An LMI condition for nonnegativity of a Quadratic Differential Form along a Behavior

T. Cotroneo and J.C. Willems
Mathematics Institute, University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands.
T.Cotroneo@math.rug.nl,J.C.Willems@math.rug.nl

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1 Introduction

In this paper we consider linear behaviors, specified as solution sets of systems of constant coefficient linear differential equations of arbitrary order; such systems are parameterized in a natural way by polynomial matrices in one indeterminate.

A very natural way of specifying functionals on such systems, is by means of quadratic forms involving the system variables and their derivatives; such functionals, also called quadratic differential forms (QDF for short), are parameterized by polynomial matrices in two indeterminates.

When using quadratic forms as a means of establishing properties of linear systems it is often crucial to be able to determine whether a given form is non-negative when computed along all trajectories of a given system. After formally defining what has to be meant by non-negativity of a QDF we proceed to show how such a property is equivalent to feasibility of a suitable LMI which can be built starting from the problem data (i.e. from the coefficients of the differential equations specifying the behavior and those of the QDF). Such a result is of great practical relevance because feasibility of LMI's can be checked by means of standard software packages such as MATLAB LMI Toolbox.

A situation in which the possibility of checking non-negativity of a QDF along trajectories in a behavior is of great relevance is when one wants to assess stability of a system. We discuss this situation in detail and show how our results generalize the well known ones for state space systems.

2 Linear differential systems

In the behavioral approach to system theory a *dynamical system* is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ where $\mathbb{T} \subseteq \mathbb{R}$ is the *time set* over which the system evolves (e.g. it will typically be $\mathbb{R}$ or $\mathbb{R}_+$ for continuous-time systems , and $\mathbb{Z}$ or $\mathbb{Z}_+$ for discrete-time), $\mathbb{W}$ is the *signal space* in which the variables of the system we are modeling take on their values and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the *behavior* of the system. The set $\mathbb{W}^{\mathbb{T}}$ consists of all possible maps from $\mathbb{T}$ to $\mathbb{W}$, and the trajectories belonging to $\mathfrak{B}$ are nothing but the subset of these which comply with the laws of the system.

In the rest of this paper we concentrate on a specific class of dynamical systems, namely *linear time-invariant differential system*. This class corresponds to systems for which the time axis is $\mathbb{R}$, the signal space is $\mathbb{R}^q$ for some q, and the behavior is specified as the set of solutions to a system of linear constant coefficients differential equations of the form:

$$R_0 w + R_1 \frac{d}{dt} w + \cdots R_L \frac{d^L}{dt^L} w = 0$$

with $R_i \in \mathbb{R}^{p \times q}$ for some p. To these equations we associate in a natural way the polynomial matrix
$R = R_0 + R_1\xi + \cdots R_L\xi^L \in \mathbb{R}^{p\times q}[\xi]$ and also write the system as $R(\frac{d}{dt})w = 0$. In this paper we restrict our attention to $\mathfrak{C}^\infty$ solutions to the above equations, therefore we take

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) | R(\frac{d}{dt})w = 0\}$$

For obvious reasons we also denote the above as a *kernel representation* of the behavior and write $\mathfrak{B} = \ker(R(\frac{d}{dt}))$.

A behavior $\mathfrak{B}$ is said to be *autonomous* if

$$(w_1, w_2 \in \mathfrak{B}), \text{ and } (w_1(t) = w_2(t) \text{ for } t < 0) \Rightarrow (w_1 = w_2)$$

in other words if the future of any system trajectory is uniquely specified by its past.

A very important subclass of autonomous behaviors is represented by *stable* behaviors, namely behaviors whose trajectories are bounded on the half-line $[0, \infty)$. *Asymptotically stable* behaviors, more in particular, are those for which $(w\ in\mathfrak{B}) \Rightarrow (\lim_{t\to\infty} w(t) = 0)$

3 System representations

Given a polynomial matrix $R \in \mathbb{R}^{p\times q}[\xi]$ the *module* spanned by its rows is denoted by $< R >$ and defined as

$$< R >= \{v \in \mathbb{R}^{1\times q}[\xi] \mid \exists v \in \mathbb{R}^{1\times p}[\xi] \text{ such that } v = pR\}$$

In other words $< R >$ is the set of all possible combinations with polynomial coefficients of the rows of R. It is not difficult to see that the same module is generated by different matrices, in other words that there exists $R' \in \mathbb{R}^{p'\times q}[\xi]$ such that $< R >=< R' >$.

