nD Markovian behaviors: the discrete finite dimensional case

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Abstract

In this paper we analyze a deterministic version of the Markov property for discrete nD systems with finite dimensional behavior. We show that in this case this property is equivalent to the existence of a description by means of decoupled first order partial difference equations.

Keywords: behavior, Markov property, first order representation

1 Introduction

This paper is concerned with the characterization of nD behaviors endowed with a deterministic version of the Markov property.

As usual within the behavioral approach to systems and control, we consider that a system is characterized by the set of its admissible signals (or system trajectories) rather than being specified by a transfer function or even a state space model; such set is known as the system behavior, $\mathcal{B}$. Moreover, in this setting, the system variables (whose evolution is described by the system trajectories) are not a priori divided into inputs and outputs.

We consider here nD discrete behaviors (i.e., whose trajectories are defined over a discrete n-dimensional domain) that can be represented as the solution set of a system of homogeneous linear partial difference equations. The question that we investigate is the connection between the fact that a behavior $\mathcal{B}$ is Markovian (in some sense to be made precise in the sequel) and the possibility of representing it by means of a system of *first order* partial difference equations.

In the one-dimensional case, say, for systems defined over $\mathbb{R}$ or $\mathbb{Z}$, a behavior $\mathcal{B}$ is said to be Markovian whenever the concatenation of two trajectories $w_1, w_2 \in \mathcal{B}$ that coincide at one point of the domain (i.e, $w_1(t) = w_2(t)$, for some t) yields a trajectory w (coinciding with w_1 in $(-\infty, t)$ and with w_2 in $[t, +\infty)$) which still is an element of $\mathcal{B}$, [5] This is a deterministic version of the stochastic independence of past and future given the present.

As referred in [5], the one-dimensional Markovian property is indeed equivalent to the representability by means of first order difference/differential equations. The extension of this result to the discrete 2D case has been studied in [2, 3], for a convenient generalization of the one-dimensional Markov property. However the obtained characterization of the corresponding representations is rather involved.

The research reported in this paper began as an attempt to answer the conjecture presented in [6], according to which in the continuous nD case the Markov property is equivalent to the representability by means of a system of first order PDEs which are decoupled in the sense that each of them only involves one partial differentiator. Although initially meant for the continuous case, the analysis that has been carried out suggested a different approach from the one used in [2] to deal with the discrete case. This has allowed us to obtained the main result of this paper, which states that, for discrete nD systems with finite-dimensional behavior, the Markov property introduced in [2, 3] is indeed equivalent to the representability by a special system of first order partial difference equations.

2 Preliminaries

As mentioned in the introduction, we consider discrete nD behavioral systems [5], whose behaviors can be described as solutions sets of systems of partial linear difference equations with constant coefficients. In other words, if $\mathcal{B}$ denotes the behavior of such a system, then

$$\mathcal{B} = \ker R(\sigma_1, \ldots, \sigma_n, \sigma_1^{-1}, \ldots, \sigma_n^{-1}),$$

where σ_i denotes the i-th nD shift defined by $\sigma_i w(t_1, \ldots, t_i, \ldots, t_n) = w(t_1, \ldots, t_i + 1, \ldots, t_n)$, and $R(z_1, \ldots, z_n, s_1^{-1}, \ldots, z_n^{-1})$ is an nD Laurent-polynomial (L-polynomial) matrix. For the sake of simplicity, we will write $(\sigma_1, \ldots, \sigma_n) = \underline{\sigma}$ and $(\sigma_1^{-1}, \ldots, \sigma_n^{-1}) = \underline{\sigma}^{-1}$ for the shift operators, and use a similar notation $\underline{z}$ and $\underline{z}^{-1}$ for the indeterminates. If $j \in \mathbb{Z}^n$ is a multi-index $j = (j_1, \ldots, j_n)$, then $\underline{\sigma}^j = \sigma_1^{j_1} \ldots \sigma_n^{j_n}$; an analogous interpretation holds for $\underline{z}^j$.

