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Concatenability of Behaviors in Hybrid System Interconnection

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Abstract

This paper is concerned with a behavioral approach to analysis of hybrid system interconnections. The hybrid system interconnection is defined as a dynamical system consisting of past and future trajectories which are switched at a certain instant by an external switching mechanism. We define the concatenability of the behaviors of the past and future interconnections, and derive a necessary and sufficient condition for the behavioral concatenability based on the notion of a state map. If the past interconnection behavior is concatenable with the future one, every past trajectory can be continued by some future trajectory without causing any impulsive phenomena. Moreover, we show that the regular feedback structure of the future interconnection guarantees the concatenability for any past interconnections.

1 Introduction

Control systems are composed of the interconnections of physical components. These interconnections are sometimes reorganized instantaneously due to switching of control strategies or component failure. We call such a dynamical system with a switching mechanism *a hybrid system interconnection.*

In recent years, modeling, analysis and control of hybrid systems have been extensively studied based on various frameworks such as linear complementarity systems [5, 10], piecewise linear systems [6, 8], mixed logical dynamical systems [1] and descriptor systems [3, 4].

It is clear that the continuity of the system variables in the hybrid system is no longer guaranteed due to this switching nature. Moreover, such discontinuity of variables may lead to an impulsive phenomenon. A typical example of an impulsive phenomenon is *the sparking* in an electrical circuit, which may occur when a circuit is suddenly shorted or connected to other electrical components. From a practical point of view, the impulsive phenomenon is not desirable because it may damage the system.

Impulsive phenomena in control systems have often been discussed in the context of descriptor or implicit system theory (e.g. [2, 11, 3, 4]). In the theory of descriptor systems, we consider the system behaviors based on the initial value problem of differential/algebraic equations without giving clear explanation about the relationship between the initial values of the variables and the system dynamics before the switching instant. In order to investigate the hybrid system interconnection in more systematically, we need a theoretical framework which explicitly takes account of the system dynamics before switching. We will use the behavioral theory [12, 7] in order to study this kind of hybrid system interconnections.

The solution of the hybrid system interconnection can be viewed as a concatenation of the past and future trajectories which are generated by the past and future interconnections, respectively. The terms 'past' and 'future' refer to the time before and after the switching instant. Based on this observation, we define the concatenability of the behaviors of past and future interconnections. If the past interconnection behavior is concatenable with the future one, every past trajectory can be continued by some future trajectory without causing any impulsive phenomena. In this paper, we will derive a necessary and sufficient condition for this behavioral concatenability in terms of the state maps [9]. We also examine the relationship between the behavioral concatenability and the feedback structure of the future interconnection.

Notations:
$\mathbb{R}[\xi]$: the set of polynomials with real coefficients
$\mathbb{R}^{m\times n}[\xi]$: the set of $m \times n$ polynomial matrices with real coefficients
$\mathfrak{C}^k(\mathbb{R},\mathbb{R}^q)$: the set of k-times differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$
$\mathfrak{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^q)$: the set of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^q$
H: the Heaviside step function
δ: the Dirac delta distribution
rowdim(R): the row dimension, i.e. the number of rows of R

$$\mathrm{col}(x_1,\ldots,x_r) = \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix}$$

The concatenation at $t = t_0$ of functions w_- and w_+ is denoted by

$$(w_- \underset{t_0}{\wedge} w_+)(t) = \begin{cases} w_-(t) & \text{for } t < t_0 \\ w_+(t) & \text{for } t \geq t_0 \end{cases}$$

2 Preliminaries

2.1 Linear differential system

In the behavioral framework[12, 9, 13, 7], a dynamical system is defined as a triple $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, $\mathbb{W}$ is the signal space, and $\mathfrak{B}$ is the behavior. The signal space $\mathbb{W}$ is the set in which the trajectories generated by the system take on their values. The behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbf{T}}$ is the set of the trajectories of system variables under consideration.

In this paper, we are mainly interested in linear differential systems described by high-order differential equation with constant coefficient matrices

$$R_0 w + R_1\frac{dw}{dt} + \cdots + R_L\frac{d^L w}{dt^L} = 0$$

or, in shorthand

$$R(\frac{d}{dt})w = 0, \quad R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L$$

where $w : \mathbb{R} \to \mathbb{R}^q$ denotes the system variables, and $R_0, R_1, \ldots, R_L$ are matrices that contain the system parameters. This system representation is called the *kernel representation.* In this case, the dynamical system Σ is defined by $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with

$$\mathfrak{B} = \left\{ w \in \mathfrak{F}(\mathbb{R}, \mathbb{R}^q) \,\middle|\, R(\frac{d}{dt})w = 0 \right\}$$

