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State Construction in Discrete Event and Continuous Systems

Jan C. Willems

Department of Electrical Engineering,
University of Leuven, Kasteelpark Arenberg 10,
B-3001 Leuven-Heverlee, Belgium

email: Jan.Willems@esat.kuleuven.ac.be.

Abstract

We discuss the formulation of dynamics in discrete-event and in continuous systems. It is argued that the behavioral framework constitutes a framework that encompasses both. This framework is applied to state construction.

Keywords: Discrete-event systems, automata, continuous systems, behaviors, latent variables, state systems, state construction.

1 Introduction

The mainstream theoretical frameworks that are used in discrete-event systems (DES) theory on the one hand, and continuous systems theory on the other, show a curious discrepancy. This discrepancy is most easily illustrated in a state space setting, i.e., by comparing automata with systems in state space form.

The usual model of an automaton involves an event alphabet $\mathbb{A}$, a state set $\mathbb{S}$, and state transition relation ϕ, a *partial* map from $\mathbb{S} \times \mathbb{A}$ to $\mathbb{S}$ (we dispense with issues having to do with initial and final states). The interpretation being that if the system is in state $s \in \mathbb{S}$, then the events $a \in \mathbb{A}$ that are possible (i.e. that the system can accept/produce) are those such that (s, a) belongs to the domain of ϕ, upon which the automaton moves to the next state $\phi(s, a) \in \mathbb{S}$.

The usual state space model for a continuous system, on the other hand, is

$$x(t+1) = f(x(t), u(t)), \quad y(t) = h(x(t), u(t)),$$

or

$$\frac{d}{dt}x(t) = f(x(t), u(t), \quad y(t) = h(x(t), u(t)),$$

depending whether we are in a discrete-time or in a continuous-time setting. The analogue of the event $a \in \mathbb{A}$ is now the input/output pair $(u(t), y(t)) \in \mathbb{U} \times \mathbb{Y}$. We see that in this case the event alphabet is a direct product of the input and output 'alphabets' $\mathbb{U}$ and $\mathbb{Y}$. Hence it is assumed that whatever state the system is in, the input event can be chosen freely from $\mathbb{U}$, while the output event then follows from h.

Thus in automata, the events that are possible when the system is in a particular current state, is in principle any subset of $\mathbb{A}$, while in the case of continuous systems, the set of possible events is always the graph of a map from $\mathbb{U}$ to $\mathbb{Y}$. Of course, the graph depends on the current state, but the domain and co-domain do not.

It is surprising that this discrepancy has not been questioned more frequently. The situation becomes almost caricatural in hybrid systems, where authors often use the automata framework for the discrete-event part, and the input/output framework for the continuous part, as if the time structure could have such a dramatic effect on the event structure. All this notwithstanding the fact that a perfectly satisfactory, well-developed, and well-motivated (by physical examples) framework, the behavioral framework, that incorporates automata and formal languages and that also applies to continuous systems, has been available since a long time.

2 The behavioral framework

We now briefly outline the behavioral framework. Details may be found in [3, 4, 5], and references therein.

A *system* Σ is defined as $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, with $\mathbb{T}$ the set of *independent variables*, $\mathbb{W}$ the set of *dependent variables*, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ the *behavior*. In this presentation, we will only consider systems with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, thought of as *time* or *sequencing* (although $\mathbb{T} = \mathbb{R}^{\mathrm{n}}$, as in PDE's, etc. is also of interest). Note that this covers DES and formal languages (the fact that 'words' are usually considered to be finite is easily accommodated for).

The behavior $\mathfrak{B}$ expresses the dynamics. Thus $w \in \mathfrak{B}$ signifies that the 'events' $w(t), t \in \mathbb{T}$ occur in orderly sequence, in accordance with the laws of the system Σ.

The specification of $\mathfrak{B}$ is an interesting issue in its own right. For $\mathbb{T} = \mathbb{Z}$, this could be through forbidden strings, grammars, automata, or difference equations, while for $\mathbb{T} = \mathbb{R}$ differential or integral equations come to mind. In this paper, we will explain the specification of $\mathfrak{B}$ through latent variables.