The interesting thing from our point of view is that it can be shown that

$$\mathfrak{B} = \ker(R(\frac{d}{dt})) = \ker(R'(\frac{d}{dt})) \Leftrightarrow < R >=< R' >$$

In other words a same behavior admits many different kernel representations, but is associated to one and only one module, namely that spanned by the rows of one, and therefore all, of its possible kernel representations.

It can be shown that among all such representations, we can always find some corresponding to matrices $R \in \mathbb{R}^{p\times q}[\xi]$ which are of *full row rank* over the polynomial ring $\mathbb{R}[\xi]$ (meaning that R has a non-singular $p \times p$ minor). In this case we also talk of a *minimal* kernel representation of $\mathfrak{B}$, because any other $R' \in \mathbb{R}^{p'\times q}[\xi]$ such that $\mathfrak{B} = \ker(R(\frac{d}{dt})) = \ker(R'(\frac{d}{dt})$ will be such that $p' \geq p$; in other words, minimal representations are defined by the property of containing as few equations as possible among all kernel representations of the same behavior $\mathfrak{B}$.

In case the behavior $\mathfrak{B}$ we are considering is autonomous, its minimal representations correspond to square (therefore also non-singular) matrices R; in other words $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ with $R \in \mathbb{R}^{q\times q}[\xi]$, $\det(R) \neq 0$. Asymptotically stable behaviors, instead, admit minimal representations with R square, non-singular and $\det(R)$ Hurwitz, meaning that all its zeroes lie in the open left half plane. In order to be asymptotically stable a behavior must therefore be autonomous.

Among minimal kernel representations, one which will turn out to be useful in the next section is the one corresponding to a row proper matrix R which we now define. Given $R \in \mathbb{R}^{p\times q}[\xi]$ we define its highest row coefficient matrix R_{hc} as the real matrix whose $i-$th row contains the coefficients of the highest power of ξ in the $i-$th row of R. R is defined to be *row proper* if R_{hc} is a full row rank matrix. If R is not row proper, its *row proper form* is defined as any matrix R' such that $< R >=< R' >$ and such that R' is row proper. Of course R' is not uniquely defined, but any row proper form can be obtained from another by taking linear combinations of the rows. Standard algorithms to build the row proper form of a given matrix are described, for example, in [2] and implemented in the function `prowred` from the Matlab Polynomial Toolbox.

4 Initial conditions for a behavior

Associated to any behavior $\mathfrak{B}$ we now define the set:

$$K^N_{\mathfrak{B}} = \left\{ k = \begin{pmatrix} k_0 \\ \vdots \\ k_N \end{pmatrix} \mid \exists w \in \mathfrak{B} \ : \ \begin{pmatrix} w(0) \\ \vdots \\ w^{(N)}(0) \end{pmatrix} = k \right\}$$

In other words $K^N_{\mathfrak{B}}$ represents all possible values that w and its derivatives up to order N can assume at time 0 given that w must belong to the given behavior. It is easily seen that in case $\mathfrak{B}$ is a linear behavior, then $K^N_{\mathfrak{B}}$ is a vector space over $\mathbb{R}$; in particular if $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ with $R \in \mathbb{R}^{p\times q}[\xi]$ then $K^N_{\mathfrak{B}}$ is a linear subspace of $\mathbb{R}^{Nq}$. For this particular case we now want to show how to build a real matrix $\tilde{R}_N$ such that $K^N_{\mathfrak{B}} = \ker(\tilde{R}_N)$.

Before doing so, we define the *coefficient matrix* $\tilde{R}$ associated to any polynomial matrix $R = R_0 + R_1\xi + \cdots R_L\xi^L \in \mathbb{R}^{p\times q}[\xi]$ as the real matrix $\tilde{R} = [R_0 \ R_1 \ \cdots R_L]$. One then has $R = \tilde{R} \begin{bmatrix} I_q \\ I_q\xi \\ \vdots \\ I_q\xi^L \end{bmatrix}$ with I_q the $q \times q$ identity matrix.

