An introduction to quadratic differential forms

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1 Introduction

In modeling and control problems it is often necessary to study certain functionals of the system variables and their derivatives; when considering linear systems, such functionals are often quadratic. The parametrization of such functionals using two-variable polynomial matrices has been studied in detail in [WT1], resulting in the definition of bilinear- and quadratic differential form (*BDF* and *QDF* respectively in the following) and in the development of a calculus for application in many areas. In this tutorial communication we review the main definitions and results regarding QDFs.

We first examine *bilinear differential forms.* These are functionals from $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}_2})$ to $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$, defined as:

$$L_\Phi(w_1,w_2) = \sum_{h,k=0}^{N} (\frac{d^h w_1}{dt^h})^T \Phi_{h,k} \frac{d^k w_2}{dt^k}.$$

where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ and N is a nonnegative integer. Let

$$\Phi(\zeta,\eta) = \sum_{h,k=0}^{N} \Phi_{h,k} \zeta^h \eta^k,$$

This two-variable polynomial matrix $\Phi(\zeta,\eta$ *induces* the bilinear differential form L_Φ defined above.

L_Φ is *symmetric*, meaning $L_\Phi(w_1,w_2) = L_\Phi(w_2,w_1)$ for all w_1,w_2, if and only if Φ is a *symmetric two-variable polynomial matrix*, i.e. if $\mathtt{w_1} = \mathtt{w_2}$ and $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^T$. The set of symmetric two-variable polynomial matrices of dimension $\mathtt{w} \times \mathtt{w}$ in the indeterminates ζ and η is denoted with $R_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

If the L_Φ is symmetric, or equivalently, if the two-variable polynomial matrix Φ is symmetric, then it induces also a quadratic functional acting on $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$

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as

$$\begin{aligned} Q_\Phi &: \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \\ Q_\Phi(w) &:= L_\Phi(w, w). \end{aligned}$$

We call Q_Φ the *quadratic differential form* associated with Φ.

Example 1 As an example of QDF, we consider power in an electrical circuit. Denote with V_k the potential, and with I_k the current at the k-th terminal of the circuit. The power is

$$P(t) = \sum_{k=1}^{N} V_k(t) I_k(t)$$

Define the external variable of the system as $(V_1 \dots V_N I_1 \dots I_N)^T =: (VI)^T$; then

$$P(t) = \begin{bmatrix} V(t) & I(t) \end{bmatrix} \begin{bmatrix} 0_{N\times N} & I_N \\ I_N & 0_{N\times N} \end{bmatrix} \begin{bmatrix} V(t) \\ I(t) \end{bmatrix}$$

Example 2 Consider a mechanical system consisting of two equal masses m connected to “walls” by springs of equal stiffness k, which are coupled together with a spring of stiffness k'. Take as external variables the displacements w_1 and w_2 of the masses from their equilibrium positions; in such case two equations describing the system are

$$\begin{aligned} m\frac{d^2 w_1}{dt^2} &= k'(w_2 - w_1) - k w_1 \\ m\frac{d^2 w_2}{dt^2} &= k'(w_1 - w_2) - k w_2 \end{aligned}$$

Because of the absence of dissipative elements, we can conclude that the total energy of the system at time t is conserved. Such quantity is induced by the two-variable polynomial matrix

$$E(\zeta, \eta) = \begin{bmatrix} m\zeta\eta + k + k' & -k' \\ -k' & m\zeta\eta + k + k' \end{bmatrix}$$

The system also admits another conserved quantity, linearly independent of $Q_E(\cdot)$. One possible choice for such conserved quantity is the functional

$$C(t) = -\frac{k'}{2} w_1(t)^2 - \frac{k'}{2} w_2(t)^2 + (k + k') w_1(t) w_2(t) + m\frac{dw_1}{dt}(t)\frac{dw_2}{dt}(t)$$

whose dimension is that of an energy. This functional is induced by the two-variable polynomial matrix

$$\Phi_{\mathfrak{C}_R,2}(\zeta, \eta) := \frac{1}{2} \begin{bmatrix} -k' & k + k' + m\zeta\eta \\ k + k' + m\zeta\eta & -k' \end{bmatrix}$$

