THE STORAGE FUNCTION FOR SYSTEMS DESCRIBED BY PDE's

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1 Introduction

In the behavioural approach, systems are often described by high-order systems of (partial) differential equations (for example as kernel, image or latent variable representations). It is often very convinient to think of these systems of differential equations in terms of matrices over polynomial rings. In control theory, a lot of problems involve the use of quadratic functionals (like linear quadratic and H_∞-control). In [4], a study of quadratic differential forms was carried out, which has greatly contributed to the study of H_∞ problems in the behavioural framework. In this paper, we wish to extend this concept of quadratic differential forms to N-D systems. This opens the way to generalization of the concept of dissipative systems, which have been well studied in the 1-D case. This generalization of the notion of dissipative systems and the construction of storage functions is the aim of this paper.

2 Quadratic Differential Forms

Throughout this paper we let ζ denote $(\zeta_1, \ldots, \zeta_n)$, η denote $(\eta_1, \ldots, \eta_n)$, and ξ denote $(\xi_1, \ldots, \xi_n)$. Let $\mathbb{R}^{q\times q}[\zeta, \eta]$ denote the set of real polynomial matrices in the $2n$ indeterminates ζ and η. We will consider quadratic forms Q_Φ induced by $\Phi \in \mathbb{R}^{q\times q}[\zeta, \eta]$. Explicitly,

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}} \tag{1}$$

The sum above ranges over the multi-indices $\mathbf{k}, \mathbf{l} \in \mathbb{N}^n$ and of course only a finite number of the $\Phi_{\mathbf{k},\mathbf{l}}$'s are nonzero, and $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{q\times q}$. Such a Φ induces a *quadratic differential form* (QDF), that is, the map

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}) \tag{2}$$

defined by

$$Q_\Phi(w)(\mathbf{x}) := \sum_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} w(\mathbf{x})\right)^T \Phi_{\mathbf{k},\mathbf{l}} \left(\frac{d^{\mathbf{l}}}{d\mathbf{x}^{\mathbf{l}}} w(\mathbf{x})\right) \tag{3}$$

where $\frac{d^{\mathbf{k}}}{d\mathbf{x}^{\mathbf{k}}} = \frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}}$ with $\mathbf{k} = (k_1, \ldots, k_n)$.

Define the * operator

$$^* : \mathbb{R}^{q\times q}[\zeta, \eta] \to \mathbb{R}^{q\times q}[\zeta, \eta]$$

by

$$\Phi^*(\zeta, \eta) := \Phi^T(\eta, \zeta) \tag{4}$$

where ^{T} denotes transposition. If $\Phi = \Phi^*$, then Φ is called *symmetric*. For the purposes of QDF's induced by polynomial matrices, it is enough to only consider the symmetric quadratic differential forms, since $Q_\Phi = Q_{\frac{1}{2}(\Phi+\Phi^*)}$.

In addition, we also consider vectors $[\Psi] \in (\mathbb{R}^{q\times q}[\zeta, \eta])^n$, i.e. $[\Psi] = (\Psi_1, \ldots, \Psi_n)$. Analoguous to the quadratic differential form Φ, $[\Psi]$ induces a *vector quadratic differential form* (VQDF)

$$Q_{[\Psi]}(w) : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q) \to (\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}))^n \tag{5}$$

defined by $Q_{[\Psi]} = (Q_{\Psi_1}, \ldots, Q_{\Psi_n})$.

Let $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ and consider the associated quadratic differential form Q_Φ. Let us call Q_Φ *nonnegative* (denoted by $Q_\Phi \geq 0$) if

$$Q_\Phi(w) \geq 0 \;\; \forall w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q) \tag{6}$$

This states that the $Q_\Phi(w)$ is pointwise nonnegative and it can be shown that this is the case if and only if

$$Q_\Phi(w) = \left| D(\frac{d}{d\mathbf{x}})w \right|^2$$

3 Path Independence

Consider the integral

$$\int_\Omega Q_\Phi(w)d\mathbf{x} \tag{7}$$

where Ω is a closed bounded subset of $\mathbb{R}^n$ with a nonempty interior. This integral is said to be independent of the "path" w (or a *path integral*), if the integral only depends on the value of w and its derivatives on the boundary of Ω, denoted by $\partial\Omega$. More precisely, if for any $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q)$ such that $\frac{d^{\mathbf{k}}w_1}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) = \frac{d^{\mathbf{k}}w_2}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$ and all $\mathbf{k} \in \mathbb{N}^n$, then

$$\int_\Omega Q_\Phi(w_1)d\mathbf{x} \;=\; \int_\Omega Q_\Phi(w_2)d\mathbf{x} \tag{8}$$