3 Latent variables

Models obtained from first principles invariably contain auxiliary variables, in addition to the 'event' variables the model aims at. We call these auxiliary variables *latent* variables, and the variables the model aims at, *manifest* variables.

A *latent variable system* is defined as

$$\Sigma_{\mathbb{L}} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}}),$$

with $\mathbb{T}$ the set of *independent variables*, $\mathbb{W}$ the set of *manifest variables*, $\mathbb{L}$ the set of *latent variables*, and

$$\mathfrak{B}_{\text{full}} \subset (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$$

the *full behavior.* $\Sigma_{\mathbb{L}}$ induces the system

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}),$$

with *manifest behavior*

$$\mathfrak{B} = \{w \mid \exists\, \ell : (w, \ell) \in \mathfrak{B}_{\text{full}}\}.$$

Examples of how such latent variable systems occur are given in [5], chapter 1.

4 State systems

We view state systems as a special case of latent variable systems. The latent variable system

$$\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$$

is said to be a *state system* if

$$\begin{aligned}
&[((w_1, x_1) \in \mathfrak{B}_{\text{full}}) \wedge ((w_2, x_2) \in \mathfrak{B}_{\text{full}}) \\
&\qquad \wedge (t \in \mathbb{T}) \wedge (x_1(t) = x_2(t))] \\
&\qquad\qquad \Rightarrow [(w_1, x_1) \wedge_t (w_2, x_2) \in \mathfrak{B}_{\text{full}}],
\end{aligned}$$

where $\wedge_t$ denotes *concatenation* at t,

$$(f_- \wedge_t f_+)(t') := \begin{cases} f_-(t') & \text{for} \quad t' < t \\ f_+(t) & \text{for} \quad t' \geq t \end{cases}$$

If $\Sigma_{\mathbb{X}}$ is a state system that induces the manifest system Σ, then we call $\Sigma_{\mathbb{X}}$ a *state representation* of Σ. From now on we assume that all systems considered are *time-invariant*. Specifically, assume $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, and $\sigma^t \mathfrak{B} = \mathfrak{B}$, or $\sigma^t \mathfrak{B}_{\text{full}} = \mathfrak{B}_{\text{full}}$ for all $t \in \mathbb{T}$, where σ^t denotes the t-shift.

It is easy to verify that automata and the systems introduced in section 1 are state systems. In fact, disregarding the behavior around time $= \pm\infty$, a latent variable DES is a state system iff it is an automaton. For continuous discrete-time systems a system of behavioral equations defines a state system iff the full behavior is (pointwise in time) described by $(w(t), x(t), x(t+1)) \in \mathfrak{B}_0$, with $\mathfrak{B}_0$ a subset of $\mathbb{W} \times \mathbb{X} \times \mathbb{X}$. For smooth continuous-time systems iff its full behavior is (pointwise in time) described by $(w(t), x(t), \frac{d}{dt}x(t)) \in \mathfrak{B}_0$, with $\mathfrak{B}_0$ a subset of $\mathbb{W} \times T\mathbb{X}$, and $T\mathbb{X}$ the tangent bundle of $\mathbb{X}$.

A state system $\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ is said to be *irreducible* iff

$$\begin{aligned}
&(\text{for } f : \mathbb{X} \to \mathbb{X}', \Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}', \mathfrak{B}'_{\text{full}}) \text{ with} \\
&\qquad \mathfrak{B}'_{\text{full}} = \{(w, f \circ x) \mid (x, w) \in \mathfrak{B}_{\text{full}}\} \\
&\qquad\qquad \text{is a state system}) \Rightarrow (f \text{ is a bijection}).
\end{aligned}$$

Two state systems $\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ and $\Sigma'_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}', \mathfrak{B}'_{\text{full}})$ are said to be *equivalent* if there exists a bijection $f : \mathbb{X} \to \mathbb{X}'$ such that

$$[(w, x) \in \mathfrak{B}_{\text{full}}] \Leftrightarrow [(w, f \circ x) \in \mathfrak{B}'_{\text{full}}].$$

Clearly equivalent state systems represent the same manifest behavior.