We now introduce the concept of symmetric canonical factorization (see [WT1], p. 1709). Let $\Phi \in R_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$; then its coefficient matrix $\tilde{\Phi}$ can be factored as $\tilde{\Phi} = \tilde{M}^T \Sigma_\Phi \tilde{M}$, where $\tilde{M}$ is a full row rank infinite matrix with $\text{rank}(\tilde{\Phi})$ rows and only a finite number of entries nonzero, and $\Sigma_\Phi \in \mathbb{R}^{\text{rank}(\tilde{\Phi})\times\text{rank}(\tilde{\Phi})}$ is a signature matrix, i.e.

$$\Sigma_\Phi = \begin{bmatrix} I_{r_+} & 0 \\ 0 & -I_{r_-} \end{bmatrix}$$

From such factorization, multiplying on the left by $\begin{pmatrix} I_{\mathtt{w}} & I_{\mathtt{w}}\zeta & I_{\mathtt{w}}\zeta^2 & \cdots \end{pmatrix}$ and on the right by $\begin{pmatrix} I_{\mathtt{w}} & I_{\mathtt{w}}\eta & I_{\mathtt{w}}\eta^2 & \cdots \end{pmatrix}^T$, we obtain the *symmetric canonical factorization* of Φ:

$$\Phi(\zeta,\eta) = M^T(\zeta)\Sigma_\Phi M(\eta).$$

2 Basic operations in the calculus of QDFs

The association of two-variable polynomial matrices with BDF's and QDF's allows to develop a calculus that has applications in dissipativity theory and H_∞-control (see [PW, WT2, TW, WT3]). One important tool in such calculus is the map

$$\begin{aligned} \partial \quad &: \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \longrightarrow \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi] \\ \partial\Phi(\xi) \quad &:= \Phi(-\xi,\xi) \end{aligned}$$

Observe that if $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is symmetric, then $\partial\Phi$ is *para-Hermitian*, i.e. $\partial\Phi = (\partial\Phi)^*$.

Another important role in the following is played by the notion of *derivative* of a QDF. Given a QDF Q_Φ, we define its *derivative* as the QDF $Q_{\overset{\bullet}{\Phi}}$ defined by

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. In terms of the two-variable polynomial matrices associated with the QDF's, the relationship between a QDF Q_Φ and its derivative $Q_{\overset{\bullet}{\Phi}}$ is expressed as

$$\frac{d}{dt}Q_\Phi(w) = Q_{\overset{\bullet}{\Phi}}(w) \text{ for all } w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \iff \overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta) \quad (1)$$

In several applications the need arises to consider integrals of QDFs. In order to make sure that those integral exist, we assume that the argument of the QDF has compact support; we denote by $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid$ has compact support$\}$.

Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$; then

$$\begin{aligned} \int Q_\Phi &: \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathbb{R} \\ \int Q_\Phi(w) &:= \int_{-\infty}^{+\infty} Q_\Phi(w)dt \end{aligned}$$

Often we consider such integrals on closed finite intervals $[t_0, t_1] \subset \mathbb{R}$. We call $\int_{t_0}^{t_1} Q_\Phi(w)$ *independent of path* if for all intervals $[t_1, t_2]$, the value of the integral depends only on the value of w and (a finite number of) its derivatives at t_1 and at t_2, but not on the intermediate path used to connect these endpoint.

The following algebraic characterization of path independence in terms of properties of two-variable polynomial matrices uses the notion of derivative of a QDF and the ∂ operator. Assume $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$; then the following statements are equivalent:

(a) $\int_{t_0}^{t_1} Q_\Phi = 0$

(b) There exists a $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ such that $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta)$;

(c) $\partial\Phi(\xi) = \Phi(-\xi, \xi) = 0$.

