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# BALANCED STATE REPRESENTATIONS FROM HIGHER ORDER DIFFERENTIAL EQUATIONS

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***Abstract*— We present an algorithm to compute a balanced state representation of a system from its description in terms of polynomial matrices and higher-order differential equations.**

***Keywords:*** **Behaviors, image representation, state representation, controllability gramian, observability gramian, balancing, model reduction.**

## I. INTRODUCTION

The physical processes and systems which are nowadays being modelled mathematically exhibit an increasing complexity, and the need to use them efficiently in order to compute control actions, to run scenarios, etc. has become critical in many applications. These requirements provide the basic motivation for the reduction of the complexity of a model and for its approximation by means of a simplified one, which captures those features of the original more relevant for the application at hand.

In the context of linear systems, complexity is usually related to the minimal number of state variables needed to represent the model. Among the various methods for model reduction developed in this area, those based on the concept of *balanced state representation* has proven to be remarkably effective. Such method computes a special state-space representation of a system, one in which each component of the state vector is roughly speaking as much controllable as it is observable. Once such a representation of the system has been computed, the components of the state vector which contribute the least to its input-output behavior can be eliminated. Among the important features of this method is that under reasonable conditions, the stability of the reduced model is assured, and the existence of a remarkable error bound.

In these algorithms, it is usually *a priori* assumed that a state-space representation or the impulse response are given. However, modeling a physical system from first principles hardly ever results in such a description, which indeed usually needs to be *constructed* from the set of higher-order differential or difference equations (possibly with auxiliary variables and with static constraints among the variables) describing the model. It is therefore of interest to develop algorithms that pass directly from such a high-complexity model to a reduced state model, without the intermediate step required to compute a (non-balanced) state representation from the first principles models. The purpose of this conference paper is to present an algorithm for the construction of a balanced state representation directly from the differential equations (or the transfer function) that describe the system, in the MIMO case. The SISO case has been dealt with in much more detail in [10].

A few words on notation. In this paper we denote the fields of real and of complex numbers respectively with $\mathbb{R}$ and $\mathbb{C}$. The space of $\mathtt{n}$ dimensional real, respectively complex, vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real, respectively complex, matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{m}\times\mathtt{n}}$. The operator $\mathrm{col}$ stacks the elements (numbers, vectors, or matrices) on which it operates. The ring of polynomials with real coefficients in the indeterminate $\xi$ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates $\zeta$ and $\eta$ is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminate $\xi$ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminates $\zeta$ and $\eta$ by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, we define $R^*(\xi) := R(-\xi)^T \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$, with $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of those compact support, and with $\mathfrak{D}^+(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all $w$'s such that $w_{|(-\infty,0]}$ has compact support.

## II. THE SYSTEM EQUATIONS

In this paper we consider continuous-time finite-dimensional linear time-invariant systems in *input/output form*, described by the set of differential equations

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \tag{1}$$

where $P \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\xi]$ is assumed to be nonsingular, $Q \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$, and the transfer function $P^{-1}Q$ is a matrix of proper rational functions. The variables $u$, $y$ are the *inputs*, respectively the *outputs* of the system. Equation (1) defines the system behavior

$$\mathfrak{B} := \{(u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}+\mathtt{y}}) \mid \text{ (1) holds } \}.$$

In the following we identify the system described by (1) with its behavior $\mathfrak{B}$.

A standing assumption in this paper is that the behavior $\mathfrak{B}$ is *controllable*, meaning that for all $(u_1, y_1), (u_2, y_2) \in \mathfrak{B}$

there exists $T > 0$ and $(u, y) \in \mathfrak{B}$ such that $(u_1, y_1)(t) = (u, y)(t)$ for $t \leq 0$ and that $(u_2, y_2)(t) = (u, y)(t+T)$ for $t > 0$. It can be shown (see sec. 5.2 of [6]) that $\mathfrak{B}$ is controllable if and only if the polynomial matrix $R := \begin{bmatrix} P & -Q \end{bmatrix}$ associated with (1) is such that the rank of the complex matrix $R(\lambda) \in \mathbb{C}^{\mathtt{y}\times(\mathtt{y}+\mathtt{u})}$ is the same for each $\lambda \in \mathbb{C}$.

