On the Markov property for continuous multidimensional behaviors

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Abstract

In this paper the relation between Markovianity and representability by means of first order PDEs is investigated. It is shown that the Markov property introduced in [6] for continuous nD behaviors is not equivalent to first order representability. In order to ensure first order representability we introduce a strong version of Markovianity in higher dimensions which (similar to the weak version) can be regarded as a generalization of the one-dimensional Markov property. For finite-dimensional behaviors, we prove that strong Markovianity is indeed equivalent to the representability by means of decoupled first order partial differential equations with particular structure.

1 Introduction

We consider multidimensional (nD) behavioral systems that can be represented as the solution set of a system of homogeneous linear partial differential equations with constant coefficients. More concretely, if $\mathcal{B}$ denotes the behavior of such a system, then

$$\mathcal{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^q) : R(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})w = 0\}, \tag{1}$$

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where $R(s_1, \ldots, s_n)$ is an nD polynomial matrix. Since $\mathcal{B}$ is the kernel of a partial differential operator, we refer to it as a *kernel behavior*. The equation $R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = 0$ is called a *representation* of $\mathcal{B}$.

The question which we investigate is the connection between the fact that a behavior $\mathcal{B}$ is Markovian (in some sense to be made precise in the sequel) and the possibility of representing it by means of a system of *first order* equations (first order representation).

In the one-dimensional case, say, for systems defined over $\mathbb{R}$, a behavior $\mathcal{B}$ is said to be Markovian whenever the concatenation of two trajectories $w_1, w_2 \in \mathcal{B}$ that coincide at one point of the domain (i.e, $w_1(t) = w_2(t)$, for some t) yields a trajectory w (coinciding with w_1 in $(-\infty, t)$ and with w_2 in $[t, +\infty)$) which still is an element of $\mathcal{B}$, [5]. This is a deterministic version of the stochastic independence of past and future given the present.

As shown in [2], the one-dimensional Markovian property is indeed equivalent to the representability by means of first order differential equations. The existing results for multidimensional systems [3, 4] concern mainly the discrete two-dimensional (2D) case. It turns out that in this case a direct generalization of the one-dimensional Markov property does not correspond to first order representations; however a stronger generalization has been introduced (the strong Markov property) which does correspond to the existence of first order representations with a special (intricate) structure.

2 Strong Markov property

The research reported here is motivated by a conjecture presented in [6], according to which in the continuous multidimensional case the Markov property is equivalent to the representability by means of a system of first order PDE's

$$R_0 w + R_1 \frac{\partial}{\partial x_1} w + \ldots + R_n \frac{\partial}{\partial x_n} w = 0. \tag{2}$$

The definition of Markovianity used in [6] is the following. Define Π to be the set of partitions (S_-, S_0, S_+) of $\mathbb{R}^n$ such that S_- and S_+ are open and S_0 is closed; given a partition $\pi = (S_-, S_0, S_+) \in \Pi$ and a pair of trajectories (w_-, w_+) that coincide on S_0, define the concatenation of (w_-, w_+) along π as the trajectory $w_- \wedge |_\pi w_+$ that coincides with w_- on S_- and with w_+ on $S_0 \cup S_+$. Then

Definition 1 [6] *A multidimensional behavior* $\mathcal{B} \subset (\mathbb{R}^q)^{(\mathbb{R}^n)}$ *is said to be Markovian if given a partition* $\pi \in \Pi$ *and a pair of trajectories* $w_-, w_+ \in \mathcal{B}$ *such that* $w_{-|S_0} = w_{+|S_0}$*, the trajectory* $w_- \wedge |_\pi w_+$ *is an element of* $\mathcal{B}$*.*

Unfortunately, similar to what happens in the discrete case, this direct generalization of the one-dimensional Markov property does not necessarily lead to the desired type of first order representations, meaning that the conjecture in [6] is false.

