Conserved- and zero-mean quadratic quantities for oscillatory systems

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1 Introduction

In this paper we consider oscillatory systems, i.e. systems whose trajectories are linear combinations of sinusoidal functions $w(t) = \sum_{k=1,\ldots,n} A_k \sin(\omega_k t + \phi_k)$, with $\omega_k, A_k, \phi_k \in \mathbb{R}$ for all k. In this context we study the structure of the set of quadratic functionals of the system variables and their derivatives, i.e. expressions of the form $Q_\Phi(w) = \sum_{i,j} (\frac{d^i w}{dt^i})^T \Phi_{ij} \frac{d^j w}{dt^j}$, where the indices i and j range over a finite set and $\Phi_{ij} = \Phi_{ji}^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. We show that these functionals are partitioned in *conserved quantities* ($Q_\Phi(w)$ is constant for all w satisfying the laws of the system) and in *zero-mean quantities* (the time average of $Q_\Phi(w)$ over the whole real axis is zero along the trajectories w of the system).

In this communication we also state a deterministic *equipartition of energy principle*: if an oscillatory system consists of symmetrically coupled identical subsystems, then the difference between the value of any quadratic functional of the variables of the one subsystem and their derivatives, and its value on the variables of the other and their derivatives is zero-mean.

The results reported here are obtained in the behavioral framework (see [2]), using the concept of quadratic differential form (QDF), introduced in [4]. In this communication we assume that the reader is familiar with the basic concepts regarding behaviors and QDFs; a tutorial paper on the latter topic is available elsewhere in these Proceedings.

The notation used in this paper is standard: the space of $\mathtt{n}$ dimensional real, respectively complex, vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{R}^{\bullet\times\mathtt{n}}$ denotes the set of real matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^\bullet$ whose elements are commonly denoted with w, we use the notation $\mathbb{R}^{\mathtt{w}}$ (note the typewriter font

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type!); similar considerations hold for matrices representing linear operators on such spaces.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta,\eta]$. We denote with $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$.

2 Basics

A *linear differential behavior* is a linear subspace $\mathfrak{B}$ of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of all solutions w of a system of linear constant-coefficient differential equations:

$$R(\frac{d}{dt})w=0, \tag{1}$$

where $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, is called a *kernel representation* of the behavior

$$\mathfrak{B}:=\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})\mid w \text{ satisfies (1) }\},$$

and w is called the *external variable* of $\mathfrak{B}$. The class of all such behaviors is denoted with $\mathfrak{L}^{\mathtt{w}}$. In this communication we consider linear differential *autonomous* systems. Intuitively, a system is autonomous if the future of every trajectory in $\mathfrak{B}$ is uniquely determined by its past, by its present "state" (see [2] for a formal definition). The behavior of an autonomous system admits kernel representations (1) in which the matrix R is square and nonsingular; moreover (see Theorem 3.6.4 in [2]), such a representation has the minimal number of equations ($\mathtt{w}$, the number of variables of the system) needed in order to describe an autonomous behavior $\mathfrak{B}$, and is consequently called a *minimal* representation.

It can be shown that all minimal representations have the same Smith form; the diagonal elements in such Smith form are called the *invariant polynomials of* $\mathfrak{B}$; their product is denoted by $\chi_{\mathfrak{B}}$, and is called the *characteristic polynomial of* $\mathfrak{B}$. The roots of $\chi_{\mathfrak{B}}$ are called the *characteristic frequencies* of $\mathfrak{B}$. It can be shown that when considering nonminimal kernel representations, the nonzero invariant polynomials in the Smith form of any matrix $R'\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\mathfrak{B}=\ker R'(\frac{d}{dt})$, also equal the invariant polynomials of $\mathfrak{B}$ (see Corollary 3.6.3 in [2]). In particular, $\chi_{\mathfrak{B}}=\det(\mathfrak{B})$ (the latter assumed monic).

We now introduce the class of linear oscillatory behaviors.

Definition 1 $\mathfrak{B}\in\mathcal{L}^{\mathtt{w}}$ *is an* oscillatory behavior *if*

$$w\in\mathfrak{B}\Longrightarrow w \textit{ is bounded on } (-\infty,+\infty)$$

From the definition it follows immediately that an oscillatory system is necessarily autonomous. The following is a characterization of oscillatory systems in terms of properties of its kernel representation.

Proposition 2 *Let* $\mathfrak{B} = \ker R(\frac{d}{dt})$*, with* $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$*. Then* $\mathfrak{B}$ *is oscillatory if and only if every nonzero invariant polynomial of* $\mathfrak{B}$ *has distinct and purely imaginary roots.*

In this communication we consider QDFs evaluated along a linear differential behavior $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$; see the tutorial paper in these Proceedings for a formal definition of the equivalence of QDFs and of the notion of R-canonical QDF. We denote the set consisting of all $\mathtt{w}$-dimensional R-canonical symmetric two-variable polynomial matrices with $\mathbb{R}_R^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. It is a matter of straightforward verification to prove that $\mathbb{R}_R^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is a vector space over $\mathbb{R}$.

