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Hidden Variables in Dissipative Systems

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Abstract— **The relevance of hidden variables in dissipative systems is examined. A definition of a dissipative dynamical system is introduced in which the storage function does not need to be observable from the variables that enter into the supply rate. It is shown that a controllable system is dissipative if and only if it has an observable storage function, but that there are uncontrollable dissipative systems that do not have an observable storage function. Finally, we argue that *unobservable storage functions are indispensable*: they occur in physical systems, for example in electrical circuits and in Maxwell's equations.**

Index Terms— **Behavioral systems, dissipative systems, storage functions, controllability, observability, unobservable storage functions.**

I. INTRODUCTION

A common assumption in the analysis of dynamical systems, particularly in control and signal processing, is the assumption of minimality, i.e. state controllability and state observability. It is well-known, of course, that these properties are not always valid. For example, traditional stability questions mainly deal with autonomous systems. Also, in control, after applying feedback, a system may loose controllability, and in problems of disturbance decoupling it is the explicit purpose to make the system unobservable.

Recently, we have discovered that in the analysis of physical systems, hidden variables (an informal term to refer mainly to lack of observability, sometimes combined with controllability) play an important, but sometimes subtle role.

It is the purpose of this paper to explain that in the theory of dissipative systems hidden variables are indispensable. In particular, we argue that a in good definition of dissipative system, one should not ask the storage function to be observable from the system variables that enter in the supply rate (or, for that matter, from the state of a particular state representation of their transfer function). Also, we will show that for systems described by partial differential equations, hidden variables occur unavoidably in potential representations, and in the storage function.

Many of the remarks and results in this paper have appeared with a different emphasis before [6], [4]. The main original result of this paper is the proof that a controllable linear differential system that is dissipative (which means that the storage function need not be observable) always allows also an observable storage function.

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Finally, we emphasize through examples (electrical circuits and Maxwell's equations) the relevance of hidden variables in physical systems.

II. 1-D DIFFERENTIAL SYSTEMS

We use the behavioral language [10], [7]. A (1-D) dynamical system is defined as $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T} \subseteq \mathbb{R}$ the time axis, $\mathbb{W}$ the signal space, and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior*. In the present paper, we deal almost exclusively with continuous time systems with time axis $\mathbb{T} = \mathbb{R}$, $\mathbb{W}$ a finite dimensional real vector space, and behavior $\mathfrak{B}$ that consists of the set of solutions of a system of linear, constant coefficient differential equations, i.e. systems $\Sigma = (\mathbb{R}, \mathbb{R}^{\bullet}, \mathfrak{B})$ (the notation $\bullet$ means that the dimension of a vector or a matrix does no need to be specified – but is, of course, finite) for which there exists a polynomial matrix $R \in \mathbb{R}^{\bullet \times \dim(w)}[\xi]$ such that

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{dim(w)}) \mid R(\tfrac{d}{dt})w = 0\},$$

equivalently $\mathfrak{B} = \ker(R(\frac{d}{dt}))$. The $\mathfrak{C}^{\infty}$ assumption, which will be used throughout for the solutions of the differential equations which are encountered, is mainly a matter of convenience of exposition.

This family of systems is denoted as $\mathfrak{L}^{\bullet}$, and $\mathfrak{L}^{\mathtt{w}}$ if the number of variables is $\mathtt{w}$, whence $\mathfrak{L}^{\bullet} = \cup_{\mathtt{w} \in \mathbb{Z}_+} \mathfrak{L}^{\mathtt{w}}$. Since the time axis equals $\mathbb{R}$, we use both notations $\Sigma \in \mathfrak{L}^{\mathtt{w}}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. If $\mathtt{w}$ is not specified, we use $\Sigma \in \mathfrak{L}^{\bullet}$ or $\mathfrak{B} \in \mathfrak{L}^{\bullet}$.

