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Dissipative distributed systems

Harish K. Pillai
ISIS Research Group, Department of Electronics and Computing Science,
University of Southampton,
SO17 1QP, Southampton, United Kingdom.
email: hp@ecs.soton.ac.uk
Jan C. Willems
Institute for Mathematics and Computing Science, University of Groningen,
P.O. Box 800, 9700 AV Groningen, The Netherlands.
email: J.C.Willems@math.rug.nl

Abstract

This paper deals with systems described by constant coefficient linear partial differential equations. We define dissipativity with respect to a quadratic differential form, i.e., a quadratic functional in the system variables and their partial derivatives. The main result states the equivalence of dissipativity and the existence of a storage function or of a dissipation rate.

Keywords: Distributed systems, partial differential equations, behaviors, supply, storage, dissipation, quadratic differential forms.

1 Introduction

One of the very useful concepts in systems theory is the notion of a dissipative system. The purpose of this paper is to develop the theory of dissipative systems for systems described by partial differential equations. The central problem in the theory of dissipative systems is the construction of an internal function called the storage function. As we shall see, results analogous to those for lumped systems may be obtained for distributed systems described by linear constant coefficient partial differential equations and with quadratic differential forms as supply rates. However, there are important differences in the resulting theory, the most important one being the fact that for distributed systems the storage functions is in general a function of unobservable ("hidden") latent variables.

A *1-D dynamical system* Σ is a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} \subset \mathbb{R}$ the time-set, $\mathbb{W}$ the signal space, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ the *behavior.* We consider here the generalization from a time-set that is a subset of $\mathbb{R}$ to domains with more independent variables (e.g., time and space). These 'dynamical' systems have $\mathbb{T} \subset \mathbb{R}^{\mathtt{n}}$, and are referred to as *n-D* systems. Define a *distributed differential system* as an *n-D* system $\Sigma = (\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, with behavior $\mathfrak{B}$ consisting of the solution set of a system of partial differential equations

$$R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = 0 \qquad (1)$$

viewed as an equation in the functions

$$(x_1, \ldots, x_n) = x \in \mathbb{R}^{\mathtt{n}} \mapsto$$
$$(w_1(x), \ldots, w_{\mathtt{w}}(x)) = w(x) \in \mathbb{R}^{\mathtt{w}}.$$

Here, $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_n]$ is a matrix of polynomials in $\mathbb{R}[\xi_1, \ldots, \xi_n]$. The behavior of this system of partial differential equations is defined as

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid (1) \text{ is satisfied}\}.$$

We denote the behavior of (1) as defined above by $\ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}))$, and the set of distributed differential systems thus obtained by $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. Note that we may as well write $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, instead of $\Sigma \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, since the set of independent variables ($\mathbb{R}^{\mathtt{n}}$) and the signal space ($\mathbb{R}^{\mathtt{w}}$) are evident from this notation. We call (1) a *kernel representation* of $\Sigma = (\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})))$. We will meet other representations later. A typical example of a distributed dynamical system is given by Maxwell's equations.

2 Elimination

Mathematical models often contain, in addition to the variables whose dynamic relation one wants to model (we call these *manifest* variables), auxiliary variables (we call these *latent* variables) that have been introduced in the modeling process. For distributed differential systems this leads to equations of the form

$$R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell, \qquad (2)$$

with R and M matrices of polynomials in $\mathbb{R}[\xi_1, \ldots, \xi_n]$. This equation relates the (vector of) manifest variables

w to the (vector of) latent variables ℓ. Define the *full behavior* of this system as

$$\mathfrak{B}_{\text{full}} = \{(w,\ell) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \times \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\dim(\ell)}) \mid (2) \text{ holds}\}$$

and the *manifest behavior* as

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mid \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\dim(\ell)}) \text{ such that (2) holds}\}$$

We call (2) a *latent variable* representation of $\mathfrak{B}$. The question occurs whether $\mathfrak{B}$ is in $\mathfrak{L}^w_n$. This is the case indeed.

Theorem 1 (Elimination theorem) : *For any real matrices of polynomials (R, M) in $\mathbb{R}[\xi_1, \xi_2, \cdots, \xi_n]$ with* $\text{rowdim}(R) = \text{rowdim}(M)$, *there exists a matrix of polynomials R' in $\mathbb{R}[\xi_1, \xi_2, \cdots, \xi_n]$ such that the manifest behavior of (2) has kernel representation* $R'(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$.