We will refer to $\mathcal{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$ as a *kernel behavior*. Since the matrix R uniquely specifies the behavior $\mathcal{B}$, we also say that R is a *representation* of $\mathcal{B}$. Clearly, every kernel behavior $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^n}$ is a linear shift-invariant subspace of $(\mathbb{R}^q)^{\mathbb{Z}^n}$.

3 Markov properties

For the sake of simplicity, in the sequel we restrict to the 2D case (i.e., take n=2); however our definitions and results go through to the general nD case with small adaptations.

A straightforward generalization of the 1D Markov property to 2D behaviors is given in definition 1 below. Before stating this definition we introduce some useful concepts.

We define an interval of $\mathbb{Z}^2$ as a set $I = ((a, b) \times (c, d)) \cap \mathbb{Z}^2$ where a and c may be $-\infty$ and b and d may be $+\infty$. Since we only consider discrete intervals, from now on we simply write $I = (a, b) \times (c, d)$. Note that I can be the whole $\mathbb{Z}^2$, a rectangle, a horizontal or vertical strip, or a horizontal or vertical line in the discrete grid. Given an interval $I \subset \mathbb{Z}^2$, let (T_-, T_0, T_+) be a partition of I. The set T_0 is said to *separate* T_- and T_+ if every path from T_- to T_+ formed by nearest neighbors intersects T_0. Given two trajectories $w_1, w_2 \in \mathcal{B}|_I$, the *concatenation* of w_1 with w_2 in I with respect to the partition (T_-, T_0, T_+), denoted by $w_1 \wedge_{T_0} w_2$, is defined as a trajectory $w \in \mathcal{B}|_I$ such that $w|_{T_-} = w_1|_{T_-}$ and $w|_{T_+ \cup T_0} = w_2|_{T_+ \cup T_0}$.

Definition 1 *Given an interval $I \subset \mathbb{Z}^2$, a 2D kernel behavior $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ is said to be* Markovian in *I if the following holds. For every partition (T_-, T_0, T_+) of I such that T_0 separates T_- and T_+, if $w_1, w_2 \in \mathcal{B}|_I$ are such that $w_1|_{T_0} = w_2|_{T_0}$ then $w_1 \wedge_{T_0} w_2 \in \mathcal{B}|_I$. The behavior $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ is simply called* Markovian *if it is Markovian in $\mathbb{Z}^2$.*

As shown in [2], the 2D behavior

$$\mathcal{B} = \ker \begin{bmatrix} \sigma_2^3 - \sigma_2^2 - \sigma_2 - 1 \\ \sigma_1 - \sigma_2^3 \end{bmatrix}$$

is Markovian, but cannot be represented by first order equations. This example led to the consideration of a stronger version of this property, [2], that we here call the strong Markov property, see also [3] .

Definition 2 *A 2D kernel behavior $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ is said to be* strong Markovian *if it is Markovian in I for every interval $I \subset \mathbb{Z}^2$.*

It turns out, [2], that a 2D kernel behavior is strong Markovian if and only if it can be represented as $\mathcal{B} = \ker R(\underline{\sigma})$, for a suitable 2D polynomial matrix $R(\underline{z}) = R_{00} + R_{10}z_1 + R_{01}z_2 + R_{11}z_1z_2$ with intricate structure. In the next section we show how this characterization can be simplified in case $\mathcal{B}$ is a finite-dimensional subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$.

4 Finite-dimensional strong Markov behaviors

A 2D kernel behavior $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ is finite-dimensional (as a subspace of $(\mathbb{R}^q)^{\mathbb{Z}^2}$) if and only if it can be represented as $\mathcal{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$ where $R(\underline{z}, \underline{z}^{-1})$ is a right-prime 2D L-polynomial matrix, [1]. In this case, if $\mathcal{B} \neq \{0\}$ it can also be represented by a latent variable model of the form

$$\begin{cases} \sigma_1 x &= A_1 x \\ \sigma_2 x &= A_2 x \\ w &= Cx \end{cases} \tag{1}$$