Notice now that $K^N_{\mathfrak{B}}$ is the set of all $k \in \mathbb{R}^{Nq}$ for which the system of differential equations with initial conditions

$$\begin{cases} R(\frac{d}{dt})w = 0 \\ S(\frac{d}{dt})w(0) = k \end{cases}$$

with $S = \begin{bmatrix} I_q \\ I_q\xi \\ \vdots \\ I_q\xi^N \end{bmatrix}$ admits a solution w. It then follows from a result shown in [3] that such a system has a solution if and only if

$$h \in \mathbb{R}^{Nq}, \ \ h^T S \in < R > \Rightarrow h^T k = 0.$$

in other words if and only if whenever a linear combination of the initial conditions is a consequence of the system equations ($h^T S \in < R >$), then such a combination is equal to 0.

Therefore, if we define the real vector space

$$\tilde{H}_N = \{ h \in \mathbb{R}^{Nq} \mid h^T S \in < R > \}$$

then $K^N_{\mathfrak{B}} = \tilde{H}^\perp_N$.

We can now define the set H_N of all differential operators of order up to N which are zero along all trajectories in the behavior; such a set is in fact given by

$$H_N = \{ h \in \mathbb{R}^{1\times q}[\xi] \mid h \in < R > \text{ and } \deg(h) \leq N \}$$

It is easily seen that this set is also a finite dimensional real vector space and that, in fact, any vector in $\tilde{H}_N$ is the coefficient vector of a polynomial vector in H_N. If we can find a polynomial matrix R_N such that its rows are a basis for H_N as a vector space over $\mathbb{R}$, then the rows of its coefficient matrix $\tilde{R}_N$ will be a basis for $\tilde{H}_N$ and $K^N_{\mathfrak{B}} = \ker(\tilde{R}_N)$.

Assume now R is in row proper form; as discussed in the previous section we can always come back to this situation without altering the $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ of interest. Let σ be the lowest degree of a row of R and $t = \max\{0, N - \sigma\}$. Because of the row proper property of R it is not difficoult to prove that R_N can be obtained by first building $\begin{pmatrix} R \\ \xi R \\ \vdots \\ \xi^t R \end{pmatrix}$

and then only considering the rows whose degree is smaller or equal to N. In other words $R_N = U \begin{pmatrix} R \\ \xi R \\ \vdots \\ \xi^t R \end{pmatrix}$ with U a real matrix that selects the rows with degree smaller or equal to N.

Example 1 : Let $R = \begin{bmatrix} \xi+1 & \xi+2 \\ \xi^3+\xi & \xi^3+1 \end{bmatrix}$. One row proper form of R is given by $R' = \begin{bmatrix} \xi+1 & \xi+2 \\ -\xi^2+\xi & -2\xi^2+1 \end{bmatrix}$ from which $R_2 = \begin{bmatrix} \xi+1 & \xi+2 \\ \xi^2+\xi & \xi^2+2\xi \\ -\xi^2+\xi & -2\xi^2+1 \end{bmatrix}$.

5 Quadratic differential forms

Quadratic differential forms, QDF's for short, are a very natural way of specifying functionals of a system's variables. A complete theory of QDF's has been developed in [4], in this section we simply recall some basic concepts and notation we shall need in the following.

Let $\mathbb{R}^{q\times q}[\zeta,\eta]$ denote the set of real polynomial matrices in the two variables ζ and η; an element $\Phi \in \mathbb{R}^{q\times q}[\zeta,\eta]$ will therefore be given by

$$\Phi(\zeta,\eta) = \sum_{k,\ell} \Phi_{kl}\zeta^k\eta^\ell$$

where $\Phi_{kl} \in \mathbb{R}^{q\times q}$ and it is assumed that only a finite number of terms in the above sum are different from zero.

To such a matrix we associate in a natural way a *quadratic differential form* $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ defined as:

$$Q_\Phi(w) = \sum_{k,\ell} \left(\frac{d^k w}{dt^k}\right)^T \Phi_{kl} \left(\frac{d^\ell w}{dt^\ell}\right)$$

In the following, without loss of generality, we will deal with QDF's defined by *symmetric* two variable polynomial matrices, meaning matrices Φ for which $\Phi(\zeta,\eta) = \Phi^T(\eta,\zeta)$. To such matrices we associate the coefficient matrix $\tilde{\Phi}$ defined as

$$\tilde{\Phi} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots \\ \vdots & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \Phi kl & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

Such a matrix is an infinite symmetric matrix with only a finite number of blocks not equal to zero. If we denote by N the index of the last non zero block, we can then define the finite coefficient matrix

$$\tilde{\Phi}_N = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \Phi_{0N} \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \Phi_{1N} \\ \vdots & \vdots & & \vdots & \vdots \\ \Phi_{N0} & \Phi_{N1} & \cdots & \cdot & \Phi_{NN} \end{bmatrix}$$

Sometimes we refer to N as the *degree* of the QDF and write $N = \deg(\Phi)$.

