functions that one obtains using the above theorem are not unique.

Finally, for an arbitrary controllable n-D behavior $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\text{cont}}$, the above theorem can be modified to obtain the following.

Theorem 9: Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{n,\text{cont}}$ and $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The following conditions are equivalent :

1) $\mathscr{B}$ is Φ-dissipative, i.e., $\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \geq 0$ for all $w \in \mathscr{B} \cap \mathscr{D}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$,
2) there exists an integer $\mathtt{l} \in \mathbb{N}$, a polynomial matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$ such that $M(\frac{d}{d\mathbf{x}})$ is an image representation of $\mathscr{B}$, a $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n)$ with $\Psi_k = \Psi_k^* \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ for $k = 1,2,\ldots,n$ and a $\Delta = \Delta^* \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ such that

$$\begin{aligned} Q_\Delta &\geq 0 \\ &\text{and} \\ \operatorname{div} Q_\Psi(\ell) &= Q_\Phi(w) - Q_\Delta(\ell) \end{aligned}$$

with $w = M(\frac{d}{d\mathbf{x}})\ell$.

For more details and proofs, we refer to [4].

9. CONCLUSIONS

In this paper, we dealt with n-D systems described by constant coefficient linear partial differential equations. We started by defining controllability for such systems, in terms of patching up of feasible trajectories. We then explained that it is exactly the controllable systems which allow an image representation, i.e., a representation in terms of what in physics is called a potential function.

Subsequently, we turned to lossless and dissipative systems. For lossless systems, we proved the equivalence with the existence of a conservation law involving the storage function. Important features of the storage function are (i) the fact that it depends on latent variables that are in general hidden (i.e., non-observable), and (ii) its non-uniqueness. For dissipative systems, we proved the equivalence with the existence a storage function and a dissipation rate. The problem of constructing a dissipation rate led to the question of factorizability of certain polynomial matrices in n variables.

We reduced this problem to Hilbert's 17-th problem, the representation of a non-negative rational function in n variables as a sum of squares of rational functions. The main point of this paper is to illustrate the immediate relevance of the sum-of-squares problem for n variable polynomials to the construction of storage functions for systems described by constant coefficient PDE's.

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