3 A decomposition theorem for QDFs

We begin this section with the definition of conserved and zero-mean quantities; among the latter we distinguish between trivially- and intrinsic zero-mean quantities. Finally, we give the main result of this section, a decomposition theorem for QDFs.

The definition of conserved quantity is as follows.

Definition 3 *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ *be an oscillatory system, and let* $\Phi \in \mathbb{R}_R^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$*. Then a QDF* Q_Φ *is a* conserved quantity *for* $\mathfrak{B}$ *if*

$$w \in \mathfrak{B} \Longrightarrow \frac{d}{dt} Q_\Phi(w) = 0$$

The definition of zero-mean quantity is as follows.

Definition 4 *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ *be an oscillatory system, and let* $\Phi \in \mathbb{R}_R^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$*. Then QDF* Q_Φ *is a* zero-mean quantity *for* $\mathfrak{B}$ *if*

$$w \in \mathfrak{B} \Longrightarrow \lim_{T \to \infty} \frac{1}{T} \int_0^T Q_\Phi(w)(t) \mathrm{d}t = 0$$

Observe that certain zero-mean quantities are such for *every* oscillatory system: their zero-mean nature has nothing to do with the dynamics of the particular oscillatory system at hand. Take for example $Q_\Phi(w) = 2 \cdot w \cdot \frac{d}{dt} w$, which is the derivative of $Q_{\Phi'}(w) = w^2$; then $\lim_{T \to \infty} \frac{1}{T} \int_0^T Q_\Phi(w) dt = \lim_{T \to \infty} \frac{1}{T} w^2 \mid_0^T$ which is zero, since w is oscillatory and consequently bounded. The following definition addresses this issue.

Definition 5 Let $\Phi \in \mathbb{R}_R^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. Then a QDF Q_Φ is a *trivially zero-mean quantity* if

$$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}), w \text{ quasi-periodic } \Longrightarrow \lim_{T \to \infty} \frac{1}{T} \int_0^T Q_\Phi(w)(t) \mathrm{d}t = 0$$

It is a matter of straightforward verification to see that given $\mathfrak{B} = \ker R(\frac{d}{dt})$ with R nonsingular, the sets of R-canonical conserved-, zero-mean, and trivially zero-mean quantities for $\mathfrak{B}$ are in one-one correspondence with linear subspaces of the vector space of $\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, the set of R-canonical symmetric quadratic differential forms. We denote such subspaces respectively with $\mathfrak{C}_R$, $\mathfrak{Z}_R$ and $\mathfrak{T}_R$, that is

$$\begin{aligned}
\mathfrak{C}_R &:= \{\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid Q_\Phi \text{ is conserved }\}\\
\mathfrak{Z}_R &:= \{\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid Q_\Phi \text{ is zero-mean }\}\\
\mathfrak{T}_R &:= \{\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \mid Q_\Phi \text{ is trivially zero-mean }\}
\end{aligned}$$

Let $\mathfrak{I}_R$ be a complement of $\mathfrak{T}_R$ in $\mathfrak{Z}_R$; then $\mathfrak{I}_R$ consists of those zero-mean quantities which are not trivial ones. We call the elements of $\mathfrak{I}_R$ the *intrinsically zero-mean quantities*, in order to emphasize that their zero-mean nature depends in an essential way on the dynamics of the system.

Parametrizations of the elements of $\mathfrak{C}_R$, $\mathfrak{Z}_R$, $\mathfrak{T}_R$ and $\mathfrak{I}_R$ in terms of algebraic properties of the corresponding two-variable polynomial matrices will be presented in detail elsewhere (see [3]).

We can now state the main result of this section, a decomposition theorem for R-canonical QDFs.

Theorem 6 *Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be oscillatory, and let $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be such that $\mathfrak{B} =$ ker $R(\frac{d}{dt})$. Assume that $\mathfrak{B}$ has no characteristic frequencies in zero. Then every $\Phi \in \mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ admits a unique decomposition as*

$$\Phi = \Phi_{\mathfrak{C}_R} + \Phi_{\mathfrak{T}_R} + \Phi_{\mathfrak{I}_R}$$

where $\Phi_{\mathfrak{C}_R} \in \mathfrak{C}_R$, $\Phi_{\mathfrak{T}_R} \in \mathfrak{T}_R$, $\Phi_{\mathfrak{I}_R} \in \mathfrak{I}_R$.

Example 7 Consider a single oscillator, described by the differential equation $m\frac{d^2w}{dt^2} + kw = 0$, i.e. $R(\xi) = m\xi^2 + k$. It can be shown that the space of R-canonical symmetric two-variable polynomials has dimension 3.