(for a proof, see Theorem 3.1 of [WT1]).

Example 3 Consider the differential equation of a simple oscillator

$$M\frac{d}{dt}2q + Kq = 0$$

The power delivered to the mass by the force $F = -Kq$ is $Q_\Pi(q) = -Kq\dot{q} = -\frac{1}{2}Kq\dot{q} - \frac{1}{2}Kq\dot{q}$, a QDF induced by the two-variable polynomial $\Pi(\zeta, \eta) := -\frac{1}{2}K(\zeta+\eta)$. Observe that $\Pi(-\xi, \xi) = 0$; it follows from the result just illustrated that there exists $\Psi(\zeta, \eta)$ s.t. $(\zeta+\eta)\Psi(\zeta, \eta) = \Pi(\zeta, \eta)$. Indeed, it is easy to verify that

$$\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta)}{\zeta + \eta} = -\frac{1}{2}K$$

This should come as no surprise, since the power in this system is the derivative of the potential energy $-\frac{1}{2}Kq^2$, induced by $\Psi(\zeta, \eta) = -\frac{1}{2}K$.

3 QDFs along a behavior

In many applications, an essential role is played by QDFs evaluated along a linear differential behavior $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. In order to introduce such notion, we briefly review the relevant concepts from behavioral system theory first, referring the reader to [PoW] for a thorough discussion of the subject.

A *linear differential behavior* is a linear subspace $\mathfrak{B}$ of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions w of a system of linear constant-coefficient differential equations:

$$R(\frac{d}{dt})w = 0, \tag{2}$$

where $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, is called a *kernel representation* of the behavior

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid w \text{ satisfies (2) }\},$$

and w is called the *external variable* of $\mathfrak{B}$. The class of all such behaviors is denoted with $\mathfrak{L}^{\mathtt{w}}$. A given behavior $\mathfrak{B}$ can be described as the kernel of different polynomial differential operators; two kernel representations $R_1(\frac{d}{dt})w = 0$ and $R_2(\frac{d}{dt})w = 0$ with $R_1, R_2 \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ represent the same behavior if and only if there exist polynomial matrices F_1, F_2 with a suitable number of columns, such that $R_1 = F_1 R_2$ and $R_2 = F_2 R_1$; in particular if R_1 and R_2 are of full row rank, this means that there exists a unimodular matrix F such that $R_1 = F R_2$.

An alternative way to represent the behavior of a linear differential system are image representations. If $M \in \mathbb{R}^{\mathtt{w} \times 1}[\xi]$ and

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \text{ there exists } \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{1}) \text{ s.t. } w = M(\frac{d}{dt})\ell\},$$

then we call

$$w = M(\frac{d}{dt})\ell \tag{3}$$

an *image representation* of $\mathfrak{B}$. Not all behaviors admit an image representation: indeed, a behavior can be represented in image form if and only if each of its kernel representations is associated with a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ such that $\text{rank}(R(\lambda))$ is constant for all $\lambda \in \mathbb{C}$; or equivalently, $\mathfrak{B}$ is controllable in the behavioral sense (see Ch. 5 of [PoW]). The image representation (3) of $\mathfrak{B}$ is called *observable* if $(M(\frac{d}{dt})\ell = 0) \Longrightarrow (\ell = 0)$. It can be shown that this is the case if and only if the matrix $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

A class of behaviors which are to some extent the opposite of controllable ones is that of *autonomous behaviors* (see [PoW]). It can be shown that such behaviors admit kernel representations (2) in which the matrix R is $\mathtt{w} \times \mathtt{w}$ and nonsingular, meaning its determinant is not the zero polynomial. Every trajectory of an autonomous behavior $\mathfrak{B}$ is a Bohl function, i.e. a finite sum of products of polynomials, real exponentials, sines and cosines, associated with the zeros of the determinant of any nonsingular representation R of $\mathfrak{B}$. Such zeros are called the *characteristic frequencies* of $\mathfrak{B}$.

