Proceedings of the 38th Conference on Decision & Control
Phoenix, Arizona USA • December 1999

Controllability and observability of linear differential behaviors

Tommaso Cotroneo
Mathematics Institute, University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands
e-mail:tommaso@math.rug.nl
Phone:+31-50-3636496
Fax:+31-50-3633976

Jan C. Willems
Mathematics Institute, University of Groningen
P.O. Box 800, 9700 AV Groningen, The Netherlands
e-mail:J.C.Willems@math.rug.nl
Phone:+31-50-3633984
Fax:+31-50-3633976

1 Linear Time Invariant Behaviors

Two central definitions in systems theory are controllability and observability. For state space systems controllability is defined as the possibility of transferring the state from any initial to any terminal value, while observability is defined as the possibility of deducing the state from an observed output. Recently (see [1]) a more general definition has been put forward that generalizes these notions to more general model classes. In [1] and related papers a number of conditions have been derived that allow to deduce controllability of the system described by the system of differential equations

$$R_0 w + R_1 \frac{d}{dt} w + \cdots R_L \frac{d^L}{dt^L} w = 0$$

with $R_i \in \mathbb{R}^{p\times q}$ and observability for

$$\begin{array}{c} R_0 w_1 + R_1 \frac{d}{dt} w_1 + \cdots R_L \frac{d^L}{dt^L} w_1 = \\ M_0 w_2 + M_1 \frac{d}{dt} w_2 + \cdots M_N \frac{d^N}{dt^N} w_2 \end{array}$$

with $R_i \in \mathbb{R}^{p\times q}$ and with $M_i \in \mathbb{R}^{p\times \ell}$. However, the conditions in [1] are not explicit in the coefficient matrices $(R_0, \ldots R_L)$ or $(M_0, \ldots M_N)$. In [2] conditions of such an explicit nature are derived, but, in the controllability case for example, they involve checking the rank of a matrix of dimension $Lpq \times L(p+1)q$.

The purpose of this paper is to derive conditions in terms of $(R_0, \ldots R_L)$ or $(M_0, \ldots M_N)$. The conditions involve explicit rank tests and will be presented as a recursive algorithm in MATLAB-style pseudo-code. The algorithm generalizes the familiar $(B, AB, \ldots, A^{n-1}B)$ or $(C^T, A^T C^T, \ldots (A^T)^{n-1} C^T)$ rank test for state space systems and the Euclidean algorithm for checking co-primeness of two polynomials.

We now formalize the notation and definitions introduced above.

In the behavioral approach to system theory a *dynamical system* is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ where $\mathbb{T} \subseteq \mathbb{R}$ is the *time set* over which the system evolves (e.g. it will typically be $\mathbb{R}$ or $\mathbb{R}_+$ for continuous-time systems , and $\mathbb{Z}$ or $\mathbb{Z}_+$ for discrete-time), $\mathbb{W}$ is the *signal space* in which the variables of the system we are modeling take on their values and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the *behavior* of the system. The set $\mathbb{W}^{\mathbb{T}}$ consists of all possible maps from $\mathbb{T}$ to $\mathbb{W}$, and the trajectories belonging to $\mathfrak{B}$ are nothing but the subset of these which comply with the laws of the system. The behavior can be specified in many ways, however the most common way of doing so is by defining it as the set of solutions of a suitable system of equations; in particular we will call *differential systems* those systems with $\mathbb{T} = \mathbb{R}$ whose behavior $\mathfrak{B}$ consists of all solutions of a system of differential equations of the form:

$$f(w, \frac{d}{dt} w, \ldots, \frac{d^N}{dt^N} w) = 0.$$

Classical notions such as linearity and time-invariance are also very naturally defined in the behavioral setting; in particular a system is said to be *linear* if $\mathbb{W}$ is a vector space and $\mathfrak{B}$ a linear subspace of $\mathbb{W}^{\mathbb{T}}$, and *time-invariant* (assuming $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$) if $\sigma^t \mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{T}$, where σ^t denotes the t-shift (defined by $(\sigma^t f)(t') := f(t' + t)$). Putting together all the above notions we obtain the class of systems we shall be interested in, namely *linear time-invariant differential sys-*

tem, systems for which

$$\mathfrak{B} = \{w \in \mathcal{L}^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) | R(\frac{d}{dt})w = 0\}$$

with $R \in \mathbb{R}^{\bullet \times q}[\xi]$ a polynomial matrix with q columns and any (finite) number of rows. The differential equation $R(\frac{d}{dt})w = 0$ that appears in the above definition must be interpreted in the sense of distributions; we also denote the above behavior as $\ker(R(\frac{d}{dt}))$. In such a setting we can now introduce the definitions of controllability and observability.