The above theorem implies that a distributed differential system $\mathfrak{B} \in \mathfrak{L}^w_n$ admits not only many kernel, but also many latent variable representations. Latent variable representations are very useful. Not only because first principles models usually come in this form, but also because latent variables routinely enter in representation questions. As we shall see in this paper, they allow to express conservation and dissipation laws in terms of local storage functions and dissipation rates.

3 Controllability and observability

An important property in the analysis and synthesis of dynamical systems is controllability. A system $\mathfrak{B} \in \mathfrak{L}^w_n$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ and for all bounded open subsets O_1, O_2 of $\mathbb{R}^n$ with disjoint closure, there exists $w \in \mathfrak{B}$ such that $w|_{O_1} = w_1|_{O_1}$ and $w|_{O_2} = w_2|_{O_2}$. We denote the set of controllable elements of $\mathfrak{L}^w_n$ by $\mathfrak{L}^w_{n,\text{cont}}$.

Note that it follows from the elimination theorem that the manifest behavior of a system in image representation, i.e., a latent variable system of the special form

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell \tag{3}$$

belongs to $\mathfrak{L}^w_n$. However, not every kernel of a constant coefficient linear partial differential operator is the image of a constant coefficient linear partial differential operator. The following theorem, obtained in [1], shows that it are precisely the controllable systems that admit an image representation.

Theorem 2 (Controllability) : *The following statements are equivalent for $\mathfrak{B} \in \mathfrak{L}^w_n$:*

1. *$\mathfrak{B}$ defines a controllable system,*
2. *$\mathfrak{B}$ admits an image representation,*
3. *The trajectories of compact support are dense in $\mathfrak{B}$.*

In the context of distributed differential systems the notion of observability is as follows.

Let $\Sigma = (\mathbb{R}^n, \mathbb{R}^{w_1} \times \mathbb{R}^{w_2}, \mathfrak{B}) \in \mathfrak{L}^{w_1+w_2}_n$. We call w_2 *observable* from w_1 in $\mathfrak{B}$ if $(w_1, w_2'), (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$.

For *1-D* systems it is easy to show that every controllable $\mathfrak{B} \in \mathfrak{L}^w_1$ admits an observable image representation. This, however, does not hold for *n-D* systems, and hence the representation of controllable systems in image representation may require the introduction of latent variables that are 'hidden', in the sense that $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell = 0$ has solutions $\ell \neq 0$.

4 Conservative and dissipative systems

We are interested in this paper in distributed dynamical systems that are conservative or dissipative with respect to a supply rate that is a quadratic function of the manifest variables and their partial derivatives. These are defined by matrices $\Phi_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} \in \mathbb{R}^{w \times w}, k_1, \ldots, k_n, \ell_1, \ldots, \ell_n \in \mathbb{Z}_+$, with all but a finite number of these matrices equal to zero. We call the map from $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$ to $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$ defined by

$$w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \mapsto \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} (\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} w)^T \Phi_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} (\frac{\partial^{\ell_1}}{\partial x_1^{\ell_1}} \cdots \frac{\partial^{\ell_n}}{\partial x_n^{\ell_n}} w)$$

a *quadratic differential form* on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w)$. Note that a quadratic differential form is completely specified by the $(w \times w)$-matrix Φ of $2n$-variable polynomials in $\mathbb{R}[\zeta_1, \cdots, \zeta_n, \eta_1, \cdots, \eta_n]$, defined by

$$\Phi(\zeta_1, \cdots, \zeta_n, \eta_1, \cdots, \eta_n) = \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} \Phi_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} \zeta_1^{k_1} \cdots \zeta_n^{k_n} \eta_1^{\ell_1} \cdots \eta_n^{\ell_n}.$$

We denote the quadratic differential form that corresponds to the matrix of polynomials Φ by Q_Φ. Define

$\Phi^\star$, the matrix of $2n-$variable polynomials, by

$$\Phi^\star(\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n) = \Phi^T(\eta_1, \dots, \eta_n, \zeta_1, \dots, \zeta_n).$$

If $\Phi = \Phi^\star$, we call Φ *symmetric.* We may (and will) assume, since obviously $Q_\Phi = Q_{\Phi^\star} = Q_{\frac{1}{2}(\Phi+\Phi^\star)}$, that in a quadratic differential form the matrix of polynomials Φ is symmetric.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\mathrm{cont}}$ and $\Phi = \Phi^\star \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n]$. Define $\mathfrak{B}$ to be *conservative* with respect to the supply rate Q_Φ if

$$\int_{\mathbb{R}^n} Q_\Phi(w) = 0$$

for all $w \in \mathfrak{B}$ of compact support, and *dissipative* if

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0$$

for all $w \in \mathfrak{B}$ of compact support.