Notice that we can define a polynomial matrix P with an infinite number of columns given by $P(\xi) = [I_q\ I_q\xi\ I_q\xi^2\ \cdots]$ and recover $\Phi(\zeta,\eta) = P(\zeta)\tilde{\Phi}P^T(\eta)$; also we can define the finite matrix $P_N(\xi) = [I_q\ I_q\xi\ I_q\xi^2\ I_q\xi^N]$ and recover $\Phi(\zeta,\eta) = P_N(\zeta)\tilde{\Phi}_N P_N^T(\eta)$.

By taking a decomposition $\tilde{\Phi}_N = \tilde{M}^T\Sigma\tilde{M}$ with $\tilde{M}$ surjective and $\Sigma = \begin{bmatrix} I_{r^+} & 0 \\ 0 & -I_{r^-} \end{bmatrix}$ a signature matrix, we see that any $\Phi(\zeta,\eta)$ can be written as $\Phi(\zeta,\eta) = M^T(\zeta)\Sigma M(\eta)$. If R is a non-singular polynomial matrix and MR^{-1} is a matrix of strictly proper rational functions, we say that Φ is $R-$canonical; such a concept will turn out to be useful when discussing stability issues. Notice how a bound on the degree of an $R-$ canonical QDF is immediately available as a consequence of its definition; we have in fact $\deg(\Phi) \leq \deg(R) - 1$ (see [2], Lemma 6.3-10).

Given Q_Φ its derivative will of course still be a quadratic differential form; in other words $\frac{d}{dt}Q_\Phi = Q_\Psi$ for some $\Psi(\zeta,\eta)$; actually it turns out that $\Psi(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$. At the level of coefficient matrices, if one defines

$$\sigma_l\tilde{\Phi} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots \\ \Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots \\ \vdots & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \Phi kl & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

and

$$\sigma_r\tilde{\Phi} = \begin{bmatrix} 0 & \Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots \\ 0 & \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots \\ 0 & \vdots & \vdots & & \vdots & \\ 0 & \cdot & \cdot & \cdots & \Phi kl & \cdots \\ 0 & \vdots & \vdots & & \vdots & \end{bmatrix}$$

then one finds

$$\tilde{\Psi} = \sigma_l\tilde{\Phi} + \sigma_r\tilde{\Phi}$$

Notice how $\deg\Psi = \deg\Phi + 1$; the finite versions $\sigma_l\tilde{\Phi}_N$ and $\sigma_r\tilde{\Phi}_N$ are of course accoridingly defined.

We can now define the main concept we intend to investigate in this paper, namely that of nonnegativity of a QDF.

Definition 2 : A QDF Q_Φ is called *nonnegative* if $Q_\Phi(w) \geq 0\ \forall w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$; if $\mathfrak{B}$ is any behavior Q_Φ is called *nonnegative along* $\mathfrak{B}$ if $Q_\Phi(w) \geq 0\ \forall w \in \mathfrak{B}$.

We also use the notation $\Phi \geq 0$ for nonnegative QDF's and $\Phi \overset{\mathfrak{B}}{\geq} 0$ for QDF's which are nonnegative along $\mathfrak{B}$.

We can now define *positive* QDF's, which are defined as being nonnegative, and moreover being identically zero only when evaluated along the zero trajectory; formally

Definition 3 : A QDF Q_Φ is called *positive* if $Q_\Phi(w) \geq 0 \; \forall w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and $Q_\Phi(w) = 0 \Leftrightarrow w = 0$. If $\mathfrak{B}$ is any behavior Q_Φ is called *positive along* $\mathfrak{B}$ if $Q_\Phi(w) \geq 0 \; \forall w \in \mathfrak{B}$ and $Q_\Phi(w) = 0 \Leftrightarrow w = 0$.

Notice the difference between nonnegative and positive forms; both can become zero at one point, but while nonnegative ones can also be identically zero along trajectories different from zero, this is not possible for positive ones.

In the case of autonomous behaviors, one can also suitably define the stronger concept of strong positivity.