where $\mathfrak{F}$ denotes an appropriate function space such as $\mathfrak{C}^\infty$ and $\mathfrak{L}_1^{loc}$. If $\mathfrak{F} = \mathfrak{C}^\infty$, the system trajectory w is *a strong solution* of $R(d/dt)w = 0$. Namely, w is L-times differentiable, and $R(d/dt)w(t) = 0$ is satisfied for all t. On the other hand, if we take $\mathfrak{F}$ as $\mathfrak{L}_1^{loc}$, then $w \in \mathfrak{B}$ should be considered as *a weak solution* since it may not be continuous. A weak solution to $R(d/dt)w = 0$ is defined as a function w satisfying

$$\int_{-\infty}^{\infty} w^T(t)(R^T(-\frac{d}{dt})f)(t)\,dt = 0$$

for all test functions f ($\mathfrak{C}^\infty$-functions of compact support).

We define $p(\Sigma)$ as the number of linear differential constraints on w that are linearly independent over $\mathbb{R}[\xi]$. If the system Σ is described by the kernel representation $R(d/dt)w = 0$, then $p(\Sigma)$ is equal to rank(R), where the "rank" is taken as the rank of a polynomial matrix. This value does not depend on any particular representations. Suppose that R has full row rank. Then, by reordering the components of w, w and R can be partitioned as

$$w = \mathrm{col}(y, u), \quad R(\xi) = [\, P(\xi) \;\; -Q(\xi)\,]$$

with P square and $\det(P) \neq 0$. The condition $\det(P) \neq 0$ implies that the variables u and y serve as the input and the output in the dynamical system Σ, respectively. Clearly, $\dim(y)$, the number of output variables in w, is equal to $p(\Sigma)$. Hence, we call $p(\Sigma)$ *the output cardinality*. The rational matrix $P^{-1}(\xi)Q(\xi)$ is called *the transfer function matrix.*

2.2 State map

Let a dynamical system Σ be given by $(\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with

$$\mathfrak{B} = \left\{ w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q) \,\middle|\, R(\frac{d}{dt})w = 0, \text{ weakly} \right\} \tag{1}$$

$$R(\xi) = R_0 + R_1\xi + \cdots + R_{L-1}\xi^{L-1} + R_L\xi^L \in \mathbb{R}^{m\times q}[\xi]$$

We consider a differential map defined by a polynomial matrix $X \in \mathbb{R}^{n\times q}[\xi]$:

$$x = X(\frac{d}{dt})w, \quad w \in \mathfrak{B} \tag{2}$$

Definition 1 A polynomial matrix X is a state map and x is a state vector if the following *axiom of state* is satisfied:

$$\begin{aligned}&[(w_1, x_1), (w_2, x_2) \in \mathfrak{B}^{\text{full}}\,] \;\&\; [x_1,\; x_2\text{: continuous at } t = 0]\\ &\quad \& \; [x_1(0) = x_2(0)]\\ &\quad \Rightarrow [(w_1, x_1) \underset{0}{\wedge} (w_2, x_2) \in \mathfrak{B}^{\text{full}}]\end{aligned}$$

where $\mathfrak{B}^{\text{full}}$ is the full behavior defined by

$$\mathfrak{B}^{\text{full}} = \left\{ (w, x) \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^n) \,\middle|\, R(\frac{d}{dt})w = 0,\; x = X(\frac{d}{dt})w, \text{ weakly} \right\} \tag{3}$$

It is well known that x is a state vector if and only if the full behavior admits a so-called *state representation* which is first-order in x and zeroth-order in w [12]. Namely, x is a state vector iff there exist constant matrices E, F and G such that

$$E\frac{d}{dt}x + Fx + Gw = 0 \;\text{ for } (w, x) \in \mathfrak{B}^{\text{full}} \tag{4}$$

Then, the full state behavior in (3) can be rewritten as

$$\mathfrak{B}^{\text{full}} = \left\{ (w, x) \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^n) \,\middle|\, E\frac{dx}{dt} + Fx + Gw = 0, \text{ weakly} \right\} \tag{5}$$

We now introduce *the shift and cut map* which plays a crucial role in construction of a state map. The shift and cut map $\sigma : \mathbb{R}[\xi] \to \mathbb{R}[\xi]$ is defined by

$$\sigma(p)(\xi) = \xi^{-1}[p(\xi) - p_0] = p_1 + p_2\xi + \cdots + p_n\xi^{n-1}$$
$$\text{for } p(\xi) = p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + p_n\xi^n \in \mathbb{R}[\xi]$$

Note that this definition can obviously be extended to vectors and matrices in a componentwise manner. Repeated application of σ is denoted by

$$\sigma^1(p) = \sigma(p), \quad \sigma^k(p) = \sigma(\sigma^{k-1}(p)), \; k = 2, 3, \ldots$$