The class of systems $\mathfrak{L}^{\bullet}$ has many nice properties, for instance $[\ \mathfrak{B}_1, \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}\ ] \Rightarrow [\ \mathfrak{B}_1 \cap \mathfrak{B}_2, \mathfrak{B}_1 + \mathfrak{B}_2 \in \mathfrak{L}^{\mathtt{w}}\ ]$. Important, but perhaps less expected, is the *elimination theorem* which states that

$$[\ \mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}\ ] \Rightarrow [\ \{w_1 \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \mid \\ \exists\, w_2 \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \text{ such that } (w_1, w_2) \in \mathfrak{B}\} \in \mathfrak{L}^{\mathtt{w}_1}\ ].$$

These systems admit many possible representations. We have already met *kernel representations*,

$$R(\frac{d}{dt})w = 0,$$

through which the family of systems $\mathfrak{L}^{\bullet}$ is actually defined. In view of the elimination theorem, we also obtain *latent variable representations*

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell, \qquad (1)$$

with $M \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ and ℓ the *latent variable*, w the *manifest variable*, and $\mathfrak{B} = \left(R(\frac{d}{dt})\right)^{-1} M(\frac{d}{dt}) \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{dim(\ell)})$ the *external*, or *manifest* (whatever is more appropriate in the context) behavior of this latent variable representation. We call (1) a *latent variable representation* of its external behavior.

Another type of representation, which is always possible, are *state representations*. Finally, we also have *input/state/output state representations*.

The system $\mathfrak{B} \in \mathfrak{L}^\bullet$ is said to be [*controllable*] $:\Leftrightarrow$ [$\forall w_1, w_2 \in \mathfrak{B}$, $\exists\ v \in \mathfrak{B}$ and $T \geq 0$, such that $v(t) = w_1(t)$ for $t < 0$, and $v(t+T) = w_2(t)$ for $t \geq 0$]. Denote by $\mathfrak{L}^\bullet_{\text{controllable}}$, respectively $\mathfrak{L}^{\mathtt{w}}_{\text{controllable}}$, the controllable elements of $\mathfrak{L}^\bullet$, respectively $\mathfrak{L}^{\mathtt{w}}$.

An important result from the behavioral theory states that a system $\mathfrak{B} \in \mathfrak{L}^\bullet$ is controllable if and only if it admits a *image representation*, that is, a latent variable representation of the following special form:

$$w = M(\frac{d}{dt})\ell.$$

Hence $\mathfrak{B} = \operatorname{im}(M(\frac{d}{dt}))$.

In this paper, we need the notion of observability in its full generality, and not only for linear differential systems, since we will consider observability of storage functions, which are nonlinear (quadratic) functions of the system variables. Let $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ be a dynamical system. Hence each element $w \in \mathfrak{B}$ consists of two components, $w = (w_1, w_2)$. Interpret the variables w_1 as 'observed' and w_2 as 'to-be-deduced' from the observed w_1. We say that [w_2 is *observable* from w_1 in Σ] $:\Leftrightarrow$ [$(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$]. That is, if and only if there exists a map $F : \mathbb{W}_1^{\mathbb{T}} \to \mathbb{W}_2^{\mathbb{T}}$ such that [$(w_1, w_2) \in \mathfrak{B}$] $\Rightarrow$ [$w_2 = F(w_1)$].

For latent variable representations we use the notion of observability to mean that the latent variables are observable from the manifest ones. Explicitly, the latent variable representation (1) is said to be *observable* if, whenever (w, ℓ_1) and (w, ℓ_2) both satisfy (1), then $\ell_1 = \ell_2$, equivalently, it turns out, if and only if there exists a polynomial matrix $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that (w, ℓ) satisfies (1) $\Rightarrow$ $\ell = F(\frac{d}{dt})w$. If a latent variable is not observable, then we think of it as *'hidden'*.