Let $\mathfrak{B}$ be a continuous-time, time invariant behavior; we will define it to be *controllable* if for any two trajectories $w_1,\ w_2 \in \mathfrak{B}$ there exists a $t_1 \geq 0$ and a third trajectory $w \in \mathfrak{B}$ such that:

$$w(t) = \begin{cases} w_1(t) & t \leq 0 \\ w_2(t - t_1) & t \geq t_1 \end{cases}$$

The intuition behind this definition is that for a behavior to be controllable one must be able to connect any admissible past trajectory to any admissible future one, through suitable steering. In the next section we will concentrate on the case $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ and derive conditions in terms of the polynomial matrix R that allow us to conclude controllability.

The notion of observability deals with systems with two types of variables: observed ones (denoted by w_1) and to-be-observed ones (denoted by w_2). Let $(\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ be such a dynamical system. Denote a typical element of the behavior by $w = (w_1, w_2)$; we say that w_2 is *observable* from w_1 if

$$((w_1, w_2), (w_1, w_2') \in \mathfrak{B}) \Rightarrow (w_2 = w_2')$$

In case $\mathfrak{B}$ is a linear behavior this is equivalent to asking

$$((0, w_2) \in \mathfrak{B}) \Rightarrow (w_2 = 0)$$

In case $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ we have to partition R conform with the partition of $w = (w_1, w_2)$. In this case it is convenient to write the behavioral equations as

$$R_1(\frac{d}{dt})w_1 = M(\frac{d}{dt})w_2$$

with (R_1, M) the aforementioned partition of R. In the next section we shall derive conditions on the polynomial matrix M which allow us to conclude observability of w_2. An extensive account of all the concepts introduced in this section can be found in [1].

2 Conditions for Controllability and Observability

A polynomial matrix $R \in \mathbb{R}^{p \times q}[\xi]$ is said to be *left prime* if $R = GR'$ with $G \in \mathbb{R}^{p \times p}[\xi]$ and $R' \in \mathbb{R}^{p \times q}[\xi]$ implies G is unimodular; the definition of *right primeness* is analogous. The *module* spanned by a set $v_1, \ldots, v_p \in \mathbb{R}^n[\xi]$ of polynomial vectors, is denoted by $< v_1, \ldots, v_p >$ and is defined as the set of all possible linear combinations with polynomial coefficients of the given vectors; in other words: $< v_1, \ldots, v_p > = \{\sum_{i=1}^{p} h_i v_i,\ h_i \in \mathbb{R}[\xi]\}$; of course $< v_1, \ldots, v_p > \subseteq \mathbb{R}^n[\xi]$.

We begin now by giving a set of conditions under which $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ is a controllable behavior under the assumption that $R(\xi)$ is of full row rank (that is, R has a submatrix of dimension $\text{rowdim}(R) \times \text{rowdim}(R)$ that has a nonzero determinant).

Theorem 1 : *Let $R(\xi) \in \mathbb{R}^{p \times q}[\xi]$ be a full row rank polynomial matrix. The following are then equivalent:*

1. *$\mathfrak{B} = \ker(R(\frac{d}{dt}))$ is controllable,*
2. *The Smith form of R is $[I\ 0]$,*
3. *$\text{rank}(R(\lambda)) = p\ \forall \lambda \in \mathbb{C}$,*
4. *R is left prime*
5. *The columns of R span the full module $\mathbb{R}^p[\xi]$*

Proof :
$1 \Leftrightarrow 2 \Leftrightarrow 3$: See [1] Theorems 5.2.5 and 5.2.9
$3 \Rightarrow 4$: Assume $R = GR'$ with $G \in \mathbb{R}^{p \times p}[\xi]$ not unimodular; then there exists $\lambda \in \mathbb{C}$ such that $\text{rank}(G(\lambda)) < p$ which implies $\text{rank} R(\lambda) < p$ thus leading to contradiction.
$4 \Rightarrow 2$: The Smith form of a polynomial matrix $R \in \mathbb{R}^{p \times q}[\xi]$ has the structure

