$\mathcal{H}_\infty$ filtering in a behavioral framework

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1 Introduction

In this paper we treat the problem of $\mathcal{H}_\infty$ filtering. This problem has been studied in the context of $\mathcal{H}_\infty$ control for linear time invariant systems before. Without exception, earlier papers on this subject assume that the plant whose signals we try to estimate in the presence of disturbances, is given by equations in the usual state space format.

One of the basic philosophies of the *behavioral approach* is, that in the analysis and synthesis of control systems, one should *not* consider the system to be identical to the set of equations by which it happens to be given. Instead, one should identify the system with the *set of all possible time-trajectories* that are compatible with these equations. This set of trajectories is called the *behavior* of the system. The idea is, of course, that the set of equations describing a system is not unique, so that there is an obvious arbitrariness in the choice of representation. Our point of view adcocates that in a theory of analysis and synthesis, one should try, as much as possible, to work with the behavior of the system, and not with one of its particular representations. Obviously, results obtained by applying this point of view will be more general than those obtained for just one particular representation. In fact, one big advantage of the behavioral approach is, that the results will apply to *any* representation in which a certain plant happens to be given.

In the present paper, we will illustrate this point of view by setting up a theory of $\mathcal{H}_\infty$ filtering in a behavioral framework. We will arrive at necessary and sufficient conditions for the existence of a filter. In line with the basic philosophy explained in the previous paragraph, these conditions will not be in terms of a particular representation of the plant, but in terms of properties of its behavior. An important role will be played by two-variable polynomial matrices, qudratic differential forms and the concept of dissipative system. It will be shown that the existence of an $\mathcal{H}_\infty$ filter is equivalent to certain dissipativity properties of the system. Furthermore, we formulate a theorem which states that under certain assumptions on the plant, the filter can be implemented as an input/output processor with a proper transfer matrix.

The outline of this paper is as follows. In section 2 we give a brief review of linear differential systems, which is the class of systems that we deal with in this paper. In section 3 we review some material on quadratic differential forms, two-variable polynomial matrices, and dissipative systems. In section 4, we formulate the $\mathcal{H}_\infty$ filtering problem and give necessary and sufficient conditions for the existence of a filter. Furthermore, we formulate a theorem which states that under certain assumptions on the plant, the filter can be implemented as an input/output processor with a proper transfer matrix. Due to space limitations, we have omitted the proofs. For these we refer to [11].

Some words on notation. We use the standard notation $\mathbb{R}^n, \mathbb{R}^{n_1 \times n_2}$, etc., for finite-dimensional vectors and matrices. When the dimension is not specified (but, of course, finite), we write $\mathbb{R}^\bullet, \mathbb{R}^{n\times\bullet}, \mathbb{R}^{\bullet\times\bullet}$, etc. The space of infinitely differentiable functions with domain $\mathbb{R}$ and co-domain $\mathbb{R}^n$ is denoted by $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^n)$, and its subspace consisting of the elements with compact support by $\mathfrak{D}(\mathbb{R},\mathbb{R}^n)$. In order to avoid convergence issues, we frequently restrict attention to compact support elements of a behavior. For this reason, we introduce the notation $\mathfrak{B}\cap\mathfrak{D} = \mathfrak{B}\cap\mathfrak{D}(\mathbb{R},\mathbb{R}^\bullet)$.

