

STORAGE FUNCTIONS FOR DISSIPATIVE LINEAR SYSTEMS ARE QUADRATIC STATE FUNCTIONS

Harry L. Trentelman and Jan C. Willems
Mathematics Institute, University of Groningen
P.O. Box 800
9700 AV Groningen
The Netherlands

Email: h.l.trentelman@math.rug.nl; j.c.willems@math.rug.nl

Abstract

This paper deals with dissipative dynamical systems. Dissipative dynamical systems can be used as models for physical phenomena in which energy exchange with their environment plays a role. In a dissipative dynamical system, the book-keeping of energy is done via the *supply rate* and a *storage function*. The supply rate is the rate at which energy flows into the system, a storage function is a function that measures the amount of energy that is stored inside the system. In this paper, we will argue that for linear dynamical systems with quadratic supply rates, *any* storage function can be represented as a *quadratic function* of *any* state variable of a linear dynamical system whose dynamics is obtained by combining the dynamics of the original system, and the dynamics of the supply rate.

1 Introduction

The concept of dissipativeness is of much interest in physics and engineering. Whereas dynamical systems are used to model physical phenomena that evolve with time, dissipative dynamical systems can be used as models for physical phenomena in which also energy exchange with their environment plays a role. Typical examples of dissipative dynamical systems are electrical circuits, in which part of the electric and magnetic energy is dissipated in the resistors in the form of heat, and (visco-)elastic mechanical systems in which friction causes a similar loss of energy. For earlier work on dissipative systems, we refer to [8], [4], [7].

In a dissipative dynamical system, the book-keeping of energy is done via the *supply rate* and a *storage function*. The supply rate is the rate at which energy flows into the system, a storage function is a function that measures the amount of energy that is stored inside the system. These functions are related via the *dissipation inequality*, which states that along time trajectories of the dynamical system the supply rate does not exceed

the increase in storage. This expresses the assumption that a system cannot store more energy than is supplied to it from the outside. The difference between the internally stored and supplied energy is the dissipated energy.

The storage function measures the amount of energy that is stored inside the system at any instant of time. In other words, storage functions do the book-keeping of internally stored energy. We expect that the value of the storage function at a particular time-instant depends only on the past of the time-trajectories through the memory of the system. A standard way to express the memory of a time trajectory of a system is by using the notion of *state*. Thus we should expect that storage functions are functions of the *state variable* of the system.

In this paper, we will indeed prove the general statement that for linear dynamical systems with quadratic supply rates, *any* storage function can be represented as a quadratic function of *any* state variable of a linear dynamical system whose dynamics is obtained by combining the dynamics of the original system, and the dynamics of the supply rate.

A few words on notation. In this paper, $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ denotes the set of all infinitely often differentiable functions $w : \mathbb{R} \rightarrow \mathbb{R}^q$; $\mathcal{D}(\mathbb{R}, \mathbb{R}^q)$ denotes the subset of those $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ that have compact support. Given two column vectors x and y , the column vector obtained by stacking x over y is denoted by $\text{col}(x, y)$. Likewise, for given matrices A and B with the same number of columns, $\text{col}(A, B)$ denotes the matrix obtained by stacking A over B .

2 Linear differential systems

We will first introduce some basic facts from the behavioral approach to linear dynamical systems. For more details we refer to [11], [10], [9].

In this paper we consider dynamical systems described by a system of linear constant coefficient differential equations

$$R\left(\frac{d}{dt}\right)w = 0 \quad (2.1)$$

in the real variables w_1, w_2, \dots, w_q , arranged as the column vector w ; R is a real polynomial matrix with, of course, q columns. This is denoted as $R \in \mathbb{R}^{q \times q}[\xi]$, where ξ denotes the indeterminate. Thus if $R(\xi) = R_0 + R_1\xi + \dots + R_N\xi^N$, then (2.1) denotes the system of differential equations

$$R_0w + R_1\frac{dw}{dt} + \dots + R_N\frac{d^Nw}{dt^N} = 0 \quad (2.2)$$

Formally, (2.1) defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$, with \mathbb{R} the time axis, \mathbb{R}^q the signal space, and \mathfrak{B} the behavior, i.e., the solution set of (2.1):