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

with $D = diag(d_1 \ldots d_\ell)$ where the d_i are monic polynomials. Having $\ell < p$ would contradict left primeness; in fact call Λ the Smith form of R and let $U,\ V$ be unimodular matrices such that $R = U\Lambda V$ then, indicating by U_1 the first l columns of U we could write $R = [U_1\ 0]\Lambda V$ with $[U_1\ 0] \in \mathbb{R}^{p \times p}[\xi]$ and not unimodular. Therefore the Smith form of R must have the structure $\Lambda = [D\ 0]$; this means we can write $R = UD[I\ 0]V$ which shows that D must be the identity, otherwise we would have $UD \in \mathbb{R}^{p \times p}[\xi]$ and not unimodular.
$2 \Rightarrow 5$: We have $R = U[I\ 0]V$ with U, V unimodular; we can find a matrix U' such that $V' = \begin{bmatrix} U & 0 \\ 0 & U' \end{bmatrix} \in \mathbb{R}^{q \times q}[\xi]$ is unimodular. Therefore $R = [I\ 0]V'V$, with $V'V$ unimodular; because the module generated by the columns of a matrix is invariant by right unimodular transformations we have that the columns of R span the same module as those of $[I\ 0]$ which is of course

the full module $\mathbb{R}^p[\xi]$.
$5 \Rightarrow 4$: Because the columns of $R(\xi)$ span the same module as I there has to exist a matrix $N(\xi)$ such that $I = R(\xi)N(\xi)$. Assume now $R = G'R$ with G square; we then have $I = G(R'N)$, equivalently $R'N$ is an inverse for G which must therefore be unimodular □

Proceeding analogously to theorem 1 (see also [1] Theorem 5.3.3) we also get conditions for observability, as expressed in the following

Theorem 2 : *Let* $\mathfrak{B} = \{w = (w_1, w_2) \in \mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) | R_1(\frac{d}{dt})w_1 = M(\frac{d}{dt})w_2\}$ *and assume* $M \in \mathbb{R}^{p\times \ell}[\xi]$. *The following are then equivalent:*

1. *w_2 is observable from w_1*
2. *The Smith Form of M is* $\begin{bmatrix} I \\ 0 \end{bmatrix}$
3. $\mathrm{rank}(M(\lambda)) = \ell \ \forall \lambda \in \mathbb{C}$
4. *M is right prime*
5. *The rows of M span the full module* $\mathbb{R}^\ell[\xi]$

Theorem 1 addresses controllability in case $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ for $R(\xi)$ of full row rank; the following theorem shows what can be said when such a rank condition does not hold.

Theorem 3 : *The following are equivalent:*

1. $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ *is controllable*
2. *The Smith Form of R is* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$
3. $\mathrm{rank}(R(\lambda))$ *is independent of* $\lambda \ \forall \lambda \in \mathbb{C}$
4. *If $N(\xi)$is a minimal generating set for the module spanned by the columns of R, then $N(\xi)$ is a right prime matrix*

Proof :
$1 \Leftrightarrow 2 \Leftrightarrow 3$: See again [1] Theorems 5.2.5 and 5.2.9
$2 \Rightarrow 4$: Let Λ be the Smith for of R. We have $R = U\Lambda V$ for unimodular U, V. Partition U as $[U_1 \ U_2]$ conform to the partition of Λ, we then have $R = [U_1 \ 0]V$ with U_1 a minimal generating set for the module spanned by the columns of R (it is a generating set because $[U_1 \ 0]$ differs from R by a right unimodular matrix, it is minimal because the columns of U_1 are columns of an unimodular matrix, therefore independent over $\mathbb{R}(\xi)$). Because U_1 are columns from an unimodular matrix we also know that there exists a matrix V_1 such that $V_1U_1 = I$ which (see the proof of theorem 1)implies that U_1 is right prime. The result then follows because any two minimal generating sets differ by a right unimodular transformation.
$4 \Rightarrow 2$: Assume Λ has the general structure $\Lambda = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ then, again partitioning U as $[U_1 \ U_2]$, we would have $R = [U_1D \ 0]V$ with U_1D a minimal generating set for the module spanned by the columns of R (one can see it exactly as above). Therefore U_1D must be right prime, which implies $D = I$ □

From a constructive point of view a minimal generating set for the module spanned by the columns of R can be built by bringing R in a canonical form such as, for example, the column proper form of R; in the next section we will show how to use this information in order to sketch an algorithm to test controllability.