A central question for state representations is: *Are all irreducible state representations with a given manifest behavior equivalent?*

5 State construction

We now address the question: Given $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, find a (irreducible) state space representation $\Sigma_{\mathbb{X}} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ for it.

There are 3 canonical constructions (introduced in [1] and [2]), leading to

1. the *past canonical* state representation
2. the *future canonical* state representation
3. the *two-sided canonical* state representation

All three constructions are based on an equivalence relation on $\mathfrak{B}$. In the past canonical case, define the equivalence relation R_- by

$$[w_1 R_- w_2] :\Leftrightarrow [(w_1 \wedge_0 w \in \mathfrak{B}) \Leftrightarrow (w_1 \wedge_0 w \in \mathfrak{B})].$$

In the future canonical case, define the equivalence relation R_+ by

$$[w_1 R_+ w_2] :\Leftrightarrow [(w \wedge_0 w_1 \in \mathfrak{B}) \Leftrightarrow (w \wedge_0 w_2 \in \mathfrak{B})].$$

In the two-sided canonical case, define the equivalence relation $R_\pm$ by

$$[w_1 R_\pm w_2] :\Leftrightarrow [((w_1 \wedge_0 w \in \mathfrak{B}) \Leftrightarrow (w_1 \wedge_0 w \in \mathfrak{B})) \wedge ((w \wedge_0 w_1 \in \mathfrak{B}) \Leftrightarrow (w \wedge_0 w_2 \in \mathfrak{B}))].$$

Obviously,

$$[w_1 R_\pm w_2] \Leftrightarrow [(w_1 R_- w_2) \wedge (w_1 R_+ w_2)].$$

We now construct the associated state representations. For the past-canonical state construction, define the state space by

$$\mathbb{X}_- = \mathfrak{B}(\text{mod } R_-)$$

and the full behavior by

$$\mathfrak{B}_{\text{full},-} = \{(w,x) \mid (w \in \mathfrak{B}) \wedge (\sigma^t w \in (\sigma^t x)(0)\ \forall t \in \mathbb{T})\}.$$

For the future-canonical state construction define the state space by

$$\mathbb{X}_+ = \mathfrak{B}(\text{mod} R_+)$$

and the full behavior by

$$\mathfrak{B}_{\text{full},+} = \{(w,x) \mid (w \in \mathfrak{B}) \wedge (\sigma^t w \in (\sigma^t x)(0)\ \forall t \in \mathbb{T})\}.$$

For the (two-sided)-canonical state construction define the state space by

$$\mathbb{X}_\pm = \mathfrak{B}(\text{mod } R_\pm)$$

and the full behavior by

$$\mathfrak{B}_{\text{full},\pm} = \{(w,x) \mid (w \in \mathfrak{B}) \wedge (\sigma^t w \in (\sigma^t x)(0)\ \forall t \in \mathbb{T})\}.$$

These canonical state representations

$$\Sigma_- := (\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_-)$$

and

$$\Sigma_+ := (\mathbb{T}, \mathbb{W}, \mathbb{X}_+, \mathfrak{B}_+)$$

have very good properties. In particular, they are irreducible.

The question when all irreducible state representations of a given system are equivalent has a very nice answer in terms of these canonical representations. Indeed, the following conditions are equivalent (see [2]):

1. All irreducible state representations of a given system $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$ are equivalent.
2. $(\mathbb{T}, \mathbb{W}, X_-, \mathfrak{B}_{\text{full},-})$ and $(\mathbb{T}, \mathbb{W}, \mathbb{X}_+, \mathfrak{B}_{\text{full},+})$ are equivalent.
3. $(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\text{full},\pm})$ is irreducible.
4. $(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\text{full},-})$ and $(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\text{full},\pm})$ are equivalent.
5. $(\mathbb{T}, \mathbb{W}, \mathbb{X}_+, \mathfrak{B}_{\text{full},+})$ and $(\mathbb{T}, \mathbb{W}, \mathbb{X}_-, \mathfrak{B}_{\text{full},\pm})$ are equivalent.

An important example of a class of systems for which all irreducible state representations are equivalent are linear systems. $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is *linear* if $\mathbb{W}$ is a vector space and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$.

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