2 Linear Differential Systems

Dynamical systems from a behavioral point of view were extensively discussed before in [4, 8, 9]. Here we review only the most basic concepts. A subset $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$ (called a *behavior*) is said to be a linear time-invariant differential system (briefly, a *differential* system) if there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^\bullet) \mid R(\frac{d}{dt})w = 0\}$. By $\mathfrak{L}^\bullet$ we denote the set of linear time-invariant differential systems, and by $\mathfrak{L}^{\mathtt{w}}$ those with $\mathtt{w}$ real variables (in other words, with behaviors $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$). This class of systems is a very general one, with nice mathematical structure. It includes finite-dimensional linear state systems, systems described by rational transfer functions, systems described by (high order) linear differential equations, etc.. Important is to note that while we *define* $\mathfrak{B} \in \mathfrak{L}^\bullet$ as the kernel of a differential operator, by $R(\frac{d}{dt})w = 0$, in actual applications, $\mathfrak{B}$ is often *not specified* in this way. We call $\mathfrak{B} \in \mathfrak{L}^\bullet$ *controllable* if for all $w_1, w_2 \in \mathfrak{B}$, there exists a $T \geq 0$ and a $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t+T) = w_2(t)$ for $t \geq 0$. By $\mathfrak{L}^\bullet_{\mathrm{cont}}, \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, we denote the controllable elements of $\mathfrak{L}^\bullet, \mathfrak{L}^{\mathtt{w}}$.

3 Quadratic Differential Forms and Dissipative Systems

In this section, we briefly review quadratic differential forms and dissipativity. For detailed treatments, we refer to [10, 5, 6]. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be a real polynomial matrix in the indeterminates ζ and η, i.e., an expression of the form

$$\Phi(\zeta,\eta) = \sum_{k,j} \Phi_{k,j}\zeta^k\eta^j \qquad (1)$$

where $\Phi_{k,j} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. The sum in (1) is a finite one, and k, $j \in \mathbb{N}$. Each $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induces a *quadratic differential form* (QDF), i.e., a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{k,j} (\frac{d^k w}{dt^k})^T \Phi_{k,j} (\frac{d^j w}{dt^j})$$

Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. The system $\mathfrak{B}$ is said to be *dissipative with respect to* Q_Φ, (briefly, Φ–dissipative) if $\int_{-\infty}^{+\infty} Q_\Phi(w)\,dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. It is said to be *dissipative on* $\mathbb{R}_-$ *with respect to* Q_Φ, (briefly, Φ– dissipative on $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Phi(w)\,dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$; dissipativity on $\mathbb{R}_+$ is analogously defined. It is easy to see that dissipativity on $\mathbb{R}_-$ or $\mathbb{R}_+$ implies dissipativity.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Then Q_Ψ is said to be a *storage function for* $\mathfrak{B}$ *with respect to the supply rate* Q_Φ if the *dissipation inequality*

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w)$$

holds for all $w \in \mathfrak{B}$.

4 The $\mathcal{H}_\infty$ Filtering problem

We consider the signal processing problem depicted in figure 1. In this set-up, the *plant* relates 3 types of variables:

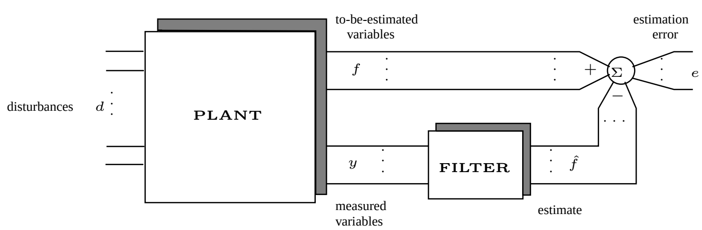

Figure 1: Plant and filter configuration

disturbances d, *to-be-estimated variables* f, and *measured variables* y. The problem is to design a *filter* that connects the measured variables y to the *estimate* $\hat{f}$, such that in the interconnected system the *estimation error* $e = f - \hat{f}$ is small in an appropriate sense. Denote the number of components of d by $\mathtt{d}$, of f (and hence $\hat{f}$ and e) by $\mathtt{f}$, and of y by $\mathtt{y}$. The variables of interest whose relationship we are trying to shape by means of a filter are d and e.