3 Algorithms

3.1 Observability

We now sketch an algorithm that allows to check whether variables w_2 are observable from w_1 in the system described by $R_1(\frac{d}{dt})w_1 = M(\frac{d}{dt})w_2$, equivalently whether matrix $M \in \mathbb{R}^{p\times\ell}[\xi]$ is right prime.

The algorithm is based on condition 5 of theorem 2, which can be restated as follows: consider the set of all row vectors of degree 0 contained in the module spanned by the rows of M; such a set is obviously an $\mathbb{R}$-vector space for which we can build a set of generators; in order for M to be right prime, this set must also be a generating set for the whole space $\mathbb{R}^\ell$. In the following algorithm we will recursively build such a generating set and check whether it has rank equal to ℓ in order to conclude right primeness of M.

Before proceeding let us introduce some notation we will use:

- Standard Matlab notation will be used to indicate rows and columns of a matrix (e.g. $M(i,:)$ is the i-th row of M, $M(i:j,:)$ are rows i to j)
- M^0 indicates the set of all degree 0 rows of matrix M. For example if $M = \begin{bmatrix} \xi^2+\xi+1 & 2 \\ 1 & 1 \end{bmatrix}$ then $M^0 = [1 \ 1]$.
- d_i indicates the row degree of the i-th row of M; for the matrix in the above point we would have $d_1 = 2$ and $d_2 = 0$

- $M_{hc} \in \mathbb{R}^{p\times l}$ is the highest row coefficient matrix of M meaning that $M_{hc}(i,:)$ is the coefficient of the highest power of ξ in $M(i,:)$. For example, looking at the same M as above, we would have $M_{hc} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

- $M_{hp} \in \mathbb{R}^{p\times l}[\xi]$ is the highest row power matrix of M, meaning that $M_{hp}(i,:)$ corresponds to the highest power of ξ in $M(i,:)$ (notice that $M_{hp}(i,:) = \xi^{d_i} M_{hc}(i,:)$). Again looking at the same M as above we would have: $M_{hp} = \begin{bmatrix} \xi^2 & 0 \\ 1 & 1 \end{bmatrix}$

- M=**Order**(M) is a procedure that reorders the rows of M in decreasing row degree order. For example if $M = \begin{bmatrix} 0 & 1 \\ \xi & 1 \\ \xi^2+\xi+1 & 2 \\ 1 & 1 \end{bmatrix}$ then **Order**(M) returns $M = \begin{bmatrix} \xi^2+\xi+1 & 2 \\ \xi & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

- Assume now we have a matrix M ordered as by **Order**(M) and let p = rowdim(M). By standard linear algebra it is easy to find, if it exists, a non-zero real vector n such that $M_{hc}(i,:) = nM_{hc}(i+1:p,:)$. For example if $M = \begin{bmatrix} \xi^2+\xi+1 & \xi^2+2 \\ \xi & 1 \\ 0 & 1 \end{bmatrix}$ then $M_{hc}(1,:) = nM_{hc}(2:3,:)$ for $n = [1\ 1]$. The function h=**polann**(n) returns a polynomial vector h such that $M_{hp}(i,:) = hM_{hp}(i+1:p,:)$. In the above case, for example, **polann**(n) would return $h = [\xi\ \ \xi^2]$. The way to build h starting from n is rather straightforward; all one needs to do is multiply the k-th entry ($k = i+1,\ldots,p$) of n times $\xi^{(d_i-d_k)}$.

- M=**Eliminate**(M,i) is a procedure that cancels the i-th row of matrix M. For example if $M = \begin{bmatrix} \xi^2+\xi+1 & \xi^2+2 \\ \xi & 1 \\ 0 & 1 \end{bmatrix}$ then **Eliminate**(M, 2) returns $M = \begin{bmatrix} \xi^2+\xi+1 & \xi^2+2 \\ 0 & 1 \end{bmatrix}$

- Let m be a polynomial vector of degree d_m and $M \in \mathbb{R}^{p\times l}[\xi]$ a matrix ordered as by **Order**; moreover let j be such that the degree of $M(i,:)$ is smaller or equal than d_m for $i \geq j$ and greater than d_m for $i \leq j-1$. Then $[M,j]$=**Insert**(M,m) is a procedure that replaces matrix M with $col(M(1:j-1,:), m, M(j:p,:))$ with, as usual, p = rowdim(M). The procedure also returns j, namely the position occupied by m in the new matrix. For example if $M = \begin{bmatrix} \xi^2+\xi+1 & \xi^2+2 \\ 1 & 1 \end{bmatrix}$ and $m = [\xi+1\ \ \xi]$ then **Insert**(M,m) returns $M = \begin{bmatrix} \xi^2+\xi+1 & \xi^2+2 \\ \xi+1 & \xi \\ 0 & 1 \end{bmatrix}$ and $j = 2$.