It can be shown that a controllable system $\mathfrak{B} \in \mathfrak{L}_{\text{controllable}}$ always admits an observable image representation. State or input/state/output representations are observable if and only if the state space is minimal (over all such representations of a given $\mathfrak{B}$). Whence in our setting minimality of a state representation is equivalent to observability. Controllability enters in the following sense: a minimal state representation of a behavior $\mathfrak{B} \in \mathfrak{L}^\bullet$ is state controllable (defined in the usual way: every state may be steered to every other state) if and only if $\mathfrak{B} \in \mathfrak{L}^\bullet_{\text{controllable}}$.

III. Quadratic differential forms

In this paper we only consider supply rates that are quadratic differential forms. These are very effectively parameterized using two-variable polynomial matrices. Let $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ denote the set of real two-variable polynomial matrices in the indeterminates ζ and η. An element of this set, say $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$, is hence a finite sum

$$\Phi(\zeta, \eta) = \sum_{\mathrm{k}',\mathrm{k}''} \Phi_{\mathrm{k}'\mathrm{k}''} \zeta^{\mathrm{k}'} \eta^{\mathrm{k}''}.$$

To each Φ, we associate the *bilinear differential form*

$$L_\Phi(v, w) := \sum_{\mathrm{k}',\mathrm{k}''} \left(\frac{d^{\mathrm{k}'}}{dt^{\mathrm{k}'}} v\right)^\top \Phi_{\mathrm{k}'\mathrm{k}''} \left(\frac{d^{\mathrm{k}''}}{dt^{\mathrm{k}''}} w\right).$$

Note that L_Φ is mapping from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. If Φ is square, $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, then it induces a *quadratic differential form* given by

$$Q_\Phi(w) := L_\Phi(w, w).$$

Note that, because of the quadratic nature of Q_Φ, we may as well assume that Φ is symmetric, i.e. $\Phi = \Phi^*$, with $\Phi^*(\zeta, \eta) := \Phi^\top(\eta, \zeta)$. Observe that the derivative of a quadratic (or bilinear) differential is also a quadratic (or bilinear) differential form. Quadratic differential forms have been studied in detail in [11].

IV. Dissipativity of 1-D systems

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ (not necessarily controllable) is said to be
[*dissipative* with respect to the supply rate Q_Φ] $:\Leftrightarrow$
[$\exists$ a latent variable representation (1) of $\mathfrak{B}$ and a $\Psi \in \mathbb{R}^{\ell\times\ell}[\zeta, \eta]$ such that the *dissipation inequality*

$$\frac{d}{dt} Q_\Psi(\ell) \leq Q_\Phi(w)$$

holds for all (w, ℓ) that satisfy (1)].
The quadratic differential form Q_Ψ that appears in the dissipation inequality is called a *storage function*. When the dissipation inequality holds as an equality, we say that $\mathfrak{B}$ is Q_Φ*-lossless* or *-conservative*.

If the storage function acts on w, i.e. $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ and

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w)$$

for all $w \in \mathfrak{B}$, then we call the storage function *observable*. Of course, if ℓ is observable from w in the latent variable representation, then the storage function $Q_\Psi(\ell)$ induces a storage function $Q_{\Psi'}(w)$. We also call such storage functions observable. But if ℓ is not observable from w in the latent variable representation, then $Q_\Psi(\ell)$ is a function of hidden variables. What we want to discuss in this paper is the rationale for using hidden variables in storage functions, unobservable storage functions.

Non-negative storage functions are very important in applications, but we will not consider them in the present paper. Our storage functions need not be sign definite.

V. Storage functions for controllable systems

The main issue discussed in this paper is the question whether unobservable storage functions are indispensable in the theory of dissipative systems. The answer to this question is, as we shall see, an unambiguous *'yes'*. However, we start with a result that shows that for dissipative controllable systems there are always observable storage functions.

