# ARMAX System Identification: First X, then AR, finally MA

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(joint work with Ivan Markovsky and Bart L.M. De Moor)

In this extended abstract, 'process' means: a zero mean, gaussian, stationary, ergodic vector process on $\mathbb{Z}$, $\perp$ means 'independence', and 'white noise' means a process $\varepsilon$ for which the $\sigma^t\varepsilon(0)$'s are all $\perp$ for $t \in \mathbb{Z}$, and $\sigma$ denotes the shift ($\sigma f(t) := f(t+1)$). Consider the difference equation

$$W(\sigma)w = E(\sigma)\varepsilon, \qquad \text{(ARMAX)}$$

with $W, E$ suitably sized polynomial matrices. The *behavior* of (ARMAX) consists of all processes $w$ such that (ARMAX) holds for some white noise process $\varepsilon$. The identification (ID) problem is to obtain estimates of $(W, E)$ from observation of a realization of $w$:

$$\tilde{w}(1), \tilde{w}(2), \ldots, \tilde{w}(T).$$

In this extended abstract, we will assume for simplicity of exposition that $T = \infty$. In the actual algorithm, we assume $T$ finite, and study the behavior of the estimates as $T \to \infty$.

Every ARMAX system admits a more refined representation

$$A(\sigma)R(\sigma)w = M(\sigma)\varepsilon \qquad \text{(AR-MA-X)}$$

with $A$ square, $\det(A)$ non-zero and without unit circle roots, and $R$ left-prime. Note that $R(\sigma)w = 0$ corresponds to the 'exogenous' part of the AR-MA-X system (obtained by setting $\varepsilon = 0$). We call $R$ the 'X' (exogenous) part, $A$ the 'AR' part,

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and $M$ the 'MA' part of the AR-MA-X system. We present an algorithm that identifies first $R$, then $A$, and finally $M$.

Many interesting problems emerge: When do two systems $(A,R,M)$ define the same behavior? Obtain canonical forms. If $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, when is $u$ a 'free input', in the sense that for any process $u$, there exists a process $y$ such that $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ belongs to the behavior of (AR-MA-X)? When is this $y$ unique? In [1] these issues are studied in depth.

It is easy to see that for all $n \in \mathbb{R}[\xi]$ in the $\mathbb{R}[\xi]$-module generated by the transposes of the rows of $R$, $n(\sigma)^\top w \perp \varepsilon$. Assume that $R = [\ P \quad Q\ ]$ with $P$ square, and correspondingly $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, with $u \perp \varepsilon$. Now look for the finite linear combinations of the rows of the observed

$$\tilde{W} = \begin{bmatrix} \tilde{w}(1) & \tilde{w}(2) & \tilde{w}(3) & \cdots & \tilde{w}(t) & \cdots \\ \tilde{w}(2) & \tilde{w}(3) & \tilde{w}(4) & \cdots & \tilde{w}(t+1) & \cdots \\ \tilde{w}(3) & \tilde{w}(4) & \tilde{w}(5) & \cdots & \tilde{w}(t+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

that are orthogonal to the rows of the observed

$$\tilde{U} = \begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(t) & \cdots \\ \tilde{u}(2) & \tilde{u}(3) & \tilde{u}(4) & \cdots & \tilde{u}(t+1) & \cdots \\ \tilde{u}(3) & \tilde{u}(4) & \tilde{u}(5) & \cdots & \tilde{u}(t+2) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Call these linear combinations 'orthogonalizers'. Obviously each orthogonalizer is a vector of the form $\pi = \mathrm{col}(\pi_0, \pi_1, \cdots, \pi_{\mathtt{n}}, \cdots)$, with the $\pi_{\mathtt{n}}$'s $\in \mathbb{R}^{\mathtt{w}}$, and all but a finite number of them non-zero. Organize the orthogonalizers as polynomial vectors $\pi(\xi) = \pi_0 + \pi_1\xi + \cdots + \pi_{\mathtt{n}}\xi^{\mathtt{n}} + \cdots \in \mathbb{R}^{\mathtt{w}}[\xi]$.

It can be shown that if $\tilde{u}$ is persistently exciting, then the orthogonalizers form exactly the $\mathbb{R}[\xi]$-module generated by the transposes of the rows of $R$. This yields an algorithm for identifying $R$ from the observations via the orthogonalizers. As we have described it here, this algorithm requires an infinite number of rows of $\tilde{W}$ and $\tilde{U}$, but if we assume that (upper bounds for) the lag $L$ and the dynamic order $n$ of the AR-MA-X system are known, we can restrict attention to the first $L$ rows of $\tilde{W}$ and the first $L+n$ rows of $\tilde{U}$.

Once $R$ has been estimated, we compute

$$\tilde{a} = \hat{R}(\sigma)\tilde{w},$$

and obtain an estimate $\hat{A}$ of $A$ from $\tilde{a}$, and proceed by computing

$$\tilde{m} = \hat{A}(\sigma)\tilde{a},$$

to obtain an estimate $\hat{M}$ of $M$, leading to an estimate $(\hat{R}, \hat{A}, \hat{M})$ for $(R, A, M)$.

This extended abstract reports on research in progress. A full paper is in preparation.

# References

[1] E.J. Hannan and M. Deistler, *The Statistical Theory of Linear Systems*, Academic Press, 1979.