Example 1 *The behavior*

$$\mathcal{B} = span\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e^y \begin{bmatrix} 0 \\ 1 \end{bmatrix}, e^{x+y} \begin{bmatrix} 1 \\ -1 \end{bmatrix}\} = \ker R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}),$$

with

$$R(s_1, s_2) = \begin{bmatrix} (s_1-1)(s_2-1) & -(s_1-1)(s_2-1) \\ 0 & s_1(s_2-1) \\ s_2(s_1-1) & 0 \\ s_1 s_2 & s_1 s_2 \end{bmatrix}$$

can be shown to be Markovian, but does not allow a first order representation of the form (2).

This suggests to consider a stronger version of the Markov property (as was done in the discrete case). In order to define such stronger property we first introduce some useful notions. Given a subspace $S \subset \mathbb{R}^n$, let Π_S be the set of partitions (S_-, S_0, S_+) of S such that S_- and S_+ are open and S_0 is closed (in S); we say that a behavior $\mathcal{B} \subset (\mathbb{R}^q)^{(\mathbb{R}^n)}$ is Markovian in S if given a partition $\pi_S \in \Pi_S$ and a pair of trajectories $w_-, w_+ \in \mathcal{B}|_S$ such that $w_{-|S_0} = w_{+|S_0}$, the trajectory $w_- \wedge |_\pi w_+$ is an element of $\mathcal{B}|_S$.

Definition 2 *A nD behavior* $\mathcal{B} \subset (\mathbb{R}^q)^{(\mathbb{R}^n)}$ *is said to be strong Markovian if it is Markovian in* S *for every subspace* S *of* $\mathbb{R}^n$*.*

Note that strong Markovianity coincides with Markovianity for one-dimensional behaviors, and in this case is therefore equivalent to first order representability. However, in higher dimensions, this equivalence is no longer true. In fact, there exist behaviors with first order representations that are not strong Markovian, for instance

$$\mathcal{B} = \ker(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})$$

is not strong Markovian (in fact it is not even Markovian). Therefore it is natural to expect that strong Markovian behaviors have first order representations with a special structure. In this paper we restrict our attention to the case of finite-dimensional behaviors with smooth trajectories, i.e., behaviors $\mathcal{B}$ that are finite-dimensional subspaces of $C^\infty(\mathbb{R}^q, \mathbb{R}^n)$.

3 Finite-dimensional behaviors: strong Markovianity and first order representations

Notice that, an nD kernel behavior $\mathcal{B} \subset C^\infty(\mathbb{R}^q, \mathbb{R}^n)$ is finite-dimensional if and only if it can be represented as $\mathcal{B} = \ker R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ where $R(s_1, s_2, \ldots, s_n)$ is a weakly zero right prime nD L-polynomial matrix, [7]. Making use of the existence of a special state model for behaviors with weakly zero right prime kernel representation we prove the following result.

Theorem 1 *A finite-dimensional kernel behavior $\mathcal{B} \subset C^\infty(\mathbb{R}^q, \mathbb{R}^n)$ is strong Markovian if and only if it can be represented by means of partial differential equations of the form*

$$\begin{bmatrix} (\frac{\partial}{\partial x_1} I_N - A_1)E \\ (\frac{\partial}{\partial x_2} I_N - A_2)E \\ \vdots \\ (\frac{\partial}{\partial x_n} I_N - A_n)E \\ F \end{bmatrix} w = 0, \tag{3}$$

where $A_1, A_2, \ldots, A_n$ are square pairwise commuting matrices and the matrix $S = [E^T \quad F^T]^T$ is invertible.

Sketch of the proof.