Equipped with these notions, we can now introduce the concept of equivalence of QDFs on a given behavior. Let $\Phi_1, \Phi_2 \in \mathbb{R}_S^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$; we say that Φ_1 is *equivalent to* Φ_2 *along* $\mathfrak{B}$, denoted

$$\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$$

if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ holds for all $w \in \mathfrak{B}$. It is a matter of straightforward verification to see that such relation is indeed an equivalence relation. This equivalence can be expressed in terms of a kernel representation (2) of $\mathfrak{B}$ as follows (see Proposition 3.2 of [WT1]): $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$ if and only if there exists $F \in \mathbb{R}^{\bullet \times \bullet}[\zeta, \eta]$ such that

$$\Phi_2(\zeta, \eta) = \Phi_1(\zeta, \eta) + R(\zeta)^T F(\zeta, \eta) + F(\eta, \zeta)^T R(\eta) \tag{4}$$

If $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is autonomous, then each equivalence class of QDF's in the equivalence $\overset{\mathfrak{B}}{=}$ admits a canonical representative. In order to see this, choose a minimal kernel representation $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ of $\mathfrak{B}$; observe that since $\mathfrak{B}$ is autonomous, then

$\det(R) \neq 0$. We call $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *R-canonical* if $(R(\zeta)^T)^{-1}\Phi(\zeta,\eta)(R(\eta))^{-1}$ is a matrix of strictly proper two-variable rational functions. It can be proved (see Proposition 4.9 p. 1716 of [WT1]) that if $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, then there exists exactly one QDF $\Phi' \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ which is R-canonical and such that $\Phi' \stackrel{\mathfrak{B}}{=} \Phi$; we call Φ' the *R-canonical representative* of Φ, denoted Φ mod R.

Example 4 As an illustration of the above definition, we consider the notion of R-equivalence for scalar systems. Assume that $\mathtt{w} = \mathtt{1}$, and let $\mathfrak{B} = \ker r(\frac{d}{dt})$, with $r \in \mathbb{R}[\xi]$ having degree n. Observe that since

$$r_0 w + r_1 \frac{dw}{dt} + \ldots + r_n \frac{d^n w}{dt^n} = 0$$

and $r_n \neq 0$, it follows that the derivatives of w of order higher than n can be rewritten as linear combinations of the derivatives of w of order less than or equal to $n-1$. Consequently, any quadratic differential form Q_Φ involving derivatives of w of order higher than or equal to n can be rewritten in an equivalent (and unique!) way as a quadratic differential form $Q_{\Phi'}$ involving the derivatives of w up to the $(n-1)$-th one. $Q_{\Phi'}$ is the r-canonical representative of Q_Φ.

We denote the set consisting of all $\mathtt{w}$-dimensional R-canonical symmetric two-variable polynomials with $\mathbb{R}_R^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

If $\mathfrak{B}$ is a controllable behavior, then it admits an image representation (3), and the equivalence of two QDFs on $\mathfrak{B}$ can be ascertained in the following way. Consider $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and define $\Phi'(\zeta,\eta) \in \mathbb{R}_S^{1\times 1}[\zeta,\eta]$ as $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$; then for every (w,ℓ) satisfying (3) it holds $Q_\Phi(w) = Q_{\Phi'}(\ell)$. Observe that $Q_{\Phi'}$ is a functional acting on "free" trajectories $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^1)$. Now let $\Phi_1, \Phi_2 \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and denote with $\Phi_1', \Phi_2' \in \mathbb{R}_S^{1\times 1}[\zeta,\eta]$ the two QDF obtained from the Φ_is and an image representation of $\mathfrak{B}$; then $\Phi_1 \stackrel{\mathfrak{B}}{=} \Phi_2$ if and only if $\Phi_1' = \Phi_2'$. If the image representation is observable, then $\Phi_1 \stackrel{\mathfrak{B}}{=} \Phi_2$ if and only if $\Phi_1 = \Phi_2$.