By applying the shift and cut map to R, we obtain

$$R_{\Xi}(\xi) := \begin{bmatrix} \sigma^1(R) \\ \sigma^2(R) \\ \vdots \\ \sigma^L(R) \end{bmatrix} = \begin{bmatrix} R_1 + R_2\xi + \cdots + R_{L-1}\xi^{L-2} + R_L\xi^{L-1} \\ R_2 + R_3\xi + \cdots + R_L\xi^{L-2} \\ \vdots \\ R_{L-1} + R_L\xi \\ R_L \end{bmatrix} \tag{6}$$

Lemma 1 [9] *A polynomial matrix X is a state map iff there exist a constant matrix A and a polynomial matrix B such that*

$$R_{\Xi}(\xi) = AX(\xi) + B(\xi)R(\xi) \tag{7}$$

Definition 2 A state map X is said to be *minimal* if rowdim$(X) \le$ rowdim(X') for any state map X' of $\mathfrak{B}$.

We henceforth denote the minimal state-space dimension of the system Σ by $n(\Sigma)$, i.e. $n(\Sigma) =$ rowdim(X) for a minimal state map X.

Lemma 2 *If X is a minimal state map, then the matrix A in (7) has full column rank.*

Lemma 3 [9] *If $w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$ is a weak solution of $R(d/dt)w = 0$, then $R_{\Xi}(\frac{d}{dt})w$ is absolutely continuous.*

Let X be a minimal state map for Σ. It then follows from Lemmas 1 and 2 that

$$R_{\Xi}(\frac{d}{dt})w = AX(\frac{d}{dt})w = Ax \;\; \forall w \in \mathfrak{B}$$

where A has full column rank. Hence, we obtain the following result.

Lemma 4 *Let $X \in \mathbb{R}^{n\times q}[\xi]$ be a minimal state map for Σ. Then, the minimal state vector $x = X(\frac{d}{dt})w$ has the same degree of smoothness as $R_{\Xi}(\frac{d}{dt})w$. Namely, for any $w \in \mathfrak{B}$, x is a $\mathfrak{C}^k$-function if and only if so $R_{\Xi}(\frac{d}{dt})w$ is.*

2.3 Consistent initial states

We consider the full state behavior $\mathfrak{B}^{\text{full}}$ in (5). A vector $x_0 \in \mathbb{R}^n$ is called *consistent* if there exists a smooth trajectory $(w, x) \in \mathfrak{B}^{\text{full}} \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^n)$ satisfying $x(0) = x_0$.

We define the space of consistent initial state vectors by

$$\Phi = \left\{x_0 \in \mathbb{R}^n \mid \exists (w, x) \in \mathfrak{B}^{\text{full}} \cap \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q \times \mathbb{R}^n) \text{ s.t. } x(0) = x_0\right\}$$

Definition 3 If the space Φ is the whole state space, namely $\Phi = \mathbb{R}^n$, then the state map X is said to be *trim*.

Lemma 5 [12] *Any minimal state map is trim.*

Given a state representation (4), we can compute the space of consistent initial state vectors as follows.

Lemma 6 *The following subspace recursion converges to Φ within finite steps.*

$$\Phi_0 = \mathbb{R}^n$$

$$\Phi_i = \{x \in \mathbb{R}^n \mid \exists w \in \mathbb{R}^q \text{ s.t. } Fx + Gw \in E\Phi_{i-1}\}, \quad i = 1, 2, \ldots$$

2.4 Interconnections

Consider two dynamical systems $\Sigma_1 = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_1)$ and $\Sigma_2 = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_2)$ with

$$\mathfrak{B}_i = \left\{w \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \middle| R_i(\frac{d}{dt})w = 0, \text{ weakly}\right\}, \; R_i \in \mathbb{R}^{m_i \times q}[\xi] \quad (i = 1, 2)$$

The interconnection of Σ_1 and Σ_2, denoted by $\Sigma_1 \wedge \Sigma_2$, is defined by

$$\Sigma_1 \wedge \Sigma_2 = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_1 \cap \mathfrak{B}_2)$$

The schematic diagram of $\Sigma_1 \wedge \Sigma_2$ is depicted in Figure 1.

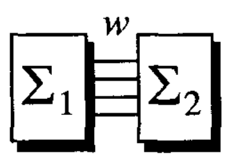


Figure 1: Interconnection $\Sigma_1 \wedge \Sigma_2$

We present the definitions of two important interconnections in terms of the output cardinality $p(\cdot)$ and the minimal state-space dimension $n(\cdot)$.

Definition 4 (i) The interconnection $\Sigma_1 \wedge \Sigma_2$ is said to be a *regular interconnection* or a *feedback interconnection* if $p(\Sigma_1 \wedge \Sigma_2) = p(\Sigma_1) + p(\Sigma_2)$.