We can now sketch in MATLAB pseudo-code our procedure for checking right primeness of a matrix M:

```
[M,obs]=RPR(M);

    M=Order(M);
    obs=(rank(M^0) == ℓ);
    p=rowdim(M);
    i=p-rowdim(M^0);

    while ((not obs) and (i ≥ 1)) do

        if (∃ real n ≠ 0 such that
          M_hc(i,:) = n M_hc(i+1:p,:)) then

            h=polann(n);
            m = M(i,:) - h M(i+1:p,:);
            M=Eliminate(M,i);

            if (m ≠ 0) then
                [M,j]=Insert(M,m)
            else p = p-1 endif;

            if (degree(m)==0) then
                obs=(rank(M^0) == l)
            else i = j endif;

        else i = i-1;
        endif;

    endwhile
```

In the above algorithm obs is a boolean variable which tells us whether the matrix we are considering is right prime or not; we call this variable obs because of the relation between right primeness and observability explored in theorem 2. As already discussed right primeness is checked by veryfing if the degree 0 vectors contained in the module spanned by the rows of M generate $\mathbb{R}^\ell$. After reordering the matrix we immediately perform such a check to see whether the degree 0 vectors in the original matrix are already enough to meet the requirement.

If this is not the case we then enter the main while loop in which we try to generate additional constant vectors by taking combinations of rows of M. This is done by replacing a row $M(i,:)$ of degree higher than 0 (in fact starting with i=p-rowdim(M^0) corresponds to taking the first row from the bottom with degree higher than 0) with a polynomial combination of $M(i,:)$ and other rows of M; such a combination has to be of lower degree than $M(i,:)$ itself. Such a degree lowering is only possible if the highest coefficient vector of the given row $M(i,:)$ is linearly dependent on the highest coefficient vectors of rows of equal or lower degree. This is exactly the condition tested in the if statement by looking for a real vector $n \neq 0$ of suitable dimension such that $M_{hc}(i,:) = nM_{hc}(i+1:p,:)$. In case such a dependence is found, starting from n we build the polynomial vector h such that $M_{hp}(i,:) = hM_{hp}(i+1:p,:)$; this ensures that the polynomial vector $m = M(i,:) - hM(i+1:p,:)$ will have degree lower than $M(i,:)$, as desired.

In case such a lowering was possible, we then eliminate $M(i,:)$ and if the new vector m is not zero, we insert it in matrix M in such a way as to mantain it ordered by degree as intially done with **Order**.

If the new vector has degree 0 we then check again if right primeness condition is fulfilled; in case it is of order greater than 0 the next iteration of the while loop will check whether the degree of this newly generated vector can itself be lowered (this is the meaning of having $i = j$).

The algorithm ends if the condition for right primeness is verified (obs is true) or if no more lowering of degree is possible; the condition $(i < 1)$ in the while statement, in fact, means we have considered all possible rows in M and no possibility of lowering was found. Because at each step we are replacing a vector with one of lower degree it is not difficoult to see that this last condition will eventually always be verified so that the stopping rule for the algorithm is well defined.

Example 4 : Assume $M = \begin{pmatrix} \xi I - A \\ -C \end{pmatrix}$, corresponding to the situation in which we want to observe state variables in a state space system. The algorithm presented above then gives an efficient way of checking the classical rank condition on the observability matrix $\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$. It can be recognized that the rows of the observability matrix are 0 degree vectors in the module spanned by the rows of M. In fact the rows of CA can be obtained as $(\xi I)C - C(\xi I - A)$ and are therefore polynomial linear combinations of degree 0 of the rows of M. By induction one then sees that the rows of CA^k obtained as $(\xi I)CA^{k-1} - CA^{k-1}(\xi I - A)$ are also 0 degree vectors in the module spanned by the rows of M. The fact that the rows of the observability matrix are a generating set for the space of all 0 degree vectors spanned by the rows of M follows from the fact that A^n is linearly dependent on $A, \dots A^{n-1}$ so that no independent row vectors would be added by considering CA^k for $k \geq n$.