4 Positive QDFs

Let $\Phi \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$; we call it *nonnegative*, denoted $\Phi \geq 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. We call Φ positive, denoted $Q_\Phi > 0$, if $\Phi \geq 0$ and $(Q_\Phi(w) = 0) \Longrightarrow (w = 0)$. Using the two-variable matrix representation of Q_Φ and the concept of symmetric canonical factorization, it can be verified that

$$\begin{aligned} Q_\Phi \geq 0 \quad &\Longleftrightarrow \quad \exists\ D \in \mathbb{R}^{\bullet\times\mathtt{w}} \text{ such that } \Phi(\zeta,\eta) = D^T(\zeta)D(\eta) \\ Q_\Phi > 0 \quad &\Longleftrightarrow \quad \exists\ D \in \mathbb{R}^{\bullet\times\mathtt{w}} \text{ such that } \Phi(\zeta,\eta) = D^T(\zeta)D(\eta), \\ &\qquad \text{and rank } (D(\lambda)) = \mathtt{w} \text{ for all } \lambda \in \mathbb{C} \end{aligned}$$

In Lyapunov stability theory for higher-order systems and in many other applications, it is important to determine whether a given QDF is zero-, nonnegative-, or positive along a behavior $\mathfrak{B}$. We call Q_Φ *zero along* $\mathfrak{B}$, denoted with

$$Q_\Phi \stackrel{\mathfrak{B}}{=} 0 \text{ or } \Phi \stackrel{\mathfrak{B}}{=} 0$$

if $Q_\Phi(w) = 0$ for all $w \in \mathfrak{B}$. We call Q_Φ *nonnegative along* $\mathfrak{B}$, denoted

$$Q_\Phi \overset{\mathfrak{B}}{\geq} 0$$

or $\Phi \overset{\mathfrak{B}}{\geq} 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$, and *positive along* $\mathfrak{B}$, denoted

$$Q_\Phi \overset{\mathfrak{B}}{>} 0$$

or $\Phi \overset{\mathfrak{B}}{>} 0$, if $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$, and $[Q_\Phi(w) = 0] \Longrightarrow [w = 0]$.

These concepts translate into properties of the one- and two-variable polynomial matrices representing $\mathfrak{B}$ and the QDFs as follows (see Proposition 3.5 p. 1712 of [WT1]). From the notion of R-equivalence and from its characterization (4) we can conclude that

$$Q_\Phi \overset{\mathfrak{B}}{=} 0 \quad \Longleftrightarrow \quad \exists\ F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that } \Phi(\zeta,\eta) = R(\zeta)^T F(\zeta,\eta) + F^T(\eta,\zeta) R(\eta)$$

Also, $\Phi \overset{\mathfrak{B}}{\geq} 0$ if and only if there exists Φ' such that $\Phi' \overset{\mathfrak{B}}{=} \Phi$ and $\Phi' \geq 0$; equivalently,

$$\begin{aligned} \Phi \overset{\mathfrak{B}}{\geq} 0 \quad \Longleftrightarrow \quad & \exists\ D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi] \text{ and } F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that} \\ & \Phi(\zeta,\eta) = D(\zeta)^T D(\eta) + R(\zeta)^T F(\zeta,\eta) + F^T(\eta,\zeta) R(\eta) \end{aligned}$$

Finally,

$$\begin{aligned} \Phi \overset{\mathfrak{B}}{>} 0 \quad \Longleftrightarrow \quad & \exists\ D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi] \text{ and } F \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta] \text{ such that} \\ & \Phi(\zeta,\eta) = D^T(\zeta) D(\eta) + F^T(\eta,\zeta) R(\eta) + R^T \zeta) F(\zeta,\eta), \\ & \text{and rank} \left(\begin{bmatrix} R(\lambda) \\ D(\lambda) \end{bmatrix} \right) = \mathtt{w} \text{ for all } \lambda \in \mathbb{C} \end{aligned}$$

Similar characterizations hold for behaviors admitting an image representation.