5 Local version of a conservation law

The following result shows in what sense a conservation law can be expressed as a local law.

Theorem 3 (Local version of a conservation law) *Consider the controllable n-D distributed dynamical system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\mathrm{cont}}$ and the* supply rate *defined by the quadratic differential form Q_Φ. Let $w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell$ be an image representation of $\mathfrak{B}$. Then $\mathfrak{B}$ is conservative with respect to Q_Φ if and only if there exist an $n-$vector of quadratic differential forms $Q_\Psi = (Q_{\Psi_1}, \dots, Q_{\Psi_n})$ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the* flux density*, such that*

$$\nabla \cdot Q_\Psi(w, \ell) = Q_\Phi(w)$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$, the full behavior of $w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell$.

When the first independent variable is time, and the others are space variables, then the local version of the conservation law can be expressed a bit more intuitively in terms of a quadratic differential form Q_S, the *storage density,* and a 3-vector of quadratic differential forms Q_F, the *spatial flux density,* as

$$\frac{\partial}{\partial t} Q_S(w, \ell) + \nabla \cdot Q_F(w, \ell) = Q_\Phi(w)$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$, the full behavior of $w = M(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ell$, an image representation of $\mathfrak{L}^{\mathtt{w}}_{4,\mathrm{cont}}$.

In the *1-D* case, the introduction of latent variables is unnecessary, and we can simply claim the existence of a quadratic differential form Q_Ψ on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, such that $\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w)$ for all $w \in \mathfrak{B}$. However, in the *n-D* case, the introduction of latent variables cannot be avoided, because not every controllable distributed parameter system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ admits an observable image representation.

The idea behind the proof of the above theorem is as follows. Using an image representation for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\mathrm{cont}}$ shows that is suffices to prove the result for the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. Next, observe that $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ is conservative with respect to Q_Φ if and only if $\partial(\Phi) = 0$. This in turn is easily seen to be equivalent to the solvability of the equation

$$\Phi(\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n) = (\zeta_1 + \eta_1)\Psi_1(\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n) + \dots + (\zeta_n + \eta_n)\Psi_n(\zeta_1, \dots, \zeta_n, \eta_1, \dots, \eta_n) \quad (4)$$

for $(\Psi_1, \dots, \Psi_n)$. Note that in the *n-D* case, contrary to the *1-D* case, the solution $(\Psi_1, \dots, \Psi_n)$ to this equation is not unique, and hence the storage function is in general not uniquely specified by the dynamics and the supply rate.

6 Local version of a dissipation law

We now discuss dissipative dynamical systems. A quadratic differential form Q_Δ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ is said to be *non-negative*, denoted $Q_\Delta \geq 0$, if $Q_\Delta(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. The following theorem gives the local version of dissipativeness for distributed differential systems.

Theorem 4 (Local version of a dissipation law) *Consider the controllable n-D distributed dynamical system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\mathrm{cont}}$ and the* supply rate *defined by the quadratic differential form Q_Φ. Then $\mathfrak{B}$ is dissipative with respect to Q_Φ if and only if there exist:*

1. *an image representation*

$$w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell \quad (5)$$

of $\mathfrak{B}$,

2. *an $n-$vector of quadratic differential forms $Q_\Psi = (Q_{\Psi_1}, \dots, Q_{\Psi_n})$ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the* flux density*,*

3. *a non-negative quadratic differential Q_Δ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the* dissipation rate*,*

such that

$$\nabla \cdot Q_\Psi(\ell) = Q_\Phi(w) + Q_\Delta(\ell)$$

for all $(w, \ell) \in \mathfrak{B}_{\text{full}}$, *the full behavior of (5).*

When the first independent variable is time, and the others are space variables, then the local version of the dissipation law can be expressed a bit more intuitively in terms of a quadratic differential form Q_S, the ***storage density***, a 3-vector of quadratic differential forms Q_F, the ***spatial flux density***, and the quadratic differential form $Q_\Delta \geq 0$, the ***dissipation rate***, such that

$$\frac{\partial}{\partial t} Q_S(\ell) + \nabla \cdot Q_F(\ell) = Q_\Phi(w) + Q_\Delta(\ell)$$

for all $(w, \ell) \in \mathfrak{B}_{\text{full}}$, the full behavior of a suitable image variable representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{4,\text{cont}}$.