where A_1 and A_2 are nonsingular commuting matrices of size N, x is the latent variable and w is the system variable, [1]. Such model is denoted by (C, A_1, A_2). This means that the trajectories $w \in \mathcal{B}$ are given by $w(i,j) = CA_1^iA_2^jx(0,0)$, for arbitrary initial values $x(0,0)$ of the latent variable. Moreover [4] the model (C, A_1, A_2), can always be taken to be observable, i.e., such that if two trajectories $w_1, w_2 \in \mathcal{B}$ coincide in the interval $[0, N-1] \times [0, N-1]$ then the corresponding initial values $x_1(0,0)$ and $x_2(0,0)$ of the latent variable coincide. This is equivalent [4] to the condition

$$\text{rank} \quad \mathcal{O} := \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2^2 \\ CA_1^3 \\ \vdots \\ CA_1^{N-1}A_2^{N-1} \end{bmatrix} = N. \tag{2}$$

The key to our problem is the following lemma.

Lemma 1 *Let $\mathcal{B}$ be a 2D finite-dimensional kernel behavior. If in addition $\mathcal{B}$ is strong Markovian then it can be represented by a latent variable model (C, A_1, A_2) where the matrix C has full column rank.*

Proof.

We start by showing that, together with observability, the strong Markov property implies that the pairs (C, A_1) and (C, A_2) are observable in the classical 1D sense. Let $x_1(0,0)$ and $x_2(0,0)$ be two initial values of the latent variable x such that

$$CA_2^j x_1(0,0) = CA_2^j x_2(0,0), \quad \text{for} \quad \text{all} j \in \mathbb{Z}.$$

Then, the corresponding system trajectories $w_1(i,j) = CA_1^iA_2^jx_1(0,0)$ and $w_2(i,j) = CA_1^iA_2^jx_2(0,0)$ are such that $w_1(0,j) = w_2(0,j)$ for all $j \in \mathbb{Z}$. Consider a partition (T_-, T_0, T_+) of $\mathbb{Z}$ given by $T_- = \{(i,j) \in \mathbb{Z}^2 : i < 0\}$, $T_0 = \{(i,j) \in \mathbb{Z}^2 : i = 0\}$ and $T_0 = \{(i,j) \in \mathbb{Z}^2 : i > 0\}$. Since $\mathcal{B}$ is strong Markovian and w_1 and w_2 coincide in T_0, the trajectory $w^* = w_1 \wedge_{T_0} w_2$ is in $\mathcal{B}$. Hence, there exists $x^*(0,0)$ such that $w^*(i,j) = CA_1^iA_2^jx^*(0,0)$. Now, since w^* coincides with w_2 in T_+, in particular these trajectories coincide in the discrete interval $[0, N-1] \times [0, N-1]$ and, due to the observability of (C, A_1, A_2), we conclude that $x_2(0,0) = x^{**}(0,0)$. On the other hand, since w^* coincides with w_1 in T_-, in particular these trajectories coincide in the discrete interval $[-(N-1), 0] \times [-(N-1), 0]$. Therefore, observability, together with the fact that the matrices A_1 and A_2 are invertible and commute, yields that $x_1(0,0) = x^{**}(0,0)$. Thus

$$x_1(0,0) = x_2(0,0).$$

This means that $\cap_{j\in\mathbb{Z}} \ker ACA_2^j = \{0\}$, which implies, by the invertibility of A_2, that

$$\operatorname{rank}\begin{bmatrix} C \\ CA_2 \\ \vdots \\ CA_2^{N-1} \end{bmatrix} = N,$$

i.e., (C, A_2) is a 1D observable pair. The proof of the observability of (C, A_1) is analogous.

It follows from the definition of the strong Markov property that $\mathcal{B}$ is Markovian in $I = T_0 = \{(i,j) \in \mathbb{Z}^2 : i = 0\}$. By arguments similar to the ones above, it is possible to prove that, together with the observability of (C, A_2), this implies that C has full column rank. Before proceeding we note that, due to the fact that A_1 and A_2 are invertible and commute, it is not difficult to see that $\mathcal{B}|_I$ is the set of all trajectories of the form $\bar{w}(j) = w(0,j) = CA_2^j x(0,0)$, for arbitrary initial values of the latent variable $x(0,0)$. Let now $x_1(0,0)$ and $x_2(0,0)$ be two initial values of the latent variable x such that