If $\mathcal{B} \neq \{0\}$ has a weakly zero right prime polynomial representation then, similar to what happens in the discrete case [1], it may also be represented by a latent variable (state) model of the form

$$\begin{cases} \frac{\partial}{\partial x_1}x &= A_1 x \\ \frac{\partial}{\partial x_2}x &= A_2 x \\ & \vdots \\ \frac{\partial}{\partial x_n}x &= A_n x \\ w &= Cx \end{cases} \tag{4}$$

where $A_1, A_2, \ldots, A_n$ are square pairwise commuting matrices of size N, x is the latent variable and w is the system variable. If in addition $\mathcal{B}$ is strong Markovian, it is possible to show that there exists a representation (4) for which the matrix C is full column rank. Let E be a left-inverse of C and F a suitable matrix such that $S = \begin{bmatrix} E \\ F \end{bmatrix}$ is invertible. Notice that equations (4) can be written in the following form

$$\begin{bmatrix} \frac{\partial}{\partial x_1}I_N - A_1 \\ \frac{\partial}{\partial x_2}I_N - A_2 \\ \vdots \\ \frac{\partial}{\partial x_n}I_N - A_n \\ C \end{bmatrix} x = \begin{bmatrix} 0_{N\times q} \\ 0_{N\times q} \\ \vdots \\ 0_{N\times q} \\ I_q \end{bmatrix} w. \tag{5}$$

Applying to both sides of this equation the invertible operator

$$U(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}) = \begin{bmatrix} I_N & 0_N & \cdots & 0_N & -(\frac{\partial}{\partial x_1}I_N - A_1)E \\ 0_N & I_N & \cdots & 0_N & -(\frac{\partial}{\partial x_2}I_N - A_2)E \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_N & 0_N & \cdots & I_N & -(\frac{\partial}{\partial x_n}I_N - A_n)E \\ 0_N & 0_N & \cdots & 0_N & S \end{bmatrix}$$

yields the equivalent equations

$$\begin{cases} \begin{bmatrix} (\frac{\partial}{\partial x_1} I_N - A_1)E \\ (\frac{\partial}{\partial x_2} I_N - A_2)E \\ \vdots \\ (\frac{\partial}{\partial x_n} I_N - A_n)E \\ F \end{bmatrix} w = 0 \\ \qquad\qquad x \qquad\qquad = \quad Ew. \end{cases} \tag{6}$$

This allows to eliminate the latent variable x from the description of $\mathcal{B}$ and obtain the desired representation

$$\begin{bmatrix} (\frac{\partial}{\partial x_1} I_N - A_1)E \\ (\frac{\partial}{\partial x_2} I_N - A_2)E \\ \vdots \\ (\frac{\partial}{\partial x_n} I_N - A_n)E \\ F \end{bmatrix} w = 0. \tag{7}$$

Conversely, let $\mathcal{B}$ have a representation as (7). Consider the transformed behavior $\hat{\mathcal{B}} := S(\mathcal{B})$. Since S corresponds to an invertible static transformation, it is clear that $\mathcal{B}$ is strong Markovian if and only if so is $\hat{\mathcal{B}}$. Now, $\hat{\mathcal{B}}$ is the set of trajectories $\hat{w} = \begin{bmatrix} x \\ z \end{bmatrix}$ such that

$$\begin{cases} \begin{bmatrix} (\frac{\partial}{\partial x_1} I_N - A_1) \\ (\frac{\partial}{\partial x_2} I_N - A_2) \\ \vdots \\ (\frac{\partial}{\partial x_n} I_N - A_n) \end{bmatrix} x = 0 \\ \qquad\qquad z \qquad\qquad = 0. \end{cases} \tag{8}$$

Taking into account that the matrices $A_1, A_2, \ldots, A_n$ commute, it is possible to prove that $\hat{\mathcal{B}}$ is strong Markovian. ■

4 Conclusion

In this paper the conjecture of [6] on the correspondence between the Markov property and first order representability for continuous nD behaviors was proved to be false. In fact, Markovianity is neither necessary nor sufficient for the existence of the first order representations proposed therein. In order to obtain equivalence with first order representability, a strong Markov property has been introduced. For finite-dimensional continuous nD behaviors this property was shown to be indeed equivalent to the representability by means of special uncoupled first order PDE's, together with a static relation.

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