It can also be shown (see Th. 6.6.1 p. 229 in [6]) that controllability of $\mathfrak{B}$ is equivalent to the existence of an *image representation* for it, meaning that there exist polynomial matrices $M \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$, $N \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$ with $M$ nonsingular and $NM^{-1}$ proper, such that the *manifest behavior* of the *latent variable system* with latent variable $\ell$

$$\begin{aligned} u &= M(\frac{d}{dt})\ell \\ y &= N(\frac{d}{dt})\ell, \end{aligned} \tag{2}$$

formally defined as

$$\{(u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}+\mathtt{y}}) \mid \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}}) \text{ such that (2) holds}\}$$

is *exactly* equal to $\mathfrak{B}$. Moreover, $M$ and $N$ in (2) can be chosen such that $\ell$ is *observable* from the manifest variable $(u, y)$, meaning that for every $(u, y) \in \mathfrak{B}$, the $\ell \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{u}})$ such that (2) holds is unique. It can be shown (see Th. 5.3.3 p. 174 of [6]) that this is the case if and only the matrix $\mathrm{col}(M(\lambda), N(\lambda)) \in \mathbb{C}^{(\mathtt{u}+\mathtt{y})\times\mathtt{u}}$ has full column rank $\mathtt{u}$ for all $\lambda \in \mathbb{C}$, equivalently, if $M$ and $N$ are right co-prime.

Besides kernel and image representations, we use state equations

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3}$$

where $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{u}}$, $C \in \mathbb{R}^{\mathtt{y}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}$ also play an important role in this paper. We say that (3) is an *input/state/output (i/s/o) representation* of $\mathfrak{B}$ if

$$\mathfrak{B} = \{(u, y) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}+\mathtt{y}}) \mid \exists\ x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}}) \text{ such that (3) holds}\}$$

## III. State construction

We now discuss how to compute an i/s/o representation for a system described in image form. Consider the set

$$\mathbb{X} := \{f \in \mathbb{R}^{1\times\mathtt{u}}[\xi] \mid fM^{-1} \text{ is strictly proper}\}$$

It is a matter of immediate verification to show that $\mathbb{X}$ is actually a finite dimensional subspace of $\mathbb{R}^{1\times\mathtt{u}}[\xi]$, the latter considered as a vector space over $\mathbb{R}$; it is also not difficult to verify that $\dim(\mathbb{X}) = \deg(\det(M)) =: \mathtt{n}$.

It is shown in section 8 of [7] that any set of vector polynomials $\{x_i\}_{i=1,\dots,\mathtt{n}'} \subset \mathbb{R}^{1\times\mathtt{u}}[\xi]$ spanning $\mathbb{X}$ defines a state representation of $\mathfrak{B}$ with state

$$x := \mathrm{col}(x_i(\frac{d}{dt})\ell)_{i=1,\dots,\mathtt{n}'}$$

i.e., the behavior of

$$\begin{aligned} u &= M(\frac{d}{dt})\ell \\ y &= N(\frac{d}{dt})\ell \\ x &= \mathrm{col}(x_i(\frac{d}{dt})\ell)_{i=1,\dots,\mathtt{n}'} \end{aligned} \tag{4}$$

satisfies the axiom of state (see p. 1058 of [7] for a formal statement of the axiom of state). The matrix $X := \mathrm{col}(x_i)_{i=1,\dots,\mathtt{n}} \in \mathbb{R}^{\mathtt{n}'\times\mathtt{u}}[\xi]$ hence induces the *state map* $X(\frac{d}{dt})\ell$. Once a state map is known, the system matrices $A \in \mathbb{R}^{\mathtt{n}'\times\mathtt{n}'}$, $B \in \mathbb{R}^{\mathtt{n}'\times\mathtt{u}}$, $C \in \mathbb{R}^{\mathtt{y}\times\mathtt{n}'}$ and $D \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}$ corresponding to the i/s/o representation (3) can be obtained from a solution

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{R}^{(\mathtt{n}'+\mathtt{y})\times(\mathtt{n}'+\mathtt{u})}$$

of the following system of linear equations in $\mathbb{R}^{1\times\mathtt{u}}[\xi]$:

$$\begin{bmatrix} \xi x_1(\xi) \\ \xi x_2(\xi) \\ \vdots \\ \xi x_{\mathtt{n}'}(\xi) \\ N(\xi) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1(\xi) \\ x_2(\xi) \\ \vdots \\ x_{\mathtt{n}'}(\xi) \\ M(\xi) \end{bmatrix}.$$

A state representation (3) associated with $A$, $B$, $C$, $D$ is called *state minimal* if the number $\mathtt{n}'$ of state variables is minimal among that of all representations (3) of $\mathfrak{B}$. It can be shown that this holds if and only if $\mathtt{n}' = \mathtt{n} = \deg(\det(M))$, which is the case if and only if $\{x_i\}_{i=1,\dots,\mathtt{n}}$ form a basis for $\mathbb{X}$. It can also be proven that in such case the solution of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to the above equation is unique.