Instead of some $\Omega \subset \mathbb{R}^n$, if we consider the integral (7) over all of $\mathbb{R}^n$, then the integral need not be well defined for all $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q)$. We can overcome this by considering the functional

$$\int Q_\Phi : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q) \rightarrow \mathbb{R} \tag{9}$$

defined by

$$\int Q_\Phi := \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \tag{10}$$

which evaluates the integral over all of $\mathbb{R}^n$. Here $\mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$ denotes the compactly supported members of $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^q)$.

The following theorem gives several conditions that are equivalent to path independence.

Theorem 1 *Let $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Then the following statements are equivalent:*

1. $\int Q_\Phi = 0$.
2. $\int_\Omega Q_\Phi$ *is independent of path for all Ω, which are closed bounded subsets of $\mathbb{R}^n$.*
3. $\Phi(-\xi,\xi) = 0$.
4. *There exist $\Psi_1, \ldots, \Psi_n \in \mathbb{R}^{q\times q}[\zeta,\eta]$, such that*
$$\Phi(\zeta,\eta) = (\zeta_1+\eta_1)\Psi_1(\zeta,\eta) + \cdots + (\zeta_n+\eta_n)\Psi_n(\zeta,\eta)$$
5. *There exists a $[\Psi] \in (\mathbb{R}^{q\times q}[\zeta,\eta])^n$ such that*
$$\text{div} Q_{[\Psi]}(w) \;=\; Q_\Phi(w)$$

By "div" in the above theorem, we mean divergence of the vector function, i.e.

$$\text{div}\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \frac{\partial f_1}{\partial x_1} + \ldots + \frac{\partial f_n}{\partial x_n} \tag{11}$$

The $[\Psi]$ in the theorem above is not unique. This non-uniqueness of $[\Psi]$ is an inherent property of N-D systems. In the 1-D case, the Ψ which satisfies the conditions of the above theorem is unique. However, it can be shown that all $[\Psi]$'s that satisfy the above theorem with respect to a given Φ are related to each other in a special form and they form an equivalence class in $(\mathbb{R}^{q\times q}[\zeta,\eta])^n$.

In [1, 3] (and elsewhere) differential systems have been studied in the behavioural framework where a behaviour $\mathfrak{B}$ is characterized as the kernel of (partial) differential operator. Now we consider the specific case of a behaviour $\mathfrak{B}$ given as a kernel of a system of partial differential equations. As shown in [1], this system of partial differential operators can be written as a polynomial matrix $R \in \mathbb{R}^{g\times q}[\xi]$ in n indeterminates. We would like to know when a QDF induced by $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ is independent of path for trajectories $w \in \mathfrak{B}$, i.e. if $w_1, w_2 \in \mathfrak{B}$ and $\frac{d^{\mathbf{k}}w_1}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x}) = \frac{d^{\mathbf{k}}w_2}{d\mathbf{x}^{\mathbf{k}}}(\mathbf{x})$ for $\mathbf{x} \in \partial\Omega$ and all $\mathbf{k} \in \mathbb{N}^n$, then

$$\int_\Omega Q_\Phi(w_1)d\mathbf{x} \;=\; \int_\Omega Q_\Phi(w_2)d\mathbf{x}$$

We first define the $\star$ operator as $X^\star(\xi) = X^T(-\xi)$.

We state below a theorem, which is applicable to controllable systems (see [1, 2] for definition of controllable behaviours for distributed systems).