Define the *full plant behavior,* $\mathcal{P}_{\text{full}}$, to be the signals (d,f,y) that the plant allows, the *manifest plant behavior,* $\mathcal{P}$, to be the signals (d,f) that the plant allows, hence with the measured variables y eliminated, and the *hidden behavior,* $\mathcal{N}$, to be those signals (d,f)'s that are compatible with the plant equations *and* with the measured variables equal to zero. Define further $\mathcal{D}$, the *disturbance behavior,* to be the signals d that are possible, whence with f and y eliminated from $\mathcal{P}_{\text{full}}$. The formal definition of these behaviors is hence

$$\begin{aligned}
\mathcal{P}_{\text{full}} &= \{(d,f,y) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{f}+\mathtt{y}}) \mid (d,f,y) \text{ satisfies} \\
&\quad \text{the plant equations}\}, \\
\mathcal{P} &= \{(d,f) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{f}}) \mid \exists\, y \text{ such that} \\
&\quad (d,f,y) \in \mathcal{P}_{\text{full}}\}, \\
\mathcal{N} &= \{(d,f) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{f}}) \mid (d,f,0) \in \mathcal{P}_{\text{full}}\}, \\
\mathcal{D} &= \{d \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}}) \mid \exists\, (f,y) \text{ such that} \\
&\quad (d,f,y) \in \mathcal{P}_{\text{full}}\}.
\end{aligned}$$

We assume throughout this section that in $\mathcal{P}_{\text{full}}$ d is free, i.e., $\mathcal{D} = \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}})$.

A *filter* is a dynamical system that relates the measured variables y to the estimate $\hat{f}$ of f. The filter imposes a relation on the variables $(y,\hat{f})$. We take this to mean $(y,\hat{f}) \in \mathcal{F}$, with $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ the behavior of the filter. Before the filter acts, the variables $d,f,y,\hat{f}$ and e are constrained to satisfy $(d,f,y) \in \mathcal{P}_{\text{full}}$ and $e = f - \hat{f}$. However, with the filter in action, they have to obey also $(y,\hat{f}) \in \mathcal{F}$. This yields the manifest behavior $\mathcal{E}$ of the variables (d,e) in the interconnected system shown in figure 1, formally defined as

$$\begin{aligned}
\mathcal{E} &= \{(d,e) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{d}+\mathtt{f}}) \mid \exists\, (f,y,\hat{f}) \\
&\quad \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{f}+\mathtt{y}+\mathtt{f}}) \text{ such that} \\
&\quad (d,f,y) \in \mathcal{P}_{\text{full}}, (y,\hat{f}) \in \mathcal{F}, e = f - \hat{f}\}.
\end{aligned}$$

The behavior $\mathcal{E}$ is called *the estimation error behavior*. Obviously, by the elimination theorem ([4], theorem 6.2.6), $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$. If, for a given element $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$, there exists $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ such that the above relation holds, then we say that the filter $\mathcal{F}$ *implements* $\mathcal{E}$. The question what $\mathcal{E}$'s are implementable is answered in the following theorem.

Theorem 1 (Filter implementability theorem) : *The behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}$ is implementable by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}$ if and only if $\mathcal{N} \subset \mathcal{E}$. Moreover, if $\mathcal{E}$ is implementable, then it can be implemented by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{y}+\mathtt{f}}_{\text{cont}}$ such that in $\mathcal{F}$, y is input and $\hat{f}$ output.*

The problem that we consider is to find a filter that renders the estimation error behavior dissipative with respect to a QDF Q_Φ in the variables (d,e). We consider only the case of $\mathcal{H}_\infty$–filtering, which corresponds to choosing Φ to be the constant two-variable polynomial matrix $\Phi(\zeta,\eta) = \Sigma$, where Σ is defined to be the signature matrix $\text{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$. The following theorem shows when such a filter exists.

Theorem 2 ($\mathcal{H}_\infty$-filtering) : *Assume that $\mathcal{N} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\text{cont}}$. Then there exists $\mathcal{E} \in \mathfrak{L}^{\mathtt{d}+\mathtt{f}}_{\text{cont}}$ such that:*

1. $\mathcal{N} \subset \mathcal{E}$ (implementability),

2. the disturbances d *remain free in* $\mathcal{E}$ (liveness),

3. $(d, e) \in \mathcal{E} \cap \mathfrak{D}$ *implies* $||e||_{\mathcal{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{f}})} \leq ||d||_{\mathcal{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{d}})}$ (disturbance attenuation),