For a general $\mathfrak{B}$, the notion of state minimality does not, contrary to the classical case, correspond to the simultaneous controllability of $(A, B)$ and observability of $(C, A)$ in an i/s/o representation (3). However, if $\mathfrak{B}$ is controllable, then it can be shown (see Prop. IX.7 of [8]) that the representation (3) of $\mathfrak{B}$ is minimal if and only if the pair $(A, B)$ is controllable and the pair $(C, A)$ is observable in the classical sense. Henceforth, we will concentrate on the minimal case $\mathtt{n}' = \mathtt{n}$.

We finally recall the definition of *balanced state space representation*. The i/s/o representation (3), assumed minimal (i.e., controllable and observable) and stable (i.e. the matrix $A$ is Hurwitz), is called *balanced* if there exist real numbers

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{n}} > 0,$$

called the *Hankel singular values*, such that

$$\begin{aligned} A\Sigma + \Sigma A^\top + BB^\top &= 0 \\ A^\top\Sigma + \Sigma A + C^\top C &= 0 \end{aligned}$$

hold, where $\Sigma = \mathrm{diag}(\sigma_i)_{i=1,\dots,\mathtt{n}}$.

Since the matrices $A$, $B$, $C$ and $D$ involved in an i/s/o representation (3) of a behavior $\mathfrak{B}$ described in image form as in (2) are determined by the choice of the state map $\mathrm{col}(x_i(\xi))_{i=1,\ldots,\mathtt{n}}$, being balanced is a property of the polynomial vectors $x_1, x_2, \ldots, x_{\mathtt{n}}$.

*The question addressed in this paper is how to choose the polynomials $\{x_1, x_2, \ldots, x_{\mathtt{n}}\}$ so that (3) defines a balanced state space system.*

## IV. The controllability and observability gramians

In the classical approach to balancing, a central role is played by two quadratic forms on the state space, namely the controllability and the observability gramians. In this section we show how they can be cast into the framework of quadratic differential forms developed in [9].

We first define quadratic differential forms. Consider the real two-variable $\mathtt{w} \times \mathtt{w}$ polynomial matrix in the indeterminates $\zeta$ and $\eta$:

$$\Phi(\zeta, \eta) = \Sigma_{i,j} \Phi_{i,j} \zeta^i \eta^j$$

where $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. In such expression $i$ and $j$ are nonnegative integers, and the sum is assumed to be finite. This polynomial matrix induces the map

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

defined by

$$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \longrightarrow \Sigma_{i,j} (\frac{d^i}{dt^i} w)^\top \ \Phi_{i,j} \ \frac{d^j}{dt^j} w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}).$$

This map is called a *quadratic differential form* (in the following often abbreviated with *QDF*) *induced by* $\Phi$, and it is denoted with $Q_\Phi$. In view of the quadratic nature of this map, we will always assume that $\Phi$ is *symmetric*, that is $\Phi_{i,j} = \Phi_{j,i}^\top$ for all $i$, $j$, or in other words $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. $Q_\Phi$ is said to be *non-negative* if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$.

The association of two-variable polynomial matrices with QDF's allows to develop a calculus that has many applications (see [9]); we now illustrate those concepts that are used in this paper. The first one is that of derivative of a QDF. Given a QDF $Q_\Psi$ we define its *derivative* as the QDF defined by $\frac{d}{dt}(Q_\Psi(w))$. In terms of the two-variable polynomial matrices associated with the QDF's, the derivative $\frac{d}{dt} Q_\Psi$ is represented by $(\zeta + \eta)\Psi(\zeta, \eta)$.

While it would be natural to consider the controllability and observability gramians as QDF's on $\mathfrak{B}$, we will consider them as QDF's acting on the latent variable $\ell$ of an observable image representation (2) of $\mathfrak{B}$. Observe that this entails no loss of generality, since there is then a one-to-one relation between $\ell$ and $(u, y) \in \mathfrak{B}$.

The *controllability gramian* $Q_K$ (equivalently, $K$) is defined as follows. Let $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}})$ and define $Q_K(\ell)$ by

$$Q_K(\ell)(0) := \inf \int_{-\infty}^{0} \|M(\frac{d}{dt})\ell'(t)\|^2 \ dt, \tag{5}$$

where the infimum is taken over all $\ell' \in \mathcal{D}^+(\mathbb{R}, \mathbb{R}^{\mathtt{u}})$ such that $\ell(t) = \ell'(t)$ for $t \geq 0$, and such that the concatenation at $t = 0$ of $(u_-, y_-) := (M(\frac{d}{dt})\ell', N(\frac{d}{dt})\ell')$ on $(-\infty, 0)$ and $(u_+, y_+) := (M(\frac{d}{dt})\ell, N(\frac{d}{dt})\ell)$ on $[0, +\infty)$ is an admissible trajectory in $\mathfrak{B}$. Note the slight difference with the classical terminology where the controllability gramian corresponds to the 'inverse' of the QDF $Q_K$.