$$\mathfrak{B} = \{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = 0\}$$

The family of dynamical systems Σ obtained in this way is denoted by \mathcal{L}^q . Instead of writing $\Sigma \in \mathcal{L}^q$, we often write $\mathfrak{B} \in \mathcal{L}^q$. For obvious reasons we refer to (2.1) as a *kernel representation* of \mathfrak{B} . In this paper we will also meet other ways to represent a given $\mathfrak{B} \in \mathcal{L}^q$, in particular using *latent variable representations* and *image representations*. We will now briefly introduce these. The system of differential equations

$$R\left(\frac{d}{dt}\right)w = M\left(\frac{d}{dt}\right)\ell \quad (2.3)$$

is said to be a *latent variable model*. We will call w the *manifest* and ℓ the *latent* variable. We assume that there are q manifest and d latent variables. R and M are polynomial matrices of appropriate dimension. Of course (2.3), being a differential equation as (2.1), defines the behavior

$$\mathfrak{B}_f = \{(w, \ell) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid (2.3) \text{ holds}\}$$

\mathfrak{B}_f will be called the *full behavior*, in order to distinguish it from the *manifest behavior* which will be introduced next. Consider the projection of \mathfrak{B}_f on the manifest variable space, i.e., the set

$$\{w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \text{there exists } \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^d) \text{ such that } (w, \ell) \in \mathfrak{B}_f\} \quad (2.4)$$

This set is called the *manifest behavior* of (2.3). If, for a given $\mathfrak{B} \in \mathcal{L}^q$, the manifest behavior (2.4) of (2.3) equals \mathfrak{B} , then (2.3) is called a *latent variable representation* of \mathfrak{B} . The latent variable representation is called *observable* if the latent variable is uniquely determined by the manifest variable, i.e., if $(w, \ell_1), (w, \ell_2) \in \mathfrak{B}_f$ implies that $\ell_1 = \ell_2$. It can be shown that (2.3) is observable iff $\text{rank}(M(\lambda)) = d$ for all $\lambda \in \mathbb{C}$.

A system $\mathfrak{B} \in \mathcal{L}^q$ is said to be *controllable* if for each $w_1, w_2 \in \mathfrak{B}$ there exists a $w \in \mathfrak{B}$ and a $t' \geq 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - t')$ for $t \geq t'$. It can be shown that \mathfrak{B} is controllable iff its kernel representation satisfies $\text{rank}(R(\lambda)) = \text{rank}(R)$ for all $\lambda \in \mathbb{C}$. Controllable systems are exactly those that admit *image representations*. More concretely, $\mathfrak{B} \in \mathcal{L}^q$ is controllable iff there exists an $M \in \mathbb{R}^{q \times q}[\xi]$ such that \mathfrak{B} is the manifest behavior of a latent variable model of the form

$$w = M\left(\frac{d}{dt}\right)\ell \quad (2.5)$$

For obvious reasons, (2.5) is called an *image representation* of \mathfrak{B} . An image representation is called *observable* if it is observable as a latent variable representation. Hence, the image representation (2.5) is observable iff $\text{rank}(M(\lambda)) = d$ for all $\lambda \in \mathbb{C}$. A controllable system always has an observable image representation.

3 Quadratic differential forms

An important role in this paper is played by quadratic differential forms and two-variable polynomial matrices. These are studied extensively in [12]. In this section we give a brief review.

We denote by $\mathbb{R}^{q \times q}[\zeta, \eta]$ the set of square, real polynomial matrices in the (commuting) indeterminates ζ and η , i.e., expressions of the form

$$\Phi(\zeta, \eta) = \sum_{k, \ell} \Phi_{k\ell} \zeta^k \eta^\ell \quad (3.1)$$

The sum in (3.1) ranges over the non-negative integers and is assumed to be finite, and $\Phi_{k\ell} \in \mathbb{R}^{q \times q}$. Such a Φ induces a *quadratic differential form* (QDF) $Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w)(t) := \sum_{k, \ell} \left(\frac{d^k w}{dt^k}(t)\right)^T \Phi_{k\ell} \left(\frac{d^\ell w}{dt^\ell}(t)\right) \quad (3.2)$$

If $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ satisfies $\Phi(\zeta, \eta) = \Phi^*(\zeta, \eta) := \Phi(\eta, \zeta)^T$ then Φ will be called *symmetric*. The symmetric elements of $\mathbb{R}^{q \times q}[\zeta, \eta]$ will be denoted by $\mathbb{R}_s^{q \times q}[\zeta, \eta]$.