Theorem: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{controllable}}$ and $\Phi = \Phi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The following are equivalent:

1. $\mathfrak{B}$ is dissipative with respect to Q_Φ, i.e. there exists a latent variable representation $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ and a $\Psi = \Psi^* \in \mathbb{R}^{\ell\times\ell}[\zeta,\eta]$ such that
$$\frac{d}{dt}Q_\Psi(\ell) \leq Q_\Phi(w)$$
for all (w,ℓ) that satisfy $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$;
2. $\mathfrak{B}$ is dissipative with respect to Q_Φ, with an observable storage function, i.e. there exists a $\Psi = \Psi^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ such that
$$\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w)$$
for all $w \in \mathfrak{B}$;
3. for all periodic $w \in \mathfrak{B}$ there holds
$$\oint Q_\Phi(w)\ dt \geq 0,$$
where $\oint$ denotes integration over a period;
4. for all $w \in \mathfrak{B}$ such that $Q_\Phi(w) \in \mathfrak{L}(\mathbb{R},\mathbb{R})$, there holds
$$\int_{-\infty}^{+\infty} Q_\Phi(w)\ dt \geq 0;$$
5. $$N(-i\omega)^\top\Phi(-i\omega,i\omega)N(i\omega) \geq 0 \quad \forall\omega \in \mathbb{R},$$
where $w = N(\frac{d}{dt})\ell$ is an image representation of $\mathfrak{B}$.

Proof: The equivalence of 2., 3., 4., and 5. are classical in the theory of dissipative systems (see [11]). We do not dwell on these. The fact that also 1. implies any of 2., 3., 4., or 5. is actually the only original part of this theorem. We prove *(1. ⇒ 5.)*.

Start from 1.: $\frac{d}{dt}Q_\Psi(\ell') \leq Q_\Phi(w)$ for all (w,ℓ') that satisfy $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell'$. By premultiplying R and M by a unimodular polynomial matrix, and by postmultiplying M by a unimodular matrix, we can write these behavioral equations as
$$R'(\frac{d}{dt})w = 0,\ \ R''(\frac{d}{dt})w = \begin{bmatrix} M''(\frac{d}{dt}) & 0\end{bmatrix} V(\frac{d}{dt})\ell',$$
with M'' square, $\det(M'') \neq 0$, and V unimodular. The equation $R'(\frac{d}{dt})w = 0$ is actually a kernel representation of $\mathfrak{B}$, and can, by controllability, be replaced by the image representation
$$w = N(\frac{d}{dt})\ell.$$

Define $\Phi'(\zeta,\eta) := N^\top(\zeta)\Phi(\zeta,\eta)N(\eta)$. Denote the first component of $V(\frac{d}{dt})\ell'$, in a partition compatible with the partition $\begin{bmatrix} M' & 0\end{bmatrix}$, by ℓ''.

We obtain that there exists Ψ' such that
$$\frac{d}{dt}Q_{\Psi'}(\ell'') \leq Q_{\Phi'}(\ell)$$
for all (ℓ,ℓ'') that satisfy $R''(\frac{d}{dt})N(\frac{d}{dt})\ell = M''(\frac{d}{dt})\ell''$, with M'' square and with non-zero determinant. This implies, in particular, that ℓ is free, hence that for any $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\dim(\ell)})$, there exists at least one corresponding $\ell'' \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\dim(\ell'')})$ satisfying these behavioral equations. Application of this, with a well-chosen class of ℓ's will give us the result that we are after.

Take $\ell(t) = e^{i\omega t}a$ with $\omega \in \mathbb{R}$ and $a \in \mathbb{C}^{\dim(\ell)}$. Note that, for convenience, we use complex-valued signals. Taking the real part yields the validity of the conclusions in the real case. Note that, for ω such that $\det(M''(i\omega)) \neq 0$, $\ell''(t) = e^{i\omega t}(M''(i\omega))^{-1}R''(i\omega)N(i\omega)a$ is a corresponding ℓ''. Now integrate, for this periodic (ℓ,ℓ''),
$$\frac{d}{dt}Q_{\Psi'}(\ell'') \leq Q_{\Phi'}(\ell)$$
over one period, and obtain $\Phi'(-i\omega,i\omega) \geq 0$. Hence
$$\Phi'(-i\omega,i\omega) \geq 0 \text{ for } \omega \in \mathbb{R}, \det(M''(i\omega)) \neq 0.$$
By continuity,
$$\Phi'(-i\omega,i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R}.$$
5. follows. □

In [9] it has been proven that every observable storage function of a controllable system is a memoryless function of the state in any minimal state space representation of a *suitable behavior. This behavior, however, depends on* Φ as well as on $\mathfrak{B}$. See [9] for the precise statement.