An intuitive interpretation of the controllability gramian is the following. $Q_K$ computes the *effort*, as measured by $\int_{-\infty}^{0} \|u(t)\|^2 \ dt = \int_{-\infty}^{0} \|M(\frac{d}{dt})\ell'(t)\|^2 \ dt$, it takes to join the latent variable trajectory $\ell$ at $t = 0$ by a trajectory $\ell'$ that is zero in the far past, and such that its concatenation at $t = 0$ with $\ell$ yields an admissible system trajectory $(u, y)$.

The *observability gramian* $Q_W$ (equivalently, $W$) is defined as follows. Let $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}})$ and define $Q_W(\ell)$ by

$$Q_W(\ell)(0) := \int_{0}^{\infty} \|N(\frac{d}{dt})\ell'(t)\|^2 \ dt, \tag{6}$$

where $\ell' \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R}^{\mathtt{u}})$ is such that $\ell(t) = \ell'(t)$ for $t < 0$, that $(M(\frac{d}{dt})\ell')(t) = 0$ for $t \geq 0$, and such that the concatenation at $t = 0$ of $(u_-, y_-) := (M(\frac{d}{dt})\ell, N(\frac{d}{dt})\ell)$ on $(-\infty, 0)$ and $(u_+, y_+) := (M(\frac{d}{dt})\ell', N(\frac{d}{dt})\ell')$ on $[0, +\infty)$ is an admissible trajectory in $\mathfrak{B}$.

An intuitive interpretation of $Q_W$ is the following: the observability gramian measures the *ease* with which it is possible to observe the effect of the latent variable trajectory $\ell$ as measured by $\int_0^{+\infty} \|y(t)\|^2 \ dt = \int_0^{+\infty} \|N(\frac{d}{dt})\ell'(t)\|^2 \ dt$, assuming that $u(t) = (M(\frac{d}{dt})\ell')(t)$ is zero for $t \geq 0$.

The computation of the two-variable polynomial matrices $K$ and $W$ is one of the central results of this paper. **Theorem 1:** *Consider the system $\mathfrak{B}$ represented in observable image form by (2), with $M$ Hurwitz (meaning that all the roots of $\det(M) \in \mathbb{R}[\xi]$ have negative real part) and $NM^{-1}$ proper. Then the controllability gramian and the observability gramian are QDF's; denote them by $Q_K$ and $Q_W$ respectively, with $K, W \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta, \eta]$.*

*The controllability gramian $K$ can be computed as follows:*

$$K(\zeta, \eta) = \frac{M^\top(\zeta)M(\eta) - A^\top(\zeta)A(\eta)}{\zeta + \eta}, \tag{7}$$

*where $A \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ is an anti-Hurwitz matrix such that $M^*M = A^*A$.*

*The observability gramian $W$ can be computed as follows. Consider the unique solution $F \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ with $FM^{-1}$ proper of the Bézout-type equation*

$$M^\top(-\xi)F(\xi) + F^\top(-\xi)M(\xi) - N^\top(-\xi)N(\xi) = 0. \tag{8}$$

*Define from such $F$ the two variable polynomial matrix*

$$W(\zeta, \eta) = \frac{M^\top(\zeta)F(\eta) + F^\top(\zeta)M(\eta) - N^\top(\zeta)N(\eta)}{\zeta + \eta}, \tag{9}$$

*Moreover, both $Q_K$ and $Q_W$ are both quadratic functions of the state of $\mathfrak{B}$, meaning that for every state map $X \in$*

$\mathbb{R}^{\mathtt{n}'\times\mathtt{u}}[\xi]$ *for* $\mathfrak{B}$ *there exist real symmetric matrices* $\bar{K},\bar{W}\in\mathbb{R}^{\mathtt{n}'\times\mathtt{n}'}$*, with both* $\bar{K},\bar{W}\geq 0$*, such that*

$$\begin{aligned} Q_K(\ell) &= (X(\frac{d}{dt})\ell)^\top \bar{K} X(\frac{d}{dt})\ell =: \|X(\frac{d}{dt})\ell\|^2_{\bar{K}} \\ Q_W(\ell) &= (X(\frac{d}{dt})\ell)^\top \bar{W} X(\frac{d}{dt})\ell =: \|X(\frac{d}{dt})\ell\|^2_{\bar{W}} \end{aligned}$$

*If in addition* $X$ *is minimal, then* $\bar{K},\bar{W}\in\mathbb{R}^{\mathtt{n}'\times\mathtt{n}'}$ *are nonsingular, and* $\bar{K},\bar{W}>0$.

The proof is given in the appendix.