(ii) The interconnection $\Sigma_1 \wedge \Sigma_2$ is said to be a *regular feedback interconnection* if it is regular and $n(\Sigma_1 \wedge \Sigma_2) = n(\Sigma_1) + n(\Sigma_2)$.

We may assume without loss of generality that the polynomial matrices R_1 and R_2 are of full row rank. Then, by Definition 4(i), $\Sigma_1 \wedge \Sigma_2$ is regular iff $\begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ has full row rank.

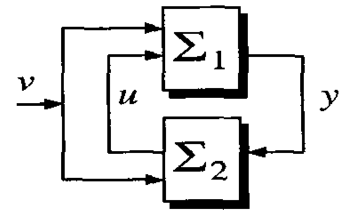


Figure 2: Feedback interconnection

In this case, after an appropriate reordering of the components of w, we obtain the equivalent kernel representations:

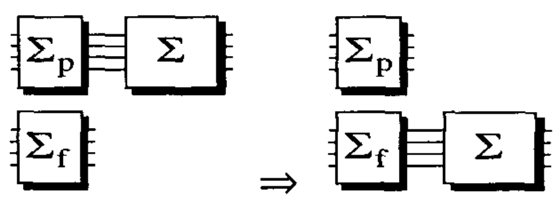
$$w = \text{col}(y, u, v)$$

$$D_1(\frac{d}{dt})y = N_1(\frac{d}{dt})u + N_1'(\frac{d}{dt})v$$

$$D_2(\frac{d}{dt})u = N_2(\frac{d}{dt})y + N_2'(\frac{d}{dt})v$$

with $\det(D_1) \neq 0$, $\det(D_2) \neq 0$ and $\det\begin{bmatrix} D_1 & -N_1 \\ -N_2 & D_2 \end{bmatrix} \neq 0$. The condition $\det(D_1) \neq 0$ implies that the first equation represents the input-output map from (u, v) to y in Σ_1. Similarly, in Σ_2, (y, v) and u can be viewed as the input and the output, respectively. This implies that a regular interconnection $\Sigma_1 \wedge \Sigma_2$ admits a feedback structure depicted in Figure 2. In general, the transfer function matrices $D_i^{-1}(\xi)N_i(\xi)$ and $D_i^{-1}(\xi)N_i'(\xi)$ $(i = 1, 2)$ may not be proper. The condition $n(\Sigma_1 \wedge \Sigma_2) = n(\Sigma_1) + n(\Sigma_2)$ in Definition 4 (ii) guarantees that $\Sigma_1 \wedge \Sigma_2$ admits a feedback structure with proper transfer function matrices.

3 Hybrid System Interconnection



(a) Past ($t < 0$) (b) Future ($t \geq 0$)

Figure 3: Hybrid System Interconnection

We consider the system interconnections illustrated in Figure 3, which consist of three sub-systems $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, $\Sigma_p = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_p)$ and $\Sigma_f = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_f)$.

The system Σ is connected with Σ_p at time $t < 0$. At time $t = 0$, Σ_p is separated from Σ, and a new interconnection is formed between Σ and Σ_f by some external switching mechanism. We assume that all this happens instantaneously. The system Σ is called *the core sub-system* since it appears both in the past and future interconnected systems. The sub-systems Σ_p and Σ_f are called the past and future constraints, respectively. The terms *'past'* and *'future'* are used for describing the time before and after the switching instant $t = 0$, respectively.

The system dynamics in the past and future are respectively described by the interconnections of two sub-systems:

$$\Sigma \wedge \Sigma_p = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_{pic}), \quad \mathfrak{B}_{pic} = \mathfrak{B} \cap \mathfrak{B}_p \tag{8}$$

$$\Sigma \wedge \Sigma_f = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_{fic}), \quad \mathfrak{B}_{fic} = \mathfrak{B} \cap \mathfrak{B}_f \tag{9}$$

The subscripts "pic" and "fic" stand for *past interconnection* and *future interconnection*, respectively. Throughout this paper, we refer to this switching system consisting of the past and future interconnections as *a hybrid interconnection*, and denote it by the triple $(\Sigma, \Sigma_p, \Sigma_f)$.

The hybrid system interconnection $(\Sigma, \Sigma_p, \Sigma_f)$ describes general situations of switching systems. For example, if we consider Σ as the plant and Σ_p, Σ_f as the controllers, then $(\Sigma, \Sigma_p, \Sigma_f)$ represents the switching of control strategies. Moreover, if Σ_p has nothing to do with Σ, then the hybrid interconnection describes the implementation of the controller Σ_f to the plant Σ.