VI. Unobservable storage functions

The following is an example of a system that is dissipative, but has no observable storage function. Consider the system with manifest variable $w = (w_1,w_2)$ and behavioral equations
$$\frac{d}{dt}x = Ax,\ \ w_1 = Cx;\ \ w_2 \text{ free}$$
and the supply rate
$$w_1^\top w_2.$$
Hence w_1 satisfies an autonomous system of differential equations, and w_2 is free. This system is obviously not controllable. The following is a latent variable representation of it:
$$\frac{d}{dt}x = Ax,\ \ w_1 = Cx;\ \ \frac{d}{dt}z = -A^\top z + C^\top w_2,$$
with latent variables (x,z). Note that this latent variable representation is obviously not observable (even when (A,C) *is what is called an observable pair of matrices).*

Now verify that
$$\frac{d}{dt}x^\top z = w_1^\top w_2.$$
Hence this system is dissipative, in fact conservative. However, it can easily be verified that there exists no Ψ such that
$$\frac{d}{dt}Q_\Psi(w_1,w_2) \leq w_1^\top w_2$$
for all (w_1,w_2) in the behavior.

The above example is, as many counterexamples, somewhat degenerate. However, the following electrical circuit shows that unobservable storage functions are a physical reality.

The following equations form a latent variable representation for the port variables (V,I) obtained from first principles modeling:

$$\begin{aligned} R_L I_L + L\frac{d}{dt}I_L &= V, \\ V_C + R_C C\frac{d}{dt}V_C &= V, \\ \frac{V-V_C}{R_C} + I_L &= I. \end{aligned}$$

Here, I_L, the current through the inductor, and V_C, the voltage across the capacitor, should be considered as latent variables. After elimination of these latent variables, we obtain, in the case $R_L = 1, R_C = 1, L = 1, C = 1$, the manifest behavioral equation

$$(1+\frac{d}{dt})V = (1+\frac{d}{dt})I. \qquad (2)$$

This system is obviously not controllable.

This system is dissipative with respect to the supply rate VI. In fact, the internally stored energy,

$$\frac{1}{2}CV_C^2 + \frac{1}{2}LI_L^2$$

is a storage function, and

$$R_L I_L^2 + \frac{1}{R_C}(V - V_C)^2$$

is the corresponding dissipation rate. In the case $R_L = 1, R_C = 1, L = 1, C = 1$, this storage function is, however, not observable.

For the system (2) with the supply rate VI, there are, however, also observable storage function. For example,

$$\frac{d}{dt}x = -x, V = \gamma x + I,$$

with $\dim(x) = 1$ is a minimal state representation of (2) for all $\gamma \neq 0$, and $\frac{1}{2}x^2$ is an observable storage function for all $\gamma \leq 1$. In fact, since the symmetric part of the associated 'system matrix' $\left[\begin{smallmatrix} -A & -B \\ C & D \end{smallmatrix}\right]$ is ≥ 0, this state representation, and hence the port behavior, can de realized as an electrical circuit containing one unit capacitor, positive resistors, gyrators, and transformers (for this, and other circuit theory results, see, for example [1]). However, there are *no* reciprocal circuit realizations (realizations without gyrators) that are minimal, in the sense that they use *only one* reactive element (a capacitor or an inductor), and further positive resistors and transformers. Indeed, for such a first order realization, the system matrix $\left[\begin{smallmatrix} -A & -B \\ C & D \end{smallmatrix}\right]$ would have to be signature symmetric, but this would imply (A,B) is controllable, contradicting the uncontrollability of (2). All minimal state representations of (2) are uncontrollable.