Example 5 We consider the problem of obtaining Lyapunov functionals for higher-order systems (see section 4 of [WT1] for a thorough treatment of this subject). Let $\mathfrak{B} = \{w \mid \frac{d^2 w}{dt^2} + 3\frac{d}{dt}w + 2w = 0\}$. The basic result of the QDF approach to Lyapunov stability is that $\mathfrak{B}$ is asymptotically stable (meaning all its trajectories vanish at $+\infty$) if and only if there exists $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that $Q_\Psi \overset{\mathfrak{B}}{\geq} 0$ and $\frac{d}{dt} Q_\Psi \overset{\mathfrak{B}}{<} 0$ (see Theorem 4.3 of [WT1]). Consider the QDF $Q_\Psi(w) := \frac{2}{3}w^2 + \frac{1}{3}(\frac{d}{dt}w)^2$, which is evidently positive along $\mathfrak{B}$. Its derivative equals $\frac{2}{3}(2w + \frac{d^2 w}{dt^2})\frac{d}{dt}w$. Since $w \in \mathfrak{B}$, it holds $2w + \frac{d^2 w}{dt^2} = -3\frac{d}{dt}w$, and consequently $\frac{d}{dt} Q_\Phi(w) = \frac{2}{3}(-3\frac{d}{dt}w)\frac{d}{dt}w = -2(\frac{d}{dt}w)^2$, which is negative for all $w \in \mathfrak{B}$. The system is asymptotically stable.

5 Average nonnegativity and half-line positivity

Questions such as when is the integral of a QDF a positive semidefinite operator arise naturally, for example when considering optimal control problems or dissipativity.

We call a QDF Q_Φ *average nonnegative*, if $\int Q_\Phi \geq 0$, i.e.,

$$\int_{-\infty}^{\infty} Q_\Phi(w)dt \geq 0 \quad \text{for all } w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$$

We call Q_Φ *average-positive* if $\int Q_\Phi \geq 0$ and $\int Q_\Phi = 0$ implies $w = 0$. The definition of average nonnegativity and positivity along a behavior follows readily from these.

A QDF can be tested for average nonnegativity and positivity by analyzing the behavior of the para-Hermitian matrix $\partial\Phi$ on the imaginary axis. Indeed, (see Proposition 5.2 in [WT1])

$$\int Q_\Phi \geq 0 \Longleftrightarrow \partial\Phi(i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}$$

and $\int Q_\Phi > 0$ if and only if $\partial\Phi(i\omega) \geq 0 \quad \forall\, \omega \in \mathbb{R}$ and $\det(\partial\Phi) \neq 0$. Using standard results in the spectral factorization of polynomial matrices, it can be shown that $\int Q_\Phi \geq 0$ if and only if there exists $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $\partial\Phi(\xi) = F^T(-\xi)F(\xi)$; and $\int Q_\Phi > 0$ if and only if there exists $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, $\det(F)$ Hurwitz, such that $\partial\Phi(\xi) = F^T(-\xi)F(\xi)$.

Finally we mention *half-line positivity*, a concept particularly important in H_∞-control problems from a behavioral point of view (see [TW, WT3]). Φ is *half-line nonnegative* (denoted by $\int^t Q_\Phi \geq 0$) if $\int_{-\infty}^0 Q_\Phi(w)\mathrm{d}t \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Φ is *half-line positive* (denoted by $\int^t Q_\Phi > 0$) if $\int^t Q_\Phi \geq 0$ and $[\int_{-\infty}^0 Q_\Phi(w)\mathrm{d}t = 0] \Longrightarrow [w = 0]$.

6 The storage function and the dissipation inequality

In the context of dissipative systems, a QDF measures the power supplied to a system: its integral over the real line then measures the net flow of energy going into the system. If such integral is positive, then the outflowing energy needs to have been stored somewhere in the system. Also, energy cannot be stored faster than it is supplied. These considerations lead to the definition of storage function and to the dissipation inequality, which we now examine (see section 5 of [WT1] for a thorough treatment).