## V. BALANCED STATE REPRESENTATION

In this section we show how to compute a balanced state representation for a system described in observable image form as in (2).

We begin by reconciling the notion of balanced state representation as introduced in section II, with the notion of state map and with the point of view introduced in section IV of the controllability and observability gramians as quadratic differential forms.

We call the minimal state representation (4) with state $(x_1,x_2,\ldots,x_\mathtt{n})$ *balanced* if

1) for $\ell_i\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\mathtt{u})$ such that $(x_j(\frac{d}{dt})\ell_i)(0)=\delta_{ij}$ ($\delta_{ij}$ denotes the Kronecker delta), there holds

$$Q_W(\ell_i)(0)=\frac{1}{Q_K(\ell_i)(0)},$$

i.e., the state components that are difficult to reach are also difficult to observe, and

2) the state components are ordered so that

$$0<Q_K(\ell_1)(0)\leq Q_K(\ell_2)(0)\leq\cdots\leq Q_K(\ell_\mathtt{n})(0),$$

and hence

$$Q_W(\ell_1)(0)\geq Q_W(\ell_2)(0)\geq\cdots\geq Q_W(\ell_\mathtt{n})(0)>0.$$

In order to perform the computation of a balanced state map, we proceed as follows. Assume that the two-variable polynomial matrices $K(\zeta,\eta)$ and $W(\zeta,\eta)$ corresponding to the controllability and the observability gramians have been computed as in (7) and (9). From the result of theorem 1 it follows that there exist matrices$X\in\mathbb{R}^{\mathtt{n}\times\mathtt{u}}[\xi]$ and $\bar{K},\bar{W}\in\mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that $K(\zeta,\eta)=X^\top(\zeta)\bar{K}X(\eta)$ and $W(\zeta,\eta)=X^\top(\zeta)\bar{W}X(\eta)$. Such equalities can be rewritten in terms of the corresponding coefficient matrices as

$$\begin{aligned} \widetilde{K} &= \widetilde{X}^\top\bar{K}\widetilde{X} \\ \widetilde{W} &= \widetilde{X}^\top\bar{W}\widetilde{X} \end{aligned}$$

Observe that it follows from theorem 1 that $\bar{K}$ and $\bar{W}$ are symmetric and nonsingular. It is a standard result in linear algebra that there exists a $\mathtt{n}\times\mathtt{n}$ nonsingular transformation matrix $T$ such that

$$\begin{aligned} T^{-T}\bar{K}T^{-1} &= \Sigma^{-1} \\ T^{-T}\bar{W}T^{-1} &= \Sigma \end{aligned}$$

with $\Sigma$ a diagonal matrix:

$$\Sigma=\mathrm{diag}(\sigma_i)_{i=1,\ldots,\mathtt{n}}, \tag{10}$$

where $\sigma_i\geq\sigma_{i+1}$, $i=1,\ldots,\mathtt{n}-1$, and $\sigma_\mathtt{n}>0$. Consequently, the following equations hold:

$$\begin{aligned} \widetilde{K} &= \widetilde{X}^\top T^\top\Sigma^{-1}T\widetilde{X} \\ \widetilde{W} &= \widetilde{X}^\top T^\top\Sigma T\widetilde{X} \end{aligned}$$

and therefore

$$\begin{aligned} K(\zeta,\eta) &= X^{\mathrm{bal},\top}(\zeta)\Sigma^{-1}X^{\mathrm{bal}}(\eta) \\ W(\zeta,\eta) &= X^{\mathrm{bal},\top}(\zeta)\Sigma X^{\mathrm{bal}}(\eta) \end{aligned} \tag{11}$$

where the polynomial matrix $X^{\mathrm{bal}}=:\mathrm{col}(x_i^{\mathrm{bal}})_{i=1,\ldots,\mathtt{n}}\in\mathbb{R}^{\mathtt{n}\times\mathtt{u}}[\xi]$ is defined as

$$X^{\mathrm{bal}}(\xi):=TX(\xi).$$

These considerations lead to the main result of this paper.

**Theorem 2:** *Assume that the QDF's* $K$ *and* $W$ *have been computed. Define the polynomial matrix* $X^{\mathrm{bal}}$ *and the real numbers* $\sigma_i$, $i=1,\ldots,\mathtt{n}$, *as (10, 11), respectively. Then the* $\sigma_i$*'s are the Hankel singular values of the system* $\mathfrak{B}$ *and*

$$u=M(\frac{d}{dt})\ell,\quad y=N(\frac{d}{dt})\ell,\quad x^{\mathrm{bal}}=X^{\mathrm{bal}}(\frac{d}{dt})\ell$$