Assume that the sub-systems Σ, Σ_p and Σ_f are described by the kernel representations.

$$\Sigma : R(\frac{d}{dt})w = 0, \quad R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L$$

$$\Sigma_p : C_p(\frac{d}{dt})w = 0, \ C_p(\xi) = C_{p0} + C_{p1}\xi + \cdots + C_{pL_p}\xi^{L_p}$$

$$\Sigma_f : C_f(\frac{d}{dt})w = 0, \quad C_f(\xi) = C_{f0} + C_{f1}\xi + \cdots + C_{fL_f}\xi^{L_f}$$

Then, we define

$$\mathfrak{B} = \left\{ w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q) \,\middle|\, R(\frac{d}{dt})w = 0, \text{ weakly} \right\} \tag{10}$$

$$\mathfrak{B}_p = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \,\middle|\, C_p(\frac{d}{dt})w = 0 \right\} \tag{11}$$

$$\mathfrak{B}_f = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \,\middle|\, C_f(\frac{d}{dt})w = 0 \right\} \tag{12}$$

to obtain

$$\mathfrak{B}_{pic} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \,\middle|\, \begin{bmatrix} R(\frac{d}{dt}) \\ C_p(\frac{d}{dt}) \end{bmatrix} w = 0 \right\} \tag{13}$$

$$\mathfrak{B}_{fic} = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \,\middle|\, \begin{bmatrix} R(\frac{d}{dt}) \\ C_f(\frac{d}{dt}) \end{bmatrix} w = 0 \right\} \tag{14}$$

Remark 1: It is natural to define the behavior of the core subsystem Σ as the set of $\mathfrak{L}_1^{loc}$-functions (weak solutions to $R(d/dt)w = 0$). This is because the continuity at $t = 0$ of system trajectories is no longer guaranteed due to the switching of interconnections.

Remark 2: It is possible to consider more general function space (e.g. $\mathfrak{L}_1^{loc}$) for $\mathfrak{B}_{pic}$ and $\mathfrak{B}_{fic}$ (or $\mathfrak{B}_p$, $\mathfrak{B}_f$). Since, however, we are interested in the jump and impulsive phenomena at the switching instant $t = 0$, we assume without loss of generality that $\mathfrak{B}_{pic}$, $\mathfrak{B}_{fic} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$.

The system trajectory of the hybrid interconnection can be viewed as a concatenation of a past trajectory $w_p \in \mathfrak{B}_{pic}$ and a future trajectory $w_f \in \mathfrak{B}_{fic}$. Since the core sub-system Σ appears both in the past and future interconnections, the past and future trajectories must be concatenated in accordance with the laws of Σ. Hence, the system trajectory of the hybrid interconnection may generally contain impulses due to this constraint. A typical example of impulsive phenomena in hybrid system interconnections is illustrated in Example 1.

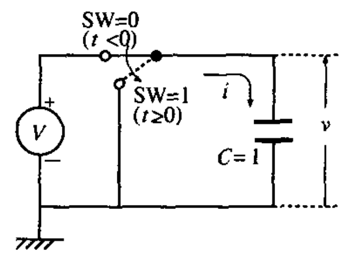


Figure 4: Electrical circuit with a switch (Example 1)

Example 1: As an example of hybrid interconnections, consider a simple electrical circuit (Figure 4) which consists of a capacitance and a energy source with constant voltage. We assume for simplicity that the capacitance is normalized. In the past ($t < 0$), the capacitance is connected to the energy source (SW = 0). At time $t = 0$, the circuit is shorted (SW = 1). Let v, i and V denote the voltage and the current across the capacitance and the source voltage. Then this electrical circuit is governed by the following physical laws.

$$\text{For } t < 0: \ \frac{d}{dt}v = i, \ \frac{d}{dt}V = 0, \ v = V$$

$$\text{For } t \geq 0: \ \frac{d}{dt}v = i, \ \frac{d}{dt}V = 0, \ v = 0$$

Since the equations $dv/dt = i$ and $dV/dt = 0$ are common to both past and future, they define the core sub-system Σ. Moreover, $v = V$ and $v = 0$ serve as the past constraint Σ_p and the future constraint Σ_f, respectively. Hence, by defining $w = \text{col}(v, \ i, \ V)$, we obtain $\Sigma = (\mathbb{R}, \mathbb{R}^3, \mathfrak{B})$, $\Sigma_p = (\mathbb{R}, \mathbb{R}^3, \mathfrak{B}_p)$ and $\Sigma_f = (\mathbb{R}, \mathbb{R}^3, \mathfrak{B}_f)$ with

$$\mathfrak{B} = \left\{ w \in \mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^3) \,\middle|\, \frac{d}{dt}w_1 = w_2, \ \frac{d}{dt}w_3 = 0, \text{ weakly} \right\}$$

$$\mathfrak{B}_p = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^3) \mid w_1 = w_3 \right\}$$

$$\mathfrak{B}_f = \left\{ w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^3) \mid w_1 = 0 \right\}$$