Clearly $Q_\Phi = Q_{\Phi^*} = Q_{\frac{1}{2}(\Phi + \Phi^*)}$. This shows that when considering quadratic differential forms we can restrict attention to Φ 's in $\mathbb{R}_s^{q \times q}[\zeta, \eta]$. It is easily seen that $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ is symmetric iff $\Phi_{k\ell}^T = \Phi_{\ell k}$ for all k and ℓ .

Associated with $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ we form the symmetric matrix

$$\tilde{\Phi} = \begin{pmatrix} \Phi_{00} & \Phi_{01} & \cdots & \cdot & \cdots \\ \Phi_{10} & \Phi_{11} & \cdots & \cdot & \cdots \\ \vdots & \vdots & & \vdots & \\ \cdot & \cdot & \cdots & \Phi_{k\ell} & \cdots \\ \vdots & \vdots & & \vdots & \end{pmatrix} \quad (3.3)$$

Note that, although $\tilde{\Phi}$ is an infinite matrix, all but a finite number of its elements are zero. We can factor $\tilde{\Phi}$ as $\tilde{\Phi} = \tilde{M}^T \Sigma_M \tilde{M}$, with \tilde{M} an infinite matrix having a finite number of rows and all but a finite number of elements equal to zero, and Σ_M a signature matrix, i.e., a matrix of the form

$$\Sigma_M = \begin{pmatrix} I_{r_+} & 0 \\ 0 & -I_{r_-} \end{pmatrix}$$

This factorization leads, after pre-multiplication by $(I_q \quad I_q \zeta \quad I_q \zeta^2 \cdots)$ and post-multiplication by $\text{col}(I_q \quad I_q \eta \quad I_q \eta^2 \cdots)$, to a factorization of Φ as $\Phi(\zeta, \eta) = M^T(\zeta) \Sigma_M M(\eta)$. This decomposition is not unique but if we take M full row rank, then Σ_M will be unique. We will denote this Σ_M as Σ_Φ . In this case, the resulting r_+ is the number of positive eigenvalues and r_- the number of negative eigenvalues of $\tilde{\Phi}$. Any factorization $\Phi(\zeta, \eta) = M^T(\zeta) \Sigma_\Phi M(\eta)$ will be called a *canonical factorization* of Φ . In such a factorization, the rows of the polynomial matrix $M(\xi)$ are linearly independent over \mathbb{R} . Of course, a canonical factorization is not unique. However, they can all be obtained from one by replacing $M(\xi)$ by $UM(\xi)$ with $U \in \mathbb{R}^{\text{rank}(\tilde{\Phi}) \times \text{rank}(\tilde{\Phi})}$ such that $U^T \Sigma_\Phi U = \Sigma_\Phi$. Also note that if $\Phi(\zeta, \eta) = M^T(\zeta) \Sigma_\Phi M(\eta)$ is a canonical factorization, and $\Phi(\zeta, \eta) = M_1^T(\zeta) \Sigma_{M_1} M_1(\eta)$ an arbitrary factorization, then there exists a real constant matrix H such that $M(\xi) = HM_1(\xi)$.