It can be also shown that this system admits only one conserved quantity, namely the total energy of the oscillator, induced by the two-variable polynomial $E(\zeta,\eta) = \frac{1}{2}m\zeta\eta + \frac{1}{2}k$. There is one intrinsically zero-mean quantity, namely the Lagrangian of the system, induced by $L(\zeta,\eta) = \frac{1}{2}m\zeta\eta - \frac{1}{2}k$. A third QDF, linearly independent from Q_E and Q_L, is induced by $I(\zeta,\eta) = (\zeta+\eta)\cdot k$.

4 An equipartition of energy principle

We begin the section by formalizing the notion of symmetry in an intrinsic way, i.e. at the level of the trajectories of the system.

Definition 8 *Let $\mathfrak{B}$ be a linear differential behavior with $\mathtt{w}$ external variables, and let $\Pi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be a linear involution, i.e. $\Pi^2 = I_{\mathtt{w}}$. $\mathfrak{B}$ is called* Π-symmetric *if $\Pi\mathfrak{B} = \mathfrak{B}$.*

In the following we use the symmetry induced by the permutation matrix

$$\Pi = \begin{pmatrix} 0 & I_{\mathtt{m}} \\ I_{\mathtt{m}} & 0 \end{pmatrix} \tag{2}$$

or equivalently, we consider systems with $2\mathtt{m}$ external variables w_i, $i = 1, \ldots, 2\mathtt{m}$ for which

$$w \in \mathfrak{B} \Longleftrightarrow \begin{pmatrix} w_{m+1} \\ \vdots \\ w_{2m} \\ w_1 \\ \vdots \\ w_m \end{pmatrix} = \Pi w \in \mathfrak{B} \tag{3}$$

In order to state the main result of this section, the notion of observability is required. Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$, with its external variable w partitioned as $w = (w_1, w_2)$; then w_2 is *observable* from w_1 if for all $(w_1, w_2), (w_1, w_2') \in \mathfrak{B}$ implies $w_2 = w_2'$. Thus, the variable w_2 is observable from w_1 if w_1 and the dynamics of the system uniquely determine w_2; in other words, the variable w_1 contains all the information about the trajectory $w = (w_1, w_2)$.

The main result of this communication is the following.

Theorem 9 *Let $\mathfrak{B}$ be an oscillatory behavior with $\mathtt{w} = 2\mathtt{m}$ external variables. Assume that $\mathfrak{B}$ is Π-symmetric, with Π given by (2), i.e. (3) holds. Moreover, assume that*

(a) $w_2, \ldots, w_{\mathtt{m}}, w_{\mathtt{m}+1}$ observable from w_1; and

(b) $w_{\mathtt{m}+2}, \ldots, w_{2\mathtt{m}}$ observable from $w_{\mathtt{m}+1}$.

Let $\Psi \in \mathbb{R}^{\mathtt{m}\times\mathtt{m}}[\zeta, \eta]$, and consider the QDF Q_Φ induced by the $2\mathtt{m}\times 2\mathtt{m}$ two-variable matrix

$$\Phi(\zeta, \eta) := \begin{pmatrix} \Psi(\zeta, \eta) & 0 \\ 0 & -\Psi(\zeta, \eta) \end{pmatrix}$$

on $\mathfrak{B}$. Then Q_Φ is a zero-mean quantity for $\mathfrak{B}$.

See also [1] for an analogous result obtained in the state-space context.

5 Example

Assume that two equal masses m connected to "walls" by springs of equal stiffness k, are coupled together with a spring of stiffness k'. We consider this as the *symmetric interconnection*, through the spring with elastic constant k', of two identical *oscillators*, each consisting of a mass m and a spring with elastic constant k. Take as external variables the displacements w_1 and w_2 of the masses from their equilibrium positions; in such case two equations describing the system are

$$
\begin{aligned}
m\frac{d^2w_1}{dt^2} &= k'(w_2 - w_1) - kw_1 \\
m\frac{d^2w_2}{dt^2} &= k'(w_1 - w_2) - kw_2
\end{aligned}
$$

From the result of Theorem 9, we can conclude that the difference between the kinetic energies of the two oscillators, represented by the two-variable polynomial matrix

$$
\begin{pmatrix} m\zeta\eta & 0 \\ 0 & -m\zeta\eta \end{pmatrix}
$$

is zero mean. Also the difference between the potential energies of the two oscillators, induced by

$$
\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}
$$

Observe that this implies that also the total energy of the two oscillators in on average the same.

6 Conclusions

In this communication we have illustrated the decomposition presented in Theorem 6 and the equipartition principle stated in Theorem 9, which are proved using the framework of quadratic differential forms. For reasons of space, we have omitted to mention other interesting results of our investigation. Prominent among these are those regarding the actual computation of conserved- and zero-mean quantities for a given system, which is reduced to the solution of polynomial matrix equations. These methods can be applied to systems described by higher-order equations, and they can be implemented easily using standard polynomial computations, thus making them available for inclusion in computer-aided modeling and simulation tools.

References

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