*is a balanced state space representation of* $\mathfrak{B}$*. The associated balanced system matrices are obtained as the solution matrix*

$$\begin{bmatrix}A^{\mathrm{bal}} & B^{\mathrm{bal}}\\ C^{\mathrm{bal}} & D^{\mathrm{bal}}\end{bmatrix}$$

*of the following system of linear equations in* $\mathbb{R}^{1\times\mathtt{u}}[\xi]$*:*

$$\begin{bmatrix}\xi x_1^{\mathrm{bal}}(\xi)\\ \xi x_2^{\mathrm{bal}}(\xi)\\ \vdots\\ \xi x_\mathtt{n}^{\mathrm{bal}}(\xi)\\ N(\xi)\end{bmatrix}=\begin{bmatrix}A^{\mathrm{bal}} & B^{\mathrm{bal}}\\ C^{\mathrm{bal}} & D^{\mathrm{bal}}\end{bmatrix}\begin{bmatrix}x_1^{\mathrm{bal}}(\xi)\\ x_2^{\mathrm{bal}}(\xi)\\ \vdots\\ x_\mathtt{n}^{\mathrm{bal}}(\xi)\\ M(\xi)\end{bmatrix}. \tag{12}$$

The proof of this theorem is given in the appendix.

We summarize the results of this section in the following algorithm to compute a balanced state representation for a behavior $\mathfrak{B}$ given in observable image form as in (2).

**ALGORITHM**

**DATA**: $M\in\mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$, $N\in\mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$ right coprime, $\deg\det(M)=:\mathtt{n}$, $M$ Hurwitz.

**COMPUTE**:

1) $K\in\mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta,\eta]$ by (7),
2) $F\in\mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ by (8) and $W\in\mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta,\eta]$ by (9),
3) $X^{\mathrm{bal}}\in\mathbb{R}^{\mathtt{n}\times\mathtt{u}}[\xi]$ and $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_\mathtt{n}>0$ by (10, 11):

$$K(\zeta,\eta)=\Sigma_{k=1}^{\mathtt{n}}\sigma_i^{-1}x_i^{\mathrm{bal}}(\zeta)x_i^{\mathrm{bal}}(\eta),$$

$$W(\zeta,\eta)=\Sigma_{i=1}^{\mathtt{n}}\sigma_i\ \ x_i^{\mathrm{bal}}(\zeta)x_i^{\mathrm{bal}}(\eta),$$

4) the system matrices $\begin{bmatrix} A^{\rm bal} & B^{\rm bal} \\ C^{\rm bal} & D^{\rm bal} \end{bmatrix}$ by solving (12).

**Remarks:**

1. Our algorithms for obtaining the controllability and observability gramians and balanced state representations, being based on polynomial computations, offer a number of advantages over the classical matrix based algorithms. In particular, they open up the possibility to involve the know-how on Bézoutians, Bézout and Sylvester matrices and equations (see for example [9],[4]), and bring 'fast' polynomial computations to bear on the problem of model reduction.

2. Instead of computing the $\sigma_i$'s and the $x_i^{\rm bal}$'s by the factorization of $K, W$ given by (10), (7), (9), we can also obtain the balanced state representation by evaluating $K$ and $W$ at $\mathtt{n}$ points of the complex plane. Such an approach results in a two-variable interpolation problem which, for special choices of the $\mathtt{n}$ interpolation points, could perhaps exhibit some advantages over the computation of such matrices as described in theorem 1. This is explained for the SISO case in [10].

3. The algorithms discussed have obvious counterparts for discrete-time systems. It is interesting to compare our algorithm for obtaining a balanced state representation with the classical SVD-based algorithm of Kung [2]. Kung's algorithm starts from the Hankel matrix formed by the impulse response and requires the computation of the SVD of an *infinite* matrix. In contrast, our algorithm requires first finding the governing difference equation, followed by finite polynomial algebra.

## Appendix: proofs

The proofs are an adaptation to the MIMO case of the proof of the SISO case given in [10].

**Proof of theorem 1:** Let $X \in \mathbb{R}^{\mathtt{n}\times\mathtt{u}}[\xi]$ be a minimal state map for $\mathfrak{B}$. We consider the claim on the controllability Gramian first. We begin the proof of the claim by showing that

$$\min \int_{-\infty}^{0} \|M(\frac{d}{dt})\ell'\|^2\, dt = Q_K(\ell)(0)$$

where $\ell' \in \mathfrak{C}^\infty(\mathbb{R}_-, \mathbb{R}^{\mathtt{u}})$ is such that $(X(\frac{d}{dt})\ell')(0) = (X(\frac{d}{dt})\ell)(0)$ and $\lim_{t\to-\infty} \ell' = 0$.

