When is a Linear System Optimal?

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The following problem was presented during the open problem session. Its title is taken from a well-known seminal paper [1] by R.E. Kalman. The questions posed in [1] and here are similar in spirit, but the setting is quite different.

Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}(\zeta,\eta), \Phi(\zeta,\eta) = \Sigma_{\mathtt{k},\ell}\,\Phi_{\mathtt{k},\ell}\zeta^{\mathtt{k}}\eta^{\ell}$, with $\Phi_{\mathtt{k},\ell} = \Phi_{\ell,\mathtt{k}}^{\top} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, hence $\Phi = \Phi^{\star}$, where $\Phi^{\star}(\zeta,\eta) := \Phi(\eta,\zeta)^{\top}$. Denote by Q_Φ the 'quadratic differential form' [5] which maps as follows

$$w \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mapsto \sum_{\mathtt{k},\ell} (\frac{d^{\mathtt{k}}}{dt^{\mathtt{k}}}w)^{\top}\Phi_{\mathtt{k},\ell}(\frac{d^{\ell}}{dt^{\ell}}w) \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}).$$

Consider for $w,\Delta \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, with Δ of compact support, the integral

$$\int_{-\infty}^{\infty} \big(Q_\Phi(w+\Delta) - Q_\Phi(w)\big)\,dt. \qquad (1)$$

Expand (1) in a term which is bilinear in w,Δ, and one which is quadratic in Δ. We obtain

$$(1) \;= \int_{-\infty}^{\infty} \Delta^{\top}\big(\Phi(-\frac{d}{dt},\frac{d}{dt})w\big)\,dt + \int_{-\infty}^{\infty} Q_\Phi(\Delta)\,dt.$$

The trajectory $w \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is said to be *stationary* with respect to Φ, relative to variations Δ, if the linear term in Δ in (1) vanishes, i.e. if

$$\int_{-\infty}^{\infty} \Delta^{\top}\Phi(-\frac{d}{dt},\frac{d}{dt})w\,dt = 0$$

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for all $\Delta \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support. It is said to be *optimal* with respect to Φ, relative to variations Δ, if (1) is non-negative for all $\Delta \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support. Hence $w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is stationary if and only if

$$\Phi(-\frac{d}{dt},\frac{d}{dt})w = 0, \tag{2}$$

and optimal if and only if in addition

$$\int_{-\infty}^{\infty} Q_\Phi(\Delta)\,dt \geq 0 \tag{3}$$

for all $\Delta \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ of compact support. It is easy to prove (see [5]) that (3) holds if and only if the following frequency domain condition is satisfied:

$$\Phi(-i\omega, i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}. \tag{4}$$

Denote by $\mathscr{L}^{\mathtt{w}}$ the set of linear time-invariant differential systems in $\mathtt{w}$ variables, i.e. $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ means that $\mathscr{B} \subseteq \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ and that there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that

$$R(\frac{d}{dt})w = 0 \tag{5}$$

has $\mathscr{B}$ as its $\mathscr{C}^\infty$ solutions. Note that, while R specifies $\mathscr{B}$, the converse is not true (see [3]). The open problem is to

Characterize the behaviors $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ that are stationary or optimal.

In other words, under what conditions on $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ does there exist $\Phi = \Phi^\star \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $\mathscr{B} = \ker\left(\Phi(-\frac{d}{dt},\frac{d}{dt})\right)$? We are looking for conditions on $\mathscr{B}$ directly, more than on representations of $\mathscr{B}$. This open problem is, of course, an high-order analogue of a well-studied problem in the calculus of variations and classical mechanics. The ideas and results from [4] are very relevant and partly solve the problem stated above.

It is straightforward to settle the case $\mathtt{w} = 1$. In this case $\mathscr{B}$ is stationary if and only if either $\mathscr{B} = \mathscr{C}^\infty(\mathbb{R},\mathbb{R})$, or $\mathscr{B}$ is a finite dimensional subset of $\mathscr{C}^\infty(\mathbb{R},\mathbb{R})$ (i) of even dimension and (ii) time-reversible (in the sense that $t \in \mathbb{R} \mapsto w(t) \in \mathbb{R}$ belongs to $\mathscr{B}$ if and only if $t \in \mathbb{R} \mapsto w(-t) \in \mathbb{R}$ belongs to $\mathscr{B}$). It is optimal if and only if in the finite dimensional case the following additional condition on the oscillatory solutions holds: (iii) whenever $t \in \mathbb{R} \mapsto t^{\mathtt{k}} \sin \omega t$ belongs to $\mathscr{B}$ for some even integer $\mathtt{k}$ and some $\omega \in \mathbb{R}$, then also $t \in \mathbb{R} \mapsto t^{\mathtt{k}+1} \sin \omega t$ belongs to $\mathscr{B}$.

References

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