Conclusion: there exist reciprocal circuit realizations of

$$(1+\frac{d}{dt})V = (1+\frac{d}{dt})I.$$

In fact, one is given in the above figure, with $R_L = 1, R_C = 1, L = 1, C = 1$. But these reciprocal realizations necessarily all have an unobservable storage functions. Hence a complete theory of dissipative physical systems must allow for unobservable storage function.

This leads to the following open problems which were announced at the 2003 CDC [4].

The most classical result of circuit theory is undoubtedly the fact that g is the driving point impedance of a circuit containing a finite number of positive resistors, capacitors, inductors, and transformers if and only if g is rational and positive real. This classic result was obtained by Brune [3]. In 1949, Bott and Duffin [2] proved that transformers are not needed.

It seems to us that a more 'complete' version of this classical problem is to ask for the realization of a differential behavior. This problem is more general than the driving point impedance problem, because of the existence of uncontrollable systems. For example, a unit resistor realizes the transfer function of the system (2) as its driving point impedance, but not its behavior (which admits, for example, the short circuit response $I(t) = e^{-t}, V(t) = 0$, not realized by the resistor).

Problem 1: *What behaviors $\mathfrak{B} \in \mathfrak{L}^2$ are realizable as the port behavior of a circuit containing a finite number of passive resistors, capacitors, inductors, and transformers?*

It is easy to see that $\mathfrak{B}$ must be single input / single output, and that the transfer function must be rational and positive real. In addition $\mathfrak{B}$ must be passive, but in general it may have a non-observable storage function, and therefore it is not clear what this says in terms of $\mathfrak{B}$.

Problem 2: *Is it possible to realize a controllable single input / single output system with a rational positive real transfer function as the behavior of a circuit containing a finite number of passive resistors, capacitors, and inductors, but no transformers?*

Note that in a sense this is the Bott-Duffin problem, the issue being that the Bott-Duffin synthesis procedure usually realizes a non-controllable system that has the correct transfer function (i.e., the correct controllable part), but not the correct behavior. In this case, there are standard synthesis procedures known that do realize the correct behavior, but they need transformers.

VII. n-D SYSTEMS

In partial differential equations, the occurrence of hidden variables is very prevalent. We illustrate this by the introduction of potentials, and by dissipative systems described by partial differential equations.

Define a linear shift-invariant n-*D differential system* as $\Sigma = (\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, with behavior $\mathfrak{B}$ consisting of the solution set of a system of partial differential equations

$$R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0 \tag{3}$$

viewed as an equation in the maps

$$(x_1, \ldots, x_{\mathtt{n}}) = x \in \mathbb{R}^{\mathtt{n}} \mapsto (w_1(x), \ldots, w_{\mathtt{w}}(x)) = w(x) \in \mathbb{R}^{\mathtt{w}}.$$

Here, $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_{\mathtt{n}}]$ is a matrix of polynomials in $\mathbb{R}[\xi_1, \ldots, \xi_{\mathtt{n}}]$. The behavior of this system of partial differential equations is defined as

$$\mathfrak{B} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (3) \text{ is satisfied}\}.$$

Important properties of these systems are their *linearity* (meaning that $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{R}^{\mathtt{n}}}$), and *shift-invariance* (meaning $\sigma^x \mathfrak{B} = \mathfrak{B}$ for all $x \in \mathbb{R}^{\mathtt{n}}$, where σ^x denotes the x–shift, defined by $(\sigma^x f)(x') = f(x'+x)$). The $\mathfrak{C}^{\infty}$-assumption in the definition of $\mathfrak{B}$ is made for convenience only, and there is much to be said for using distributions instead. We denote the behavior of (3) as defined above by $\ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}}))$, and the set of distributed differential systems thus obtained by $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. Note that we may as well write $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, instead of $\Sigma \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, since the set of independent variables ($\mathbb{R}^{\mathtt{n}}$) and the signal space ($\mathbb{R}^{\mathtt{w}}$) are evident from this notation. Of course, also here, the system allows many other representations.