Theorem 2 *Let $\mathfrak{B}$ be a controllable system, $\mathfrak{B} = \ker R(\frac{d}{d\mathbf{x}}) = \text{im} M(\frac{d}{d\mathbf{x}})$, and let $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ induce a QDF on $\mathfrak{B}$. Then the following conditions are equivalent :*

1. *QDF induced by Φ is independent of path on $\mathfrak{B}$;*
2. *there exists $X \in \mathbb{R}^{\bullet\times q}[\xi]$ such that*
$$X^\star(\xi)R(\xi) + R^\star(\xi)X(\xi) = \Phi(-\xi,\xi)$$
3. *the QDF corresponding to Φ' is a path integral, where Φ' is given by $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$;*
4. *$\Phi'(-\xi,\xi) = 0$;*
5. *there exists a VQDF $Q_{[\Psi']}$, with $[\Psi'] \in (\mathbb{R}^{m\times m}[\zeta,\eta])^n$, where m is the number of columns of M, such that*
$$\text{div}Q_{[\Psi']}(\ell) = Q_{\Phi'}(\ell) = Q_\Phi(w) \qquad (12)$$
for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^m)$ and $w = M(\frac{d}{d\mathbf{x}})\ell$.

From the above theorem, it is seen that the VQDF acts on the latent variables associated to the image representation of the given controllable behaviour. In 1-D systems, every controllable system has an observable image representation. This is not true in the N-D case. As a result, in the 1-D case, we can actually find a quadratic differential form Ψ such that
$$\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w)$$
for all $w \in \mathfrak{B}$, whereas in the N-D case, we can find a VQDF $[\Psi]$ such that
$$\text{div } Q_{[\Psi]}(w) = Q_\Phi(w)$$
for all $w \in \mathfrak{B}$ if $\mathfrak{B}$ is a controllable behaviour which has an observable image representation. In [2], we give which controllable behaviours have an observable image representation. However, in general we have to deal with the latent variables ℓ in the conservation equation (12).

4 Dissipative systems

In the last section, we considered QDF's such that $\int Q_\Phi$ was the zero map. Such QDF's define conservation laws for a given behaviour. In this section, we consider QDF's where the integral mentioned above is not equal to zero, but is non-negative. We refer to these as dissipative systems.

Given a $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$, we call the QDF induced by Φ *average non-negative* if
$$\int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \geq 0$$
for all $w \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^q)$. We denote an average non-negative QDF Φ by $\int Q_\Phi \geq 0$. If the inequality in the above equation is strict for $w \neq 0$, then we call the QDF *average positive* and denote it by $\int Q_\Phi > 0$.

Proposition 3 *Let $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Then*

- $(\int Q_\Phi \geq 0) \Leftrightarrow \Phi(-i\omega,i\omega) \geq 0\ \forall\omega \in \mathbb{R}^n$
- $(\int Q_\Phi > 0) \Leftrightarrow \Phi(-i\omega,i\omega) \geq 0\ \forall\omega \in \mathbb{R}^n$ *and the matrix $\Phi(-\xi,\xi)$ is non-singular.*

Intuitively, we could think of these average nonnegative QDF's as measuring the power going into the system. In many practical examples, the power is indeed a quadratic differential form of some system variables. For example, if we consider Maxwell's equations, the power supplied to the system is given as the $\sum_k E_k J_k$, where E_k is the electric field component in the k-direction and J_k is the current in the k-direction. Average non-negativity would imply that the net power flowing into a system is non-negative. This means that the system dissipates energy. Of course, locally the flow of energy could be positive or negative, leading to variations in energy density and fluxes. The energy density and fluxes could be thought of as a storage function for the energy. If the system is dissipative, then the rate of change of energy density and fluxes cannot exceed the power delivered into the system. This interaction between supply, storage and dissipation has been formalized for 1-D systems in [4]. We now formalize the interaction between these concepts for distributed systems.

Definition 4 *Let $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ induce a QDF Q_Φ. A vector quadratic differential form (VQDF) $[\Psi] \in (\mathbb{R}^{q\times q}[\zeta,\eta])^n$ is said to be a* storage function *for Φ if*
$$\text{div}Q_{[\Psi]}(\ell) \leq Q_\Phi(w) \qquad (13)$$
for all $w \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^q)$ and some latent variables ℓ, such that there exists a partial differential relation between the w's and the ℓ's of the form $w = M'(\frac{d}{d\mathbf{x}})\ell$.