$$Cx_1(0,0) = Cx_2(0,0).$$

Then the corresponding trajectories $\bar{w}_1(j) = w_1(0,j) = CA_2^j x_1(0,0)$ and $\bar{w}_2(j) = w_2(0,j) = CA_2^j x_2(0,0)$ (of $\mathcal{B}|_I$) coincide at $(0,0)$. Since $\mathcal{B}$ is Markovian in I, $\bar{w}^{**} = \bar{w}_1 \wedge_{(0,0)} \bar{w}_2$ is still a trajectory of $\mathcal{B}|_I$, implying that $\bar{w}^{**}(j) = CA_2^j x^{**}(0,0)$ for some $x^{**}(0,0)$. Since $\bar{w}^{**}$ coincides with $\bar{w}_1$ in $I_- := \{(i,j) \in \mathbb{Z}^2 : i = 0, j < 0\}$ and with $\bar{w}_2$ in $I_+ := \{(i,j) \in \mathbb{Z}^2 : i = 0, j > 0\}$ the observability of the pair (C, A_2), together with the invertibility of A_2, allows to conclude that

$$x_1(0,0) = x^{**}(0,0) = x_2(0,0).$$

This means that C has full column rank. ■

Remark. Note that (1) is in fact a 2D state space model for $\mathcal{B}$. Thus the previous lemma means that, as should be expected, in case this model is observable and $\mathcal{B}$ is strong Markovian, the components of the state variable x at each point (i, j) can be obtained as linear combinations of the system variable values $w(i, j)$, i.e., x can be obtained by means of linear *static* relations on w.

In the sequel a latent variable model (C, A_1, A_2) where C has full column rank is referred to as an FCR model. Assume that $\mathcal{B} \subset \mathbb{R}^{q\mathbb{Z}^2}$ has an FCR model (C, A_1, A_2), then equations (1) can be written in matrix form as

$$\begin{bmatrix} \sigma_1 I_N - A_1 \\ \sigma_2 I_N - A_2 \\ C \end{bmatrix} x = \begin{bmatrix} 0_{N\times q} \\ 0_{N\times q} \\ I_q \end{bmatrix} w. \tag{3}$$

Applying to both sides of this equation the invertible operator

$$U(\underline{\sigma}) = \begin{bmatrix} I_N & 0_N & -(\sigma_1 I_N - A_1)E \\ 0_N & I_N & -(\sigma_2 I_N - A_2)E \\ 0_N & 0_N & S \end{bmatrix}$$

where E is a left-inverse of C and $S = [E^T \;\; F^T]^T$, for some suitable matrix F, is invertible, we obtain the equivalent equations

$$\begin{cases} \begin{bmatrix} (\sigma_1 I_N - A_1)E \\ (\sigma_2 I_N - A_2)E \\ F \end{bmatrix} w &= \quad 0 \\ \qquad\qquad x &= \quad Ew. \end{cases} \tag{4}$$

This allows to eliminate the latent variable x from the description of $\mathcal{B}$ and to obtain the representation

$$\begin{bmatrix} (\sigma_1 I_N - A_1)E \\ (\sigma_2 I_N - A_2)E \\ F \end{bmatrix} w = 0, \tag{5}$$

yielding the following result.

Proposition 1 *Let $\mathcal{B}$ be a 2D finite-dimensional kernel behavior. Then, if $\mathcal{B}$ is strong Markovian, it can be represented by means of partial difference equations of the form (5), where*

(A_1, A_2) *is a pair of invertible commuting matrices and the matrix* $S = [E^T \quad F^T]^T$ *is invertible.*

We next show that the converse is also true.