A typical example is given by Maxwell's equations, which describe the possible realizations of the fields $\vec{E} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \vec{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \vec{j} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, and $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. Maxwell's equations

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}, \end{aligned}$$

with ε_0 the dielectric constant of the medium and c^2 the speed of light in the medium, define a distributed differential system

$$\Sigma = \left(\mathbb{R}^4, \mathbb{R}^{10}, \ker\left(R(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\right)\right) \in \mathfrak{L}^{10}_4,$$

with the matrix of polynomials $R \in \mathbb{R}^{8 \times 10}[\xi_1, \xi_2, \xi_3, \xi_4]$ easily deduced from the above equations. This defines the system $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \mathfrak{B})$, with $\mathfrak{B}$ the set of all $(\vec{E}, \vec{B}, \vec{j}, \rho)$ that satisfy Maxwell's equations.

Many of the results for 1-D systems generalize, often with non-trivial proofs, to n-D systems. For a study of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, we refer to the fundamental paper [5], where, for instance, the elimination theorem is proven. As an illustration of the elimination theorem, consider the elimination of $\vec{B}$ and ρ from Maxwell's equations. The following equations describe the possible realizations of the fields $\vec{E}$ and $\vec{j}$:

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot \vec{E} + \nabla \cdot \vec{j} &= 0, \\ \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{j} &= 0. \end{aligned}$$

We now explain the generalization of controllability to linear constant-coefficient partial differential equations. A system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ and for all bounded open subsets O_1, O_2 of $\mathbb{R}^{\mathtt{n}}$ with disjoint closure, there exists $w \in \mathfrak{B}$ such that $w|_{O_1} = w_1|_{O_1}$ and $w|_{O_2} = w_2|_{O_2}$. We denote the set of controllable elements of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ by $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\text{controllable}}$. Here again it has been shown [5] that it are precisely the controllable systems that admit an image representation. The notion of observability carries over unchanged from the 1-D to the n-D case. An important difference between the 1-D and the n-D case is that, contrary to the 1-D case, there may not exist an observable image representation of a controllable behavior in the n-D case..

Note that an image representation corresponds to what in mathematical physics is called a *potential function*. An interesting aspect is the fact that it ties the existence of a potential function with the system theoretic property of controllability: concatenability of trajectories in the behavior. In the case of Maxwell's equations, an image representation is given by

$$\begin{aligned} \vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\ \vec{B} &= \nabla \times \vec{A}, \\ \vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A}, \\ \rho &= \frac{\varepsilon_0}{c^2}\frac{\partial^2}{\partial t^2}\phi - \varepsilon_0 \nabla^2 \phi, \end{aligned}$$

where $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a scalar, and $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ a vector potential. This image representation is not observable. Maxwell's equations, in fact, do not admit an observable image representation. Hence, in mathematical physics, potentials often involve hidden variables [8].

VIII. Dissipative n-D systems

Quadratic differential forms and their notation readily generalizes to the n-D case.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\text{controllable}}$ and consider the 2n-variable polynomial matrix $\Phi = \Phi^{\star} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta_1, \ldots, \zeta_{\mathtt{n}}, \eta_1, \ldots, \eta_{\mathtt{n}}]$. Define $\mathfrak{B}$ to be *globally conservative* with respect to the supply rate Q_{Φ} if

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\ dx_1 \cdots dx_{\mathtt{n}} = 0$$

for all $w \in \mathfrak{B}$ of compact support, and *globally dissipative* if

$$\int_{\mathbb{R}^{\mathtt{n}}} Q_{\Phi}(w)\ dx_1 \cdots dx_{\mathtt{n}} \geq 0$$

for all $w \in \mathfrak{B}$ of compact support.