In the *1-D* case, the introduction of latent variable is once again unnecessary, and we can simply claim the existence of quadratic differential forms (Q_Ψ, Q_Δ) on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, with $Q_\Delta \geq 0$, such that $\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) + Q_\Delta(w)$ for all $w \in \mathfrak{B}$.

In order to see where the introduction of latent variables enters in the n-*D* case, we will briefly sketch the proof of the above theorem in the *1-D* case. It is easy to see, using an observable image representation, that it suffices to consider the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Next, use Fourier transforms to prove that the integral $\int_{-\infty}^{+\infty} Q_\Phi(w)\, dt$ is non-negative for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ of compact support if and only if the hermitian matrix $\Phi(i\omega, -i\omega) \in \mathbb{C}^{\mathtt{w} \times \mathtt{w}}$ is non-negative definite for all $\omega \in \mathbb{R}$. This in turn implies that the matrix of polynomials $\Phi(\xi, -\xi) \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ be factored as $\Phi(\xi, -\xi) = D^T(-\xi) D(\xi)$ with $D(\xi) \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ a matrix of polynomials. The result then follows by taking

$$\begin{aligned} \Psi(\zeta, \eta) &= \frac{\Phi(\zeta, \eta) - D^T(\zeta) D(\eta)}{\zeta + \eta}, \text{ and} \\ \Delta(\zeta, \eta) &= D^T(\zeta) D(\eta). \end{aligned}$$

For dissipative systems the storage function Q_Ψ, and hence the dissipation rate, Q_Δ, are, in general, not unique, not even in the *1-D* case, since the factorization $\Phi(\xi, -\xi) = D^T(-\xi) D(\xi)$ is not unique.

The generalization of this proof to the n-*D* case fails on two accounts. Firstly, because there may not exist an observable image representation for $\mathfrak{B}$. Secondly, because the polynomial matrix factorization

$$\begin{aligned} &\Phi(\xi_1, \ldots, \xi_n, -\xi_1, \ldots, -\xi_n) = \\ &D^T(-\xi_1, \ldots, -\xi_n) D(\xi_1, \ldots, \xi_n) \end{aligned} \tag{6}$$

with $D \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_n]$ may not be possible, whereas we still have that dissipativeness of $\mathfrak{B}$ *with respect to the supply rate* Q_Φ is equivalent to non-negative definiteness of the hermitian matrix $\Phi(i\omega_1, \ldots, i\omega_n, -i\omega_1, \ldots, -i\omega_n) \in \mathbb{C}^{\mathtt{w} \times \mathtt{w}}$ for all $\omega_1, \ldots, \omega_n \in \mathbb{R}$. However, it turns out that a factorization as (6), with $D \in \mathbb{R}^{\bullet \times \mathtt{w}}(\xi_1, \ldots, \xi_n)$ a matrix of rational functions in the variables $\xi_1, \ldots, \xi_n$ does exist. This factorizability is a consequence of Hilbert's 17-th problem on the factorizability of nonnegative multivariable rational functions as a sum of squares. The fact that we have to introduce rational functions accounts for the need to introduce an image representation of $\mathfrak{B}$.

7 Conclusions

In this paper, we studied conservative and dissipative systems in the context of distributed dynamical systems described by constant-coefficient partial differential equations. For such systems, it is possible to express a global conservation or dissipation law as a local one, involving the flux density and the dissipation rate. There are two interesting aspects of the construction of the flux density and the dissipation rate. The first one is the relation with Hilbert's 17-th problem on the factorization of real non-negative rational functions in many variables as a sum of squares of real rational functions. The second interesting aspect is that local conservation or dissipation laws necessarily involve 'hidden' latent variables.

References

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