It follows that

$$w_1 = v = V(1 - H), \ \ w_2 = i = -V\delta, \ \ w_3 = V \text{ (constant)}$$

The derivative dv/dt is taken in the sense of *distributions* since v is not continuous at $t = 0$ if $V \neq 0$. If V is not equal to zero, then the current i has an impulse; the circuit will *spark*. Clearly, this trajectory with $V \neq 0$ does not belong to $\mathfrak{B}$ because impulsive distributions are not $\mathfrak{L}_1^{loc}$-functions.

The impulsive phenomenon is undesirable from a practical viewpoint because such a phenomenon may result in damage of the physical system; for example, the circuit will burn out if the sparking takes place in the system of Example 1. Therefore, it is important to consider the concatenability conditions under which the past and future trajectories can be concatenated without impulses.

We give the precise definitions of the concatenabilities.

Definition 5 (concatenability of trajectories)
Two smooth trajectories $w_p \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ and $w_f \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ are said to be *concatenable* if $w_p \wedge_0 w_f \in \mathfrak{B}$ is satisfied.

Definition 6 (concatenability of behaviors)
The behavior $\mathfrak{B}_{pic}$ is said to be *concatenable with* $\mathfrak{B}_{fic}$ if, for any past trajectory $w_p \in \mathfrak{B}_{pic}$, there exists a future trajectory $w_f \in \mathfrak{B}_{fic}$ such that $w_p \wedge_0 w_f \in \mathfrak{B}$.

In this paper, we will derive necessary and sufficient conditions for the concatenabilities defined above. Moreover, we will consider the relation among the concatenability conditions, the state maps and the regularity of interconnection.

Remark 3: In order to deal with impulsive trajectories as in Example 1, we may need to extend the set of solutions of $R(d/dt)w = 0$ to a class of distributions including the delta distribution and its derivatives. However, for the purpose of this paper, it suffices to restrict the behavior of Σ to the set of $\mathfrak{L}_1^{loc}$-functions, since we consider the impulse-free concatenation of the past and future trajectories, that is, we have not used any impulses in the definitions of the concatenabilities.

4 Concatenability of Trajectories

In this section, we derive a necessary and sufficient condition for the concatenability of two smooth trajectories.

Let $w_p,\ w_f \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ be given. After simple but lengthy calculation including partial integration, we obtain

$$\int_{-\infty}^{\infty} (R^T(-\frac{d}{dt})f)^T(t)(w_p \underset{0}{\wedge} w_f)(t)dt = \sum_{i=1}^{L}(-1)^{i-1}(\frac{d^{i-1}f}{dt^{i-1}})^T(0)\left[\sigma^i(R)(\frac{d}{dt})(w_f - w_p)\right](0)$$

for an arbitrary test function f. It then follows from the definition of a weak solution that $w_p \wedge_0 w_f$ is a weak solution of $R(d/dt)w = 0$ iff

$$\left[\sigma^i(R)(\frac{d}{dt})(w_f - w_p)\right](0) = 0 \quad \text{for } i = 1,2,\dots,L$$

Consequently, we obtain the following theorem from the above discussion and Lemmas 3 and 4.

Theorem 1 *The following statements are equivalent.*

(i) $w_p,\ w_f \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$ *are concatenable.*

(ii) $\left[R_\Xi(\frac{d}{dt})(w_f - w_p)\right](0) = 0$, *where R_Ξ is defined by(6).*

(iii) *The minimal state vectors of Σ corresponding to w_p and w_f coincide at $t = 0$, namely*

$$(w_p, x_p), (w_f, x_f) \in \mathfrak{B}^{full} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q \times \mathbb{R}^{n(\Sigma)}) \implies x_p(0) = x_f(0)$$

where $\mathfrak{B}^{full}$ is the full state behavior associated with the minimal state vector x.