From the definition of $K(\zeta,\eta)$ it follows that $\frac{d}{dt}Q_K(\ell')(t) + \|(A(\frac{d}{dt})\ell')(t)\|^2 = \|(M(\frac{d}{dt})\ell')(t)\|^2$; integrating between $-\infty$ and $0$, we obtain:

$$\int_{-\infty}^{0} \|(M(\frac{d}{dt})\ell')(t)\|^2\, dt = Q_K(\ell')(0) + \int_{-\infty}^{0} \|(A(\frac{d}{dt})\ell')(t)\|^2\, dt$$

We now show that $Q_K$ is a quadratic function of the state. Observe that from the equality $M^*M = A^*A$ and from the definition of $K(\zeta,\eta)$ it follows that $M^{-T}(\zeta)K(\zeta,\eta)M^{-1}(\eta)$ is a matrix of strictly proper rational functions. Conclude from this (see section 2 of [9]) that $Q_K$ is a quadratic function of the state of $\mathfrak{B} = \text{im}\,(\text{col}(M(\frac{d}{dt}), N(\frac{d}{dt}))$. Consequently, there exists a matrix $\bar{K}$ such that $K(\zeta,\eta) = X^\top(\zeta)\bar{K}X(\eta)$, so that we can write

$$\int_{-\infty}^{0} \|(M(\frac{d}{dt})\ell')(t)\|^2 {\rm d}t$$

$$= (X(\frac{d}{dt})\ell')(0)^\top \bar{K}(X(\frac{d}{dt})\ell')(0) + \int_{-\infty}^{0} \|(A(\frac{d}{dt})\ell')(t)\|^2 {\rm d}t$$

$$= (X(\frac{d}{dt})\ell)(0)^\top \bar{K}(X(\frac{d}{dt})\ell)(0) + \int_{-\infty}^{0} \|(A(\frac{d}{dt})\ell')(t)\|^2 {\rm d}t$$

Conclude from this expression that for a fixed $a := (X(\frac{d}{dt})\ell)(0) \in \mathbb{R}^{\mathtt{n}}$, the minimum of $\int_{-\infty}^{0} \|(M(\frac{d}{dt})\ell')(t)\|^2 {\rm d}t$ is taken for those trajectories such that $(X(\frac{d}{dt})\ell')(0) = a$ and $A(\frac{d}{dt})\ell' = 0$ on $(-\infty, 0]$; this implies also $Q_K(\ell)(0) \geq 0$. It can also be proved that since $X$ is minimal, then for every choice of $a$ there exists exactly one such trajectory; an argument by contradiction then yields that $\bar{K} > 0$. In order to complete the proof and prove the claim for $\ell' \in \mathcal{D}^+(\mathbb{R}, \mathbb{R}^{\mathtt{u}})$, and use an approximation argument.

We proceed proving the claim regarding the observability gramian. The claim on the existence of a unique solution $F$ to the Bézout equation $F^*M + M^*F = N^*N$ such that $FM^{-1}$ is strictly proper follows from the fact that $M$ is Hurwitz, that $NM^{-1}$ is strictly proper, and from Proposition 4.4 p. 120 of [4]. Thus $W(\zeta,\eta)$ as defined in (9) is well-defined. Now apply Proposition 4.1 of [4] in order to conclude that $M^{-T}(\zeta)W(\zeta,\eta)M^{-1}(\eta)$ is a strictly proper rational function in $\zeta$ and $\eta$, and consequently that for every state map $X$ there exists a matrix $\bar{W}$ such that $W(\zeta,\eta) = X^\top(\zeta)\bar{W}X(\eta)$.

Now observe that for every $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}})$ here holds

$$\frac{d}{dt}Q_W(\ell) = 2(M(\frac{d}{dt})\ell)^\top (F(\frac{d}{dt})\ell) - \|N(\frac{d}{dt})\ell\|^2. \quad (13)$$

Therefore, if $\ell' \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R}^{\mathtt{u}})$ is such that $(M(\frac{d}{dt})\ell')(t) = 0$ for $t \geq 0$, and moreover $(X(\frac{d}{dt})\ell')(0) = (X(\frac{d}{dt})\ell)(0)$, then by integrating (13) and using the fact that $M$ is Hurwitz, it holds that

$$\int_{0}^{\infty} \|N(\frac{d}{dt})\ell'\|^2\ dt = Q_W(\ell')(0)$$

$$= (X(\frac{d}{dt})\ell')(0)^\top \bar{W}(X(\frac{d}{dt})\ell')(0)$$

$$= (X(\frac{d}{dt})\ell)(0)^\top \bar{W}(X(\frac{d}{dt})\ell)(0) = Q_W(\ell)(0).$$

This proves that $Q_W$ is the observability gramian. From the last equation it also follows that $Q_W \geq 0$; then, using the observability of the image representation and the minimality of the state map $X$, an argument by contradiction yields that $\bar{W} > 0$.