Proposition 2 *Let* $\mathcal{B}$ *be a 2D finite-dimensional kernel behavior. Then, if it can be represented by means of partial difference equations of the form (5), where* (A_1, A_2) *is a pair of invertible commuting matrices and the matrix* $S = [E^T \quad F^T]^T$ *is invertible,* $\mathcal{B}$ *is strong Markovian.*

Proof.
Let $\mathcal{B}$ have a representation as stated in the proposition. Consider the transformed behavior $\hat{\mathcal{B}} := S(\mathcal{B})$. Since S corresponds to an invertible static transformation, it is clear that $\mathcal{B}$ is strong Markovian if and only if so is $\hat{\mathcal{B}}$. Now, $\hat{\mathcal{B}}$ is the set of all trajectories $\hat{w} = [x^T z^T]^T$ such that

$$\begin{cases} \begin{bmatrix} \sigma_1 I_N - A_1 \\ \sigma_2 I_N - A_2 \end{bmatrix} x & = & 0 \\ \qquad z & = & 0. \end{cases} \tag{6}$$

Taking into account that (A_1, A_2) is a pair of commuting invertible matrices, we easily conclude that the restriction $\hat{\mathcal{B}}|_I$ of $\hat{\mathcal{B}}$ to an interval $I \subset \mathbb{Z}^2$ is the set of all trajectories $\hat{w} = [x^T z^T]^T$ such that

$$z(i, j) = 0$$

and

$$x(i, j) = A_1^{i-i_0} A_2^{j-j_0} x(i_0, j_0),$$

for all $(i, j) \in I$, where (i_0, j_0) is a particular arbitrary point in I and the value of $x(i_0, j_0) \in \mathbb{R}^N$ is arbitrary. Clearly, if such two trajectories coincide at one single point $(i^*, j^*) \in I$ then they coincide everywhere in I (and in $\mathbb{Z}^2$). This implies that $\hat{\mathcal{B}}$ is Markovian in every interval $I \subset \mathbb{Z}^2$, and is hence strong Markovian. ■

Combining Propositions 1 and 2 we obtain the main result of this paper.

Theorem 1 *Let $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^2}$ be 2D finite-dimensional kernel behavior. Then the following are equivalent.*

1. *$\mathcal{B}$ is strong Markovian*

2. *$\mathcal{B}$ can be represented by means of partial difference equations of the form*

$$\begin{bmatrix} (\sigma_1 I_N - A_1)E \\ (\sigma_2 I_N - A_2)E \\ F \end{bmatrix} w = 0, \tag{7}$$

   where (A_1, A_2) is a pair of invertible commuting matrices and the matrix $S = [E^T \;\; F^T]^T$ is invertible.

Note that the representation of Theorem 1 consists in two decoupled first order partial difference matrix equations (one of which only involving the shift σ_1 and the other involving only σ_2) together with a static relation. In case the behavior $\mathcal{B}$ is trim, i.e. if given $v \in \mathbb{R}^q$ there exist a trajectory $w \in \mathcal{B}$ and a point $(i, j) \in \mathbb{Z}^2$ such that $w(i, j) = v$, then no static relation is present and $\mathcal{B}$ is simply represented by two decoupled first order equations.

Theorem 1 easily generalizes to the nD case, yielding the following characterization.

Theorem 2 *An nD finite-dimensional kernel behavior $\mathcal{B} \subset (\mathbb{R}^q)^{\mathbb{Z}^n}$ is strong Markovian if and only if it can be represented by means of partial difference equations of the form*

$$\begin{bmatrix} (\sigma_1 I_N - A_1)E \\ (\sigma_2 I_N - A_2)E \\ \vdots \\ (\sigma_n I_N - A_n)E \\ F \end{bmatrix} w = 0, \tag{8}$$

where $A_1, A_2, \ldots, A_n$ are invertible pairwise commuting matrices and the matrix $S = [E^T \;\; F^T]^T$ is invertible.

5 Conclusion

Taking advantage of the existence of a first order latent variable representation with special structure for discrete 2D finite-dimensional behaviors, we have shown that for such behaviors the strong Markov property is equivalent to the representability by means of two special decoupled first order equations, together with a static relation. This gives a complete characterization of this property in representation terms. However, a question which remains here unanswered (but is the subject of our current investigation) is the following: is every finite-dimensional behavior represented by two decoupled first order matrix difference equations a strong Markovian behavior? A positive answer to this question would provide a full solution to the representability conjecture of [6].

It is our conviction that the obtained results go through to the continuous case. The research concerning this case will be reported in due time.

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