Let $\Phi \in \mathbb{R}_s^{q\times q}[\zeta, \eta]$; the QDF Q_Ψ is said to be a *storage function* for Q_Φ if the following *dissipation inequality* holds

$$\frac{d}{dt}Q_\Psi \leq Q_\Phi$$

Storage functions are related to dissipation functions, which we now define. A QDF Q_Δ is a *dissipation function* for Q_Φ if $Q_\Delta \geq 0$ and $\int Q_\Phi = \int Q_\Delta$. There is a close relationship between storage functions, average nonnegativity, and dissipation functions, expressed in the following result.

Proposition 6 *Let $\Phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$. The following conditions are equivalent:*

1. $\int Q_\Phi \geq 0$,
2. Φ *admits a storage function,*
3. Φ *admits a dissipation function.*

Moreover, there exists a one-one relation between storage functions Ψ and dissipation functions Δ for Φ, defined by

$$\frac{d}{dt}Q_\Psi = Q_\Phi - Q_\Delta$$

or, equivalently,

$$(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - \Delta(\zeta,\eta).$$

Given an average nonnegative QDF, in general there exist an infinite number of storage function. As the following result shows, all such storage functions lie between two extremal ones.

Proposition 7 *Let $\int Q_\Phi \geq 0$. Then there exist storage functions Ψ_- and Ψ_+ such that any other storage function Ψ for Φ satisfies*

$$Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}.$$

Q_{Ψ_-} is called the *smallest* and Q_{Ψ_+} the *largest storage function* of Q_Φ.

In many cases it is of interest to compute explicitly a storage function for a given QDF. The following result suggests a procedure to compute the extremal storage functions Q_{Ψ_-} and Q_{Ψ_+} introduced in the previous theorem.

Proposition 8 *Let $\Phi(\zeta,\eta) \in \mathbb{R}_s^{\bullet\times\bullet}[\zeta,\eta]$. Assume* $\det(\partial\Phi) \neq 0$ *and $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then the smallest and the largest storage functions Ψ_- and Ψ_+ of Φ can be constructed as follows: let H and A be semi-Hurwitz, respectively semi-anti-Hurwitz, polynomial spectral factors of $\partial\Phi$. Then*

$$\Psi_+(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - A^T(\zeta)A(\eta)}{\zeta+\eta},$$
$$\Psi_-(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - H^T(\zeta)H(\eta)}{\zeta+\eta}.$$

Example 9 Consider the QDF induced by $\Phi(\zeta,\eta) = 1 + \zeta\eta$. Since $\partial\Phi(i\omega) = 1 + \omega^2$ we conclude that $\int Q_\Phi \geq 0$, i.e. Q_Φ is average positive.

Since $\partial\Phi(\xi) = 1 - \xi^2$, it admits the two (Hurwitz, resp. anti-Hurwitz) spectral factorizations $\partial\Phi(\xi) = (1-\xi)(1+\xi) = (1+\xi)(1-\xi)$. Now define $\Delta_-(\zeta,\eta) = (1+\zeta)(1+\eta)$, then the corresponding storage function is $\Psi_-(\zeta,\eta) = -1$, the smallest storage function for Q_Φ. On the other hand, if we define $\Delta_+(\zeta,\eta) = (1-\zeta)(1-\eta)$, then the largest storage function is induced by $\Psi_+(\zeta,\eta) = 1$.

7 Conclusions

In this communication we illustrated the basic features of QDFs, including their calculus and various concepts such as nonnegativity, average nonnegativity, and half-line positivity, which have particular relevance in several fields of application, such as Lyapunov theory (see [WT1, PR]), dissipativity theory, and H_∞-control (see [WT2, WT3, TW]). For reasons of space, we have limited our treatment to the continuous-time case, without illustrating the analogous of QDFs for systems described by partial differential equations (see [PS, PW]), and for systems in discrete-time (see [KF]).

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