A QDF Q_Δ induced by $\Delta \in \mathbb{R}^{q\times q}[\zeta,\eta]$ is said to be a dissipation function for Φ if
$$Q_\Delta \geq 0 \text{ and } \int_{\mathbb{R}^n} Q_\Delta(\ell)d\mathbf{x} = \int_{\mathbb{R}^n} Q_\Phi(w)d\mathbf{x} \qquad (14)$$
for all $w \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R}^q)$ and some latent variables ℓ which are related to the w's by some partial differential relation
$$w = D(\frac{d}{d\mathbf{x}})\ell. \qquad (15)$$

The next proposition shows the relationship between a given supply function and the associated storage and dissipation functions. This means that average non-negativity can be interpreted by a local non-negativity condition involving the rate of change of storage function and the supply rate.

Proposition 5 *The following conditions are equivalent :*

1. $\int Q_\Phi \geq 0$
2. Φ *admits a storage function*
3. Φ *admits a dissipation function*

In addition, the VQDF associated to storage $Q_{[\Psi]}$ *and the QDF associated to dissipation* Q_Δ *with respect to a given supply QDF* Q_Φ *are related by*

$$\text{div } Q_{[\Psi]}(\ell) = Q_\Phi(w) - Q_\Delta(\ell) \quad (16)$$

where ℓ*'s are the associated latent variables, related to* w *by the equation* $w = M'(\frac{d}{d\mathbf{x}})\ell$.

Definition 6 *QDF's* Φ *and* Δ *alongwith a VQDF* $[\Psi]$, *that are related by the dissipation equation (16) for all* w *in* $\mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$ *would be called a matched triple and denoted by* $(\Phi, [\Psi], \Delta)$.

A triple $(\Phi, [\Psi], \Delta)$ *that are related by the dissipation equation (16) for all* $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q)$ *would be called a matched triple on* $\mathfrak{B}$.

Note that unlike the 1-D case, there is no one-to-one correspondence between storage and dissipation functions. Given a supply QDF and an associated dissipation QDF, one can find several VQDFs that would satisfy (16). However, as mentioned before, one can find an equivalence class in $(\mathbb{R}^{q\times q}[\zeta,\eta])^n$ which can be uniquely associated to a given supply and dissipation QDFs. Thus, for a matched triple $(\Phi, [\Psi], \Delta)$, the $[\Psi]$ essentially represents an equivalence class in $(\mathbb{R}^{\bullet\times\bullet}[\zeta,\eta])^n$.

So far in this section, we have been considering QDFs Q_Φ which are average non-negative. Now we would like to consider a behaviour $\mathfrak{B}$ and a supply rate associated to this behaviour. So for a QDF to be a supply rate associated to a behaviour, it is enough for the QDF to be average non-negative on all $w \in \mathfrak{B}$. So we define a QDF Q_Φ to be *average non-negative with respect to a behaviour* $\mathfrak{B}$ if

$$\int_{\mathbb{R}^n} Q_\Phi(w) d\mathbf{x} \geq 0 \quad \forall w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}^n, \mathbb{R}^q) \quad (17)$$

These QDFs could now represent supply rates for a given behaviour.

The main purpose of this paper is to announce the following result.

Theorem 7 *Let* $\mathfrak{B}$ *be a controllable behaviour,* $\mathfrak{B} = \ker R(\frac{d}{d\mathbf{x}}) = \text{im} M(\frac{d}{d\mathbf{x}})$ *and* $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$. *If the QDF* Q_Φ *is average non-negative with respect to the behaviour* $\mathfrak{B}$, *then there exists dissipation QDF* Q_Δ *and storage VQDF* $Q_{[\Psi]}$ *such that*

$$\text{div } Q_{[\Psi]}(\ell) = Q_\Phi(w) - Q_\Delta(\ell) \quad (18)$$

where $w \in \mathfrak{B}$ *and* ℓ*'s are the latent variables related to* w *by the equation* $w = M'(\frac{d}{d\mathbf{x}})\ell$.

Summarizing, we have obtained the following result. For controllable systems described by constant coefficient linear partial differential equations with supply rates that are QDF's in the system variables, assume that the system is dissipative in the sense that (17) holds. Then it is indeed possible to express dissipativeness as the existence of a vector storage function $Q_{[\Psi]}$ and a dissipation rate Q_Δ, such that the triple $(\Phi, [\Psi], \Delta)$ is a matched triple on $\mathfrak{B}$. One should note the unavoidable emergence of latent variables in the dissipation equation (17) for the N-D case. In the 1-D case, the dissipation equation can be written in terms of manifest variables alone, whereas in the N-D case, this is only possible if the latent variables that appear in the dissipation equation are observable.

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