Checking that the observability matrix has rank n is therefore equivalent to checking that the 0 degree vectors spanned by the rows of M generate the vector space $\mathbb{R}^n$. As discussed above this is exactly equivalent to what our algorithm checks in a, in general, more efficient way (e.g. just consider the case in which C is a non-singular matrix, then observability follows immediately without needing to compute the rest of the observability matrix; in this case, in fact, our algorithm would stop without even entering the main while loop).

Example 5 : Assume $M = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ with m_1, m_2 polynomials and d_1 =degree$(m_1) \geq d_2$ =degree(m_2). Applying division algorithm for polynomials we know we can write $m_1 = q_2 m_2 + m_3$ with d_2 =degree$(m_2) > d_3$ =degree(m_3). Going through our algorithm we see that after at most $d_1 - d_2 + 1$ steps we will end up with $M = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix}$. We can then again write $m_2 = q_3 m_3 + m_4$ and after at most $d_2 - d_3 + 1$ steps we will have $M = \begin{pmatrix} m_3 \\ m_4 \end{pmatrix}$. At this point it is easily recognized that our algorithm corresponds to the classical euclidean algorithm for computing the gratest common divisor of two polynomials. This means that after the last step we will have $M = m$ with m =GCD(m_1, m_2) and that our condition for observability is equivalent to asking that m is a constant, equivalently that m_1 and m_2 are coprime polynomials.

3.2 Controllability

A matrix is left prime if and only if its transpose is right prime; given the algorithm from the preceding section it is therefore easy to build a procedure **LPR** which checks whether a matrix R is left prime (equivalently whether $\ker(R(\frac{d}{dt}))$ is controllable under the assumptions of theorem 1). We would have

[R,ctr]=**LPR**(R);

> $M = R^T$;
> [M,ctr]=**RPR**(M);
> $R = M^T$;

We know however that left primeness is equivalent to controllability only in case R is a full rank matrix; to check controllability in the general case we will have to verify the conditions of theorem 3. In order to sketch an algorithm that does so let us make two remarks:

1. If **LPR** returns ctr=false than the matrix R which is returned is very close to the column

proper form of the original R. In fact, we see that in case left primeness is not verified than the algorithm stops when the highest column coefficient vectors of all columns with degree higher than 0 are linearly independent; as for degree 0 vectors, instead, we always check their rank but not their independence so we can't be assured that they are a linearly independent set.

The columns with degree higher than 0 therefore already satisfy the property that defines the column proper form of a matrix R and all one needs to do to actually get the column proper form is replace the degree 0 columns of the returned R with a basis for the $\mathbb{R}$-vector space they generate. We call R=**COLPRP**(R) a procedure that brings the returned R into column proper form in the way described above.

2. If R is in column proper form than the number of its columns is equal to the rank of the original matrix R. In case such a rank is equal to the row dimension of R than it means that R is of full row rank and that the test performed by **LPR** was necessary and sufficient for controllability.

   In case the rank is smaller than the rowdimension of R than we apply condition 4 of theorem 3 to check controllability. Having in hand the column proper form of R, equivalently a minimal generating set for the module spanned by the columns of R, we just need to check whether such a matrix is right prime and conclude controllability.

The above remarks lead to the following algorithm for checking controllability:

```
[R,ctr]=CTRB(R);

    [R,ctr]=LPR(R);
    if (not ctr) then

        R=COLPRP(R);
        if (rowdim(R)>coldim(R)) then

            [R,ctr]=RPR(R);

        endif

    endif
```

Example 6 : Assume $R = \begin{pmatrix} \xi I - A & -B \end{pmatrix}$, corresponding to the situation in which we want to test controllability of a state space system. In much the same way as done for observability one sees that our algorithm is equivalent to an efficient check of the usual rank condition on the controllability matrix $(B, AB, \dots A^{n-1}B)$.

Example 7 : Assume $R = (r_1\ r_2)$ for r_1, r_2 polynomials. Again as in the observability case our algorithm will end up building r =GCD(r_1, r_2) and conclude controllability if the two polynomials are coprime.

4 Bibliography

References

[1] J.W.Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Springer Verlag, 1998.

[2] J.Hoffmann, Algebraic aspects of controllability for AR-systems, *Int. Journal of Control* 60(5): 715-732, 1994