5 Concatenability of Behaviors

We introduce the minimal state representation for each sub-system:

$$\Sigma : E\frac{d}{dt}x + Fx + Gw = 0, \qquad n = \dim(x) = n(\Sigma)$$

$$\Sigma_p : E_p\frac{d}{dt}y + F_py + G_pw = 0, \quad n_p = \dim(y) = n(\Sigma_p)$$

$$\Sigma_f : E_f\frac{d}{dt}z + F_fz + G_fw = 0, \quad n_f = \dim(z) = n(\Sigma_f)$$

Then, $\mathrm{col}(x,y)$ and $\mathrm{col}(x,z)$ are respectively (possibly non-minimal) state vectors of $\Sigma \wedge \Sigma_p$ and $\Sigma \wedge \Sigma_f$, and their full state behaviors are given by

$$\mathfrak{B}_{pic}^{full} = \left\{(w, \mathrm{col}(x,y)) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q \times \mathbb{R}^{n+n_p}) \,\middle|\, \begin{bmatrix} E & 0 \\ 0 & E_p \end{bmatrix}\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} F & 0 \\ 0 & F_p \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} G \\ G_p \end{bmatrix} w = 0 \right\}$$

$$\mathfrak{B}_{fic}^{full} = \left\{(w, \mathrm{col}(x,z)) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q \times \mathbb{R}^{n+n_f}) \,\middle|\, \begin{bmatrix} E & 0 \\ 0 & E_f \end{bmatrix}\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} F & 0 \\ 0 & F_f \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} G \\ G_f \end{bmatrix} w = 0 \right\}$$

Define the spaces of consistent initial state vectors as

$$\Phi_{pic} = \left\{\mathrm{col}(x_0, y_0) \in \mathbb{R}^{n+n_p} \,\middle|\, \exists (w,x,y) \in \mathfrak{B}_{pic}^{full} \text{ s.t. } x(0) = x_0,\ y(0) = y_0 \right\}$$

$$\Phi_{fic} = \left\{\mathrm{col}(x_0, z_0) \in \mathbb{R}^{n+n_f} \,\middle|\, \exists (w,x,z) \in \mathfrak{B}_{fic}^{full} \text{ s.t. } x(0) = x_0,\ z(0) = z_0 \right\}$$

and the projection matrices

$$\Pi_p = [\, I_n \ \ 0_{n\times n_p} \,], \quad \Pi_f = [\, I_n \ \ 0_{n\times n_f} \,]$$

Note that we can easily compute Φ_{pic} and Φ_{fic} by the recursive method described in Lemma 6.

Let $\mathrm{col}(x_p, y)$ and $\mathrm{col}(x_f, z)$ be the state trajectories for $w_p \in \mathfrak{B}_{pic}$ and $w_f \in \mathfrak{B}_{fic}$, respectively, i.e. $(w_p, \mathrm{col}(x_p,y)) \in \mathfrak{B}_{pic}^{full}$, $(w_f, \mathrm{col}(x_f,z)) \in \mathfrak{B}_{fic}^{full}$. Recall that x_p and x_f are the minimal state trajectories for Σ corresponding to w_p and w_f, respectively. Then, by Theorem 1, $w_p \in \mathfrak{B}_{pic}$ and $w_f \in \mathfrak{B}_{fic}$ are concatenable iff $x_p(0) = x_f(0)$. It follows from this observation that $\mathfrak{B}_{pic}$ is concatenable with $\mathfrak{B}_{fic}$ if and only if

$$\forall (w_p, \mathrm{col}(x_p, y)) \in \mathfrak{B}_{pic}^{full},\quad \exists (w_f, \mathrm{col}(x_f, z)) \in \mathfrak{B}_{fic}^{full} \text{ s.t. } x_p(0) = x_f(0)$$

By rewriting this condition in terms of the spaces of consistent initial states, we obtain a main result of this paper.

Theorem 2 *$\mathfrak{B}_{pic}$ is concatenable with $\mathfrak{B}_{fic}$ if and only if $\Pi_p\Phi_{pic} \subseteq \Pi_f\Phi_{fic}$ holds.*

If $\Pi_f\Phi_{fic} = \mathbb{R}^n$ holds, then $\mathfrak{B}_{pic}$ is concatenable with $\mathfrak{B}_{fic}$ for any choice of the past constraint Σ_p (or $C_p(\xi)$). Hence, we are interested in the question "What type of future interconnection $\Sigma \wedge \Sigma_f$ will satisfy $\Pi_f\Phi_{fic} = \mathbb{R}^n$?".

It is trivial that $\Pi_f\Phi_{fic} = \mathbb{R}^n$ holds if there does not exist any essential constraints in the future, namely $\mathfrak{B}_{fic} = \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$. In terms of kernel representation, the latter condition is equivalent to the existence of a polynomial matrix K such that $C_f = KR$.

In the case of regular interconnection, $\Pi_f\Phi_{fic} = \mathbb{R}^n$ is not necessarily satisfied. An example of regular interconnections such that $\Pi_f\Phi_{fic} \neq \mathbb{R}^n$ is given in the next section.

In contrast with regular interconnection, regular feedback interconnection has the nice property related to the behavioral concatenability shown in the next theorem. This justifies application of a regular feedback controller in hybrid system interconnections.