Definition 4 : Let $\mathfrak{B}$ be an autonomous behavior. A QDF Q_Φ is called *strongly positive along* $\mathfrak{B}$ (indicated by $\Phi \overset{\mathfrak{B}}{>>} 0$) if

1. $\Phi \overset{\mathfrak{B}}{\geq} 0$
2. $(w \in \mathfrak{B}, \;\; Q_\Phi(w)(0) = 0) \Rightarrow (w = 0)$

In other words, a strongly positive QDF can become 0 at one point only if evaluated along the zero trajectory (in which case it is identically equal to zero).

As shown in [4] the concepts defined above are crucial in defining concepts such as stability and dissipativity for systems described by high order differential equations like the ones we are interested into.

6 Testing nonnegativity of a QDF

We now come to the main issue of this paper, namely introducing an LMI test which allows to check whether for a given Φ and a given $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ there holds $\Phi \overset{\mathfrak{B}}{\geq} 0$ or $\Phi \overset{\mathfrak{B}}{>>} 0$.

A crucial remark at this point is that if $\mathfrak{B}$ is a linear time invariant behavior and Φ a QDF corresponding to a finite coefficient matrix

$$\tilde{\Phi}_N = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \Phi_{0N} \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \Phi_{1N} \\ \vdots & \vdots & & \vdots & \vdots \\ \Phi_{N0} & \Phi_{N1} & \cdots & \cdot & \Phi_{NN} \end{bmatrix}$$

then

$$\Phi \overset{\mathfrak{B}}{\geq} 0 \Leftrightarrow \tilde{\Phi}_N \geq 0 \text{ on } K_{\mathfrak{B}}^N$$

where the $\geq$ sign on th right has to be intended in the usual sense of positive definite matrices and $K_{\mathfrak{B}}^N$ is defined as in section 2. Thus the problem of studying nonnegativity of a QDF along a behavior has been reduced to that of studying nonnegativity of a real symmetric matrix on a given linear subspace.

In section 3 it has been shown how to build a real matrix $\tilde{R}_N$ such that $K_{\mathfrak{B}}^N = \ker(\tilde{R}_N)$. ¿From easy linear algebra argument it then follows that $\tilde{\Phi}_N \geq 0$ on $K_{\mathfrak{B}}^N \Leftrightarrow \exists M$ such that

$$\tilde{\Phi}_N + M^T \tilde{R}_N + \tilde{R}_N^T M \geq 0 \qquad (1)$$

The above is an LMI in the unknown M whose feasibility is equivalent to nonnegativity of Q_Φ along $\mathfrak{B}$; it is therefore exactly the kind of condition we had been looking for.

The parameters $\tilde{\Phi}_N$ and $\tilde{R}_N$ can be built starting from Φ and R as discussed in sections 2,3, and 4; the existence of a solution M to the above LMI can be checked with the command `feasp` from the Matlab LMI Toolbox.

In the special that $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, equivalently that $\mathfrak{B}$ is the kernel of the 0 matrix, LMI (1) returns the known fact

$$\Phi \geq 0 \Leftrightarrow \tilde{\Phi}_N \geq 0$$

If we now indicate by $\tilde{\Psi}$ the left hand side of LMI 1, we can then regard it as the coefficient matrix of a $\Psi(\zeta, \eta)$. By what just said, if LMI 1 is feasible, then $\Psi \geq 0$, in other words nonnegativity of a QDF along trajectories in a behavior ($\Phi \overset{\mathfrak{B}}{\geq} 0$) is equivalent to nonnegativity over all of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ of another suitably defined QDF ($\Psi \geq 0$).

We now wish to investigate a little bit further the structure of such a $\Psi(\zeta, \eta)$. Premultiplying the left hand side of LMI (1) by matrix $P_N(\zeta)$ and postmultiply it by $P_N^T(\eta)$ as shown in section 4, we obtain

$$\Psi(\zeta, \eta) = \Phi(\zeta, \eta) + G^T(\zeta) R_N(\eta) + R_N^T(\zeta) G(\eta)$$

with $G(\eta) = MP(\eta)$. As shown in section 3, $R_N(\xi) = V(\xi) R_r(\xi)$ with R_r a row proper form of the original matrix R we start with, and V a suitable polynomial matrix corresponding to the construction discussed in section 3. Moreover there exists a matrix T such that $R_r = TR$, finally yielding

$$\Psi(\zeta, \eta) = \Phi(\zeta, \eta) + F^T(\eta, \zeta) R(\eta) + R^T(\zeta) F(\zeta, \eta)$$

with $F(\zeta, \eta) = T^T(\zeta) V^T(\zeta) G(\eta)$.