**Proof of theorem ??:** Using the factorization (11) of $K(\zeta,\eta)$ and $W(\zeta,\eta)$ and the definition (10) of $\Sigma$, we obtain

$$Q_K(\ell)=\Sigma_{i=1}^{\mathtt{n}}\sigma_i^{-1}|x_i^{\mathrm{bal}}(\frac{d}{dt})\ell|^2,$$

and

$$Q_W(\ell)=\Sigma_{i=1}^{\mathtt{n}}\sigma_i|x_i^{\mathrm{bal}}(\frac{d}{dt})\ell|^2.$$

Hence, if $\ell_i\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{u}})$ is such that $(x_i^{\mathrm{bal}}(\frac{d}{dt})\ell_i)(0)=\delta_{i,j}$, then

$$Q_K(\ell_i)(0)=\sigma_i^{-1},\text{ and }Q_W(\ell_i)(0)=\sigma_i.$$

This shows that the polynomial vectors $x_i^{\mathrm{bal}}$, $i=1,\ldots,\mathtt{n}$, define a balanced state representation. That the $\sigma_i$'s are the Hankel singular values of $\mathfrak{B}$ is a standard consequence of the theory of balanced state representations.

## VI. Example

We took example 2 of [3]. The transfer function is $\frac{(\xi+4)}{(\xi+1)(\xi+3)(\xi+5)(\xi+10)}$ In our notation,

$$\begin{aligned}p(\xi)&=150+245\xi+113\xi^2+19\xi^3+\xi^4\\ q(\xi)&=\xi+4\end{aligned}$$

For $K$, the anti-Hurwitz factorization of $p(-\xi)p(\xi)=a(-\xi)a(\xi)$, with $a(\xi)=150-245\xi+113\xi^2-19\xi^3+\xi^4$, yields

$$\begin{aligned}K(\zeta,\eta)&=73500+5700\eta^2+5700\zeta^2+49670\zeta\eta\\&\quad+490\eta^3\zeta+490\zeta\eta^3+3804\zeta^2\eta^2+38\zeta^3\eta^3\end{aligned}$$

In order to obtain $M(\zeta,\eta)$ we need to solve the Bézout equation. This yields

$$f(\xi)=\frac{4}{75}+\frac{102181}{3432000}\xi+\frac{35131}{6864000}\xi^2+\frac{1849}{6864000}\xi^3$$

is such a solution. From it we obtain

$$\begin{aligned}M(\zeta,\eta)&=\frac{1}{6864000}(92887900+46636690(\zeta+\eta)\\&+7232870(\eta^2+\zeta^2)+366080(\zeta^3+\eta^3)+24467131\zeta\eta\\&+3969803\zeta\eta^2+3969803\zeta^2\eta+204362\zeta\eta^3\\&+204362\zeta^3\eta+672064\zeta^2\eta^2+35131\zeta^2\eta^3\\&+35131\zeta^3\eta^2+1849\zeta^3\eta^3)\end{aligned}$$

We next obtain the $\sigma$'s and the $X^{\mathrm{bal}}$. The Hankel singular values obtained through our procedure are

$$0.01593838752113, 0.00272425189843,\\ 0.00012720366224, 0.00000800595148.$$

which are indeed those given in Moore's paper.

The $x_i^{\mathrm{bal}}$ polynomials obtained are

$$\begin{aligned}x_1^{\mathrm{bal}}(\xi)&=29.0903+14.7840\xi+2.3226\xi^2+0.1181\xi^3\\x_2^{\mathrm{bal}}(\xi)&=-4.0562+5.4494\xi+2.0930\xi^2+0.1307\xi^3\\x_3^{\mathrm{bal}}(\xi)&=0.5526-0.5565\xi-0.0296\xi^2+0.0563\xi^3\\x_4^{\mathrm{bal}}(\xi)&=0.3095-0.4256\xi+0.1217\xi^2-0.0069\xi^3\end{aligned}$$

In order to find the matrices corresponding to a balanced i/s/o representation, we solve the equations (12). This yields

$$A^{\mathrm{bal}}=\begin{bmatrix}-0.43781&1.1685&-0.41426&-0.05098\\-1.1685&-3.1353&2.8352&0.32885\\-0.41426&-2.8352&-12.475&-3.2492\\0.05098&0.32885&3.2492&-2.9516\end{bmatrix},$$

$$B^{\mathrm{bal}}=\begin{bmatrix}0.11814\\0.1307\\0.056337\\-0.0068746\end{bmatrix},$$

$$C^{\mathrm{bal}}=\begin{bmatrix}0.11814&-0.1307&0.056337&0.0068746\end{bmatrix}.$$

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