Consider the controllable n-D distributed dynamical system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n,controllable}}$ and the *supply rate* defined by the quadratic differential form Q_Φ. Let $w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell$ be an image representation of $\mathfrak{B}$. Then $\mathfrak{B}$ is conservative with respect to Q_Φ if and only if there exist an $n-$vector of quadratic differential forms $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_n})$ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the *flux density*, such that

$$\nabla \cdot Q_\Psi(w,\ell) = Q_\Phi(w)$$

for all (w,ℓ), that satisfy $w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell$. Here $\nabla = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n}$.

When the first independent variable is time, and the others are space variables, then the local version of the conservation law can be expressed a bit more intuitively in terms of a quadratic differential form Q_S, the *storage density*, and a 3-vector of quadratic differential forms Q_F, the *spatial flux density*, as

$$\frac{\partial}{\partial t} Q_S(w,\ell) + \nabla \cdot Q_F(w,\ell) = Q_\Phi(w)$$

for all (w,ℓ) that satisfy $w = M(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ell$, an image representation of $\mathfrak{L}^{\mathtt{w}}_{4,\mathrm{controllable}}$. Here $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

In the *1-D* case, as we have that the introduction of latent variables in the storage sense is, in a sense, unnecessary in the controllable case. However, in the *n-D* case, the introduction of latent variables cannot be avoided, because not every controllable distributed parameter system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{n}$ admits an observable image representation.

As an illustration of this result, consider Maxwell's equations. This defines a conservative distributed dynamical system with respect to the supply rate $-\vec{E} \cdot \vec{j}$, the rate of electric energy supplied to the electro-magnetic field. In other words, for all $(\vec{E}, \vec{j}) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \times \mathbb{R}^3$ of compact support that satisfy the partial differential equations obtained from Maxwell's equations after elimination of the fields $\vec{B}$ and ρ, there holds

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} \vec{E} \cdot \vec{j} \; dxdydz \; dt = 0.$$

A local version of the law of conservation of energy is provided by introducing the *stored energy density*, S, and the *energy flow* (the flux density), $\vec{F}$, the *Poynting vector*. These are related to $\vec{E}$ and $\vec{B}$ by

$$\begin{aligned} S(\vec{E},\vec{B}) &= \frac{\varepsilon_0}{2}\vec{E} \cdot \vec{E} + \frac{\varepsilon_0 c^2}{2}\vec{B} \cdot \vec{B}, \\ \vec{F}(\vec{E},\vec{B}) &= \varepsilon_0 c^2 \vec{E} \times \vec{B}. \end{aligned}$$

As is well-known, there holds,

$$\frac{\partial}{\partial t} S(\vec{E},\vec{B}) + \nabla \cdot \vec{F}(\vec{E},\vec{B}) + \vec{E} \cdot \vec{j} = 0$$

along the behavior defined by Maxwell's equations.

Note that the local version of conservation of energy involves $\vec{B}$ in addition to $\vec{E}$ and $\vec{j}$, the variables that define the rate of energy supplied. Whence $\vec{B}$ plays the role of a latent variable, and it is not possible to express conservation of energy in terms of $\vec{E}, \vec{j}$, and their partial derivatives. Hence the local energy involves hidden variables already in Maxwell's equations.

We finally discuss dissipative n-D systems. The following result gives the local version of dissipativeness for distributed differential systems.

Consider the controllable n-D distributed dynamical system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n,controllable}}$ and the *supply rate* defined by the quadratic differential form Q_Φ. Then $\mathfrak{B}$ is dissipative with respect to Q_Φ if and only if there exist:

1) a latent variable representation

$$R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell \qquad (4)$$

of $\mathfrak{B}$,

2) an $n-$vector of quadratic differential forms $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_n})$ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the *flux density*,

such that

$$\nabla \cdot Q_\Psi(w,\ell) \leq Q_\Phi(w)$$

for all (w,ℓ), that satisfy (4).

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