Assuming $\mathfrak{B}$ is an autonomous behavior, we can now use arguments similar to the ones shown above to obtain a necessary condition for strong positivity of a QDF Q_Φ along $\mathfrak{B}$. It can in fact be seen that

$$\Phi \overset{\mathfrak{B}}{>>} 0 \Rightarrow \tilde{\Phi}_N > 0 \text{ on } K_{\mathfrak{B}}^N$$

and therefore that $\Phi \overset{\mathfrak{B}}{>>} 0$ only if there exists an M such that

$$\tilde{\Phi}_N + M^T \tilde{R}_N + \tilde{R}_N^T M > 0 \qquad (2)$$

Notice how LMI (2) is nothing but the strict version of LMI (1); its feasibility, however, is only a necessary condition for strict positivity as the following example shows

Example 5 : Consider the autonomous behavior $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ with $R(\xi) = \xi^2 + \xi + 1$ and let $Q_\Phi(w) = w^2$. Then $\tilde{\Phi}_0 = 1$ and $K^0_{\mathfrak{B}} = \mathbb{R}$, therefore $\tilde{\Phi}_0 > 0$ on $K^0_{\mathfrak{B}}$, there are however non-zero trajectories in $\mathfrak{B}$ with $w(0) = 0$, showing that Q_Φ is not strongly positive along $\mathfrak{B}$.

7 Lyapunov Theory

We now wish to apply the results from the previous section to the problem of establishing asymptotic stability of a behavior. We have, in fact

Theorem 6 :
Consider $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ *with* $\deg(R) = N + 1$ *and let* $\tilde{\Phi}_N = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \Phi_{0N} \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \Phi_{1N} \\ \vdots & \vdots & & \vdots & \vdots \\ \Phi_{N0} & \Phi_{N1} & \cdots & \cdot & \Phi_{NN} \end{bmatrix}$. *Then* $\mathfrak{B}$ *is asymptotically stable if and only if the following system of LMI in the unknowns* $\tilde{\Phi}_N$, N, M *is feasible*

$$\begin{array}{l} \tilde{\Phi}_N + M^T \tilde{R}_N + \tilde{R}_N^T M > 0 \\ \sigma_l \tilde{\Phi}_N + \sigma_r \tilde{\Phi}_N + N^T \tilde{R}_{N+1} + \tilde{R}_{N+1}^T N \leq -I \end{array} \quad (3)$$

with I *the identity matrix of suitable dimensions*

Proof : $\Rightarrow$) If $\mathfrak{B}$ is asymptotically stable, then by theorem 4.12 of [4] we know that given any $\Psi \overset{\mathfrak{B}}{>} 0$ there exists an $R-$ canonical Φ which is strongly positive along $\mathfrak{B}$ and such that its derivative is smaller or equal than Ψ along $\mathfrak{B}$. Because Φ is $R-$ canonical, we know it can be taken of degree N, therefore corresponding to a coefficient matrix $\tilde{\Phi}_N$ as shown above. By taking $Q_\Psi(w) = w^2 + \cdots + \left(w^{N+1}\right)^2$ we then obtain that the system of LMI (3) must be feasible, the first equation corresponding to $\Phi \overset{\mathfrak{B}}{>>} 0$, the second one to $\Phi - \Psi \overset{\mathfrak{B}}{<} 0$
$\Leftarrow$) If the first LMI is feasible it means we can find a $\Phi \overset{\mathfrak{B}}{>} 0$; if the second one is feasible it means that the derivative of Q_Φ is, lanog $\mathfrak{B}$ smaller or equal to $w^2 + \cdots + (w^{N+1})^2$, therefore negative along $\mathfrak{B}$. By theorem 4.3 of [4] we can then conclude asymptotic stability of $\mathfrak{B}$. $\square$

With a little bit of computations it can be seen that, when looking at state space systems $\dot{x} = Ax$, corresponding in our notation to $R = \xi I - A$, the above theorem returns the fact that the systems is asymptotically stable if and only if a positive definite matrix P can be found such that $A^T P + PA < -I$.

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