4. $(d, e) \in \mathcal{E}$ *and* $d(t) = 0$ *for* $t \geq 0$ *implies* $e(t) \to \infty$ (stability),

if and only if $\mathcal{N}$ *is* Σ*–dissipative on* $\mathbb{R}_-$*, with* $\Sigma = \operatorname{diag}(I_{\mathtt{d}}, -I_{\mathtt{f}})$*, equivalently, if and only if there exists a two-variable polynomial matrix* $\Psi_{\mathcal{N}} \in \mathbb{R}^{(\mathtt{d}+\mathtt{f})\times(\mathtt{d}+\mathtt{f})}[\zeta, \eta]$*, such that* $Q_{\Psi_{\mathcal{N}}}(d, e) \geq 0$ *and* $\frac{d}{dt} Q_{\Psi_{\mathcal{N}}}(d, e) \leq |d|^2 - |e|^2$ *for* $(d, e) \in \mathcal{N}$*, i.e.,* $\mathcal{N}$ *is a* Σ*-dissipative system and has at least one non-negative storage function.*

Let us explain the meaning of these conditions. The idea is that before the filter acts, the variables (d, e) are free: for e, this is trivially so, and for d, it holds by assumption. With the filter put into place, as shown in figure 1, the variables (d, e) are constrained to belong to $\mathcal{E}$. The first condition is thus merely the implementability condition of theorem 1. The second condition states that the filter is not allowed to restrict the free exogenous disturbances d: the interconnected systems should still be allowed to accept arbitrary d's. The third condition expresses disturbance attenuation: for all $(d, e) \in \mathcal{E} \cap \mathfrak{D}$ there should hold $\int_{-\infty}^{+\infty} |e|^2\, dt \leq \int_{-\infty}^{+\infty} |d|^2\, dt$. The fourth condition states that without the disturbances acting, the estimation error must go to zero. Actually conditions 3 and 4 combined are equivalent to Σ–dissipativity of $\mathcal{E}$ on $\mathbb{R}_-$: for all $(d, e) \in \mathcal{E} \cap \mathfrak{D}$ there should hold $\int_{-\infty}^{0} |e|^2\, dt \leq \int_{-\infty}^{0} |d|^2\, dt$. The theorem states that dissipativity of $\mathcal{N}$ on $\mathbb{R}_-$, an obvious necessary condition (since $\mathcal{N} \subset \mathcal{E}$), is also sufficient.

Assume that the dissipativity requirement of theorem 2 is satisfied. Theorem 1 then implies that there exists a filter which implements $\mathcal{E}$, and has y as input and $\hat{f}$ as output. In other words, the filter can be viewed as a signal processor that accepts any input signal $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{y}})$ and produces as output the estimate $\hat{f}$ of f. There is no a priori reason, of course, for the transfer function of this signal processor from y to $\hat{f}$ to be proper, since singular filtering is very much part of our set-up. However, properness may be obtained by imposing some additional structure on the plant. In the next theorem, we assume that in the plant d is input and y and f are output. Denote the transfer functions from d to (f, y) in $\mathcal{P}_{\text{full}}$ by $G_{d \mapsto f}$ and $G_{d \mapsto y}$, respectively.

Theorem 3 : *Assume that in the plant* d *is input and* y *and* f *output, with the transfer functions* $G_{d\mapsto f}$ *and* $G_{d\mapsto y}$ *proper. Assume further that* $G_{d\mapsto y}^{\infty} := \lim_{s\to\infty} G_{d\mapsto y}(s)$*, the feedthrough term of* $G_{d\mapsto y}$*, is surjective. Then the estimation error behavior* $\mathcal{E}$ *of theorem 2 (assuming it exists) can be implemented by a filter* $\mathcal{F} \in \mathfrak{L}_{\text{cont}}^{\mathtt{y}+\mathtt{f}}$ *such that in* $\mathcal{F}$*,* y *is input and* $\hat{f}$ *output, and* $G_{y\mapsto \hat{f}}$*, the transfer function of the* $\mathcal{H}_\infty$*–filter, is also proper.*

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