Theorem 3 *Assume that $\Sigma \wedge \Sigma_f$ is a regular feedback interconnection. Then, $\Pi_f \Phi_{fic} = \mathbb{R}^n$ holds. Therefore, for any past constraint Σ_p, $\mathfrak{B}_{pic}$ is concatenable with $\mathfrak{B}_{fic}$.*

Proof: Recall that x and z are minimal state vectors of Σ and Σ_f, respectively. It follows from Definition 4(ii) that $\Sigma \wedge \Sigma_f$ is regular feedback iff (x, z) is its minimal state vector. The latter condition implies the state-trimness $\Phi_{fic} = \mathbb{R}^n \times \mathbb{R}^{n_f}$. Hence, we obtain $\Pi_f \Phi_{fic} = \mathbb{R}^n$. ∎

6 Examples

Example 1: Consider the electrical circuit in Example 1 again. As shown in Section 3, $\mathfrak{B}_{pic}$ is not concatenable with $\mathfrak{B}_{fic}$ since there exists an impulsive trajectory for non-zero source voltage V. The polynomial matrices associated with the kernel representations for the sub-systems are given by

$$R(\xi) = \begin{bmatrix} \xi & -1 & 0 \\ 0 & 0 & \xi \end{bmatrix}, \quad \begin{array}{l} C_p(\xi) = [\ 1 \quad 0 \quad -1\ ] \\ C_f(\xi) = [\ 1 \quad 0 \quad 0\ ] \end{array}$$

Hence, we get

$$p(\Sigma \wedge \Sigma_f) = 3, \quad p(\Sigma) = 2, \quad p(\Sigma_f) = 1$$
$$n(\Sigma \wedge \Sigma_f) = 1, \quad n(\Sigma) = 2, \quad n(\Sigma_f) = 0$$

This implies that $\Sigma \wedge \Sigma_f$ is a regular interconnection, but *not* a regular feedback interconnection.

Since C_p and C_f are constant matrices, we have

$$\Pi_p = \Pi_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad n_p = n_f = 0$$

The spaces of consistent initial state vectors are computed through the minimal state representations and the subspace recursion of Lemma 6.

$$\Phi_{pic} = \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Phi_{fic} = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Consequently, we obtain $\Pi_p \Phi_{pic} \not\subset \Pi_f \Phi_{fic}$.

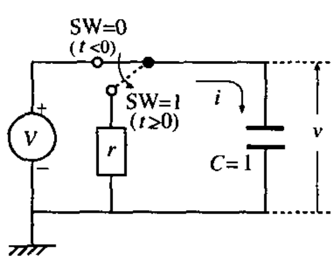


Figure 5: Electrical circuit with a switch (Example 2)

Example 2: Next, consider the electrical circuit in Figure 5. This circuit is almost the same as the previous one except for the existence of a resistor r in the future interconnection.

By defining $w = \mathrm{col}(v, i, V)$ as in Example 1, we obtain

$$R(\xi) = \begin{bmatrix} \xi & -1 & 0 \\ 0 & 0 & \xi \end{bmatrix}, \quad \begin{array}{l} C_p(\xi) = [\ 1 \quad 0 \quad -1\ ] \\ C_f(\xi) = [\ 1 \quad r \quad 0\ ] \end{array}$$

$\Sigma \wedge \Sigma_f$ is a regular feedback interconnection because

$$p(\Sigma \wedge \Sigma_f) = 3, \quad p(\Sigma) = 2, \quad p(\Sigma_f) = 1$$
$$n(\Sigma \wedge \Sigma_f) = 2, \quad n(\Sigma) = 2, \quad n(\Sigma_f) = 0$$

Therefore, by Theorem 3, $\mathfrak{B}_{pic}$ is concatenable with $\mathfrak{B}_{fic}$. In fact, any past trajectory $w_p = \mathrm{col}(V, 0, V) \in \mathfrak{B}_{pic}$ $(V \in \mathbb{R})$ can be concatenated, without causing any impulses, with the future trajectory $w_f = \mathrm{col}(e^{-t/r}V, -\frac{1}{r}e^{-t/r}V, V) \in \mathfrak{B}_{fic}$.

7 Conclusion

In this paper, we considered the analysis of the hybrid system interconnection based on the behavioral approach. We first defined concatenability of the behaviors of the past and future interconnections. Then, a necessary and sufficient condition for the concatenability of the behaviors $\mathfrak{B}_{pic}$ and $\mathfrak{B}_{fic}$ was derived in terms of the state maps of the core sub-system Σ. Moreover, we showed that, if the future interconnection is regular feedback, the past interconnection behavior $\mathfrak{B}_{pic}$ can be concatenated with $\mathfrak{B}_{fic}$ for any choice of the past constraint Σ_p. This justifies the use of a regular feedback controller for the hybrid system interconnections.

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