The main motivation for identifying QDF's with two-variable polynomial matrices is, that they allow a very convenient calculus. One example of this is differentiation. If Q_Φ is a QDF, so will be $\frac{d}{dt} Q_\Phi$ defined by $(\frac{d}{dt} Q_\Phi)(w) := \frac{dQ_\Phi(w)}{dt}$. It is easily checked that $\frac{d}{dt} Q_\Phi = Q_{\dot{\Phi}}$ with $\dot{\Phi}(\zeta, \eta) := (\zeta + \eta)\Phi(\zeta, \eta)$. Suppose now that $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is given. An important question is: does there exist $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ such that $\dot{\Psi} = \Phi$, equivalently $\frac{d}{dt} Q_\Psi = Q_\Phi$? Obviously such Ψ

exists iff Φ contains a factor $\zeta + \eta$. Under this condition we can simply take $\Psi(\zeta, \eta) = \frac{1}{\zeta + \eta} \Phi(\zeta, \eta)$. It was shown in [12] that Φ contains a factor $\zeta + \eta$ iff $\partial\Phi = 0$, where $\partial\Phi$ is the one-variable polynomial matrix defined by $\partial\Phi(\xi) := \Phi(-\xi, \xi)$. It was also proven in [12] that $\partial\Phi = 0$ iff $\int_{-\infty}^{\infty} Q_\Phi(w) dt = 0$ for all $w \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$.

If $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, we will say $\Phi \geq 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. It was shown in [12] that $\Phi \geq 0$ iff there exists $D \in \mathbb{R}^{q \times q}[\xi]$ such that $\Phi(\zeta, \eta) = D^T(\zeta)D(\eta)$, equivalently $Q_\Phi(w) = \|D(\frac{d}{dt})w\|^2$ for all $w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. In addition, we need the concept of *average non-negativity*: we will say $\int Q_\Phi \geq 0$ if $\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0$ for all $w \in \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$. Again, it was shown in [12] that $\int Q_\Phi \geq 0$ iff $(\partial\Phi)(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. In turn, this condition is equivalent with the existence of polynomial spectral factorizations of $\partial\Phi$: $(\partial\Phi)(i\omega) \geq 0$ iff there exists $D \in \mathbb{R}^{q \times q}[\xi]$ such that $(\partial\Phi)(\xi) = D^T(-\xi)D(\xi)$ (see [1], [2]).

4 Dissipative systems

Let $\mathfrak{B} \in \mathcal{L}^q$ be a controllable linear differential system. Let $R(\frac{d}{dt})w = 0$ and $w = M(\frac{d}{dt})\ell$ be a kernel and an observable image representation, respectively, of \mathfrak{B} . Here, $R \in \mathbb{R}^{q \times q}[\xi]$ and $M \in \mathbb{R}^{q \times d}$. In addition, consider the quadratic differential form $Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ induced by the symmetric two-variable polynomial matrix Φ ; Q_Φ is called the *supply rate*. Intuitively, we think of $Q_\Phi(w)$ as the power going into the physical system \mathfrak{B} . In many applications, the power will indeed be a quadratic expression involving the system variables and its higher order derivatives. For example, in mechanical systems, it is $\sum_k F_k \frac{dq_k}{dt}$ with F_k the external force acting on, and q_k the position of the k -th pointmass; in electrical circuits it is $\sum_k V_k I_k$ with V_k the potential and I_k the current into the circuit at the k -th terminal. The system \mathfrak{B} is called *dissipative* with respect to the supply rate Q_Φ if along trajectories that start at rest and bring the system back to rest, the total amount of energy flowing into the system is non-negative: the system dissipates energy.

Definition 4.1 : (\mathfrak{B}, Q_Φ) is called dissipative if $\int_{-\infty}^{\infty} Q_\Phi(w) dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$.

Of course, at some times t the power $Q(w)(t)$ might be positive: energy is flowing into the system; at other times, it might be negative, energy is flowing out of the system. This outflow is possible because energy is stored. However, because of dissipation, the rate of increase of the storage can not exceed the supply. The interaction between supply, storage, and dissipation is formalized as follows:

Definition 4.2 : The QDF Q_Ψ induced by $\Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is called a *storage function* for (\mathfrak{B}, Q_Φ) if for all $w \in \mathfrak{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$

$$\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w) \quad (4.1)$$

The QDF Q_Δ induced by $\Delta \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ is called a *dissipation function* for (\mathfrak{B}, Q_Φ) if $Q_\Delta(w) \geq 0$ for all $w \in \mathfrak{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and

$$\int_{-\infty}^{\infty} Q_\Phi(w) dt = \int_{-\infty}^{\infty} Q_\Delta(w) dt$$

for all $w \in \mathfrak{B} \cap \mathcal{D}(\mathbb{R}, \mathbb{R}^q)$. If the supply rate Q_Φ , the dissipation function Q_Δ , and the storage function Q_Ψ satisfy

$$\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w) \quad (4.2)$$

for all $w \in \mathfrak{B} \cap \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$, then we call the triple $(Q_\Phi, Q_\Psi, Q_\Delta)$ *matched on \mathfrak{B}* .

Theorem 4.3 : *The following conditions are equivalent*

1. (\mathfrak{B}, Q_Φ) is dissipative
2. $M(-i\omega)^\top \Phi(-i\omega, i\omega) M(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$
3. (\mathfrak{B}, Q_Φ) admits a storage function
4. (\mathfrak{B}, Q_Φ) admits a dissipation function.

Furthermore, for any dissipation function Q_Δ there exists a storage function Q_Ψ , and for any storage function Q_Ψ there exists a dissipation function Q_Δ such that $(Q_\Phi, Q_\Psi, Q_\Delta)$ is matched on \mathfrak{B} .

Example 4.4 : Consider the system

$$M \frac{d^2 q}{dt^2} + D \frac{dq}{dt} + Kq = F \quad (4.3)$$

with $K, D, M \in \mathbb{R}^{k \times k}$, $K = K^\top \geq 0$, $D + D^\top \geq 0$, and $M = M^\top \geq 0$. The position vector q and force vector F take their values in \mathbb{R}^k . Such second order equations occur frequently as models of (visco-)elastic mechanical systems. As manifest variable take $w = \text{col}(q, \dot{q})$. As supply rate we take $Q_\Phi(q, F) = F^\top \dot{q}$. This corresponds to taking

$$\Phi(\zeta, \eta) = \frac{1}{2} \begin{pmatrix} 0 & \zeta I \\ \eta I & 0 \end{pmatrix}$$

An image representation of the system is given by $\text{col}(q, F) = M \left(\frac{d}{dt} \right) \ell$, with M equal to

$$M(\xi) = \begin{pmatrix} I \\ M\xi^2 + D\xi + K \end{pmatrix}$$

Obviously, due to damping, the system is dissipative. This indeed follows from the fact that $M^\top(-i\omega) \Phi(-i\omega, i\omega) M(i\omega) = (D + D^\top) \omega^2 \geq 0$. A storage function is given by $Q_\Psi(q, F) = \frac{1}{2} \left(\frac{dq}{dt} \right)^\top M \frac{dq}{dt} + \frac{1}{2} q^\top K q$. This corresponds to taking

$$\Psi(\zeta, \eta) = \frac{1}{2} \begin{pmatrix} K & 0 \\ 0 & \zeta \eta M \end{pmatrix}$$

Indeed, for all (q, F) satisfying (4.3) we have $\frac{d}{dt} \left(\frac{1}{2} \left(\frac{dq}{dt} \right)^\top M \frac{dq}{dt} + \frac{1}{2} q^\top K q \right) = F^\top \frac{dq}{dt} - \frac{1}{2} \left(\frac{dq}{dt} \right)^\top (D + D^\top) \frac{dq}{dt} \leq F^\top \frac{dq}{dt}$. It also follows that a dissipation function is given by $Q_\Delta(q, F) = \frac{1}{2} \left(\frac{dq}{dt} \right)^\top (D + D^\top) \frac{dq}{dt}$. This corresponds to taking

$$\Delta(\zeta, \eta) = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \zeta \eta (D + D^\top) \end{pmatrix}$$

Obviously, $(Q_\Phi, Q_\Psi, Q_\Delta)$ is matched on \mathfrak{B} .

5 State representations

A latent variable model $R' \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) x$ (with the latent variable denoted by x this time) is said to be a *state model* if whenever (w_1, x_1) and (w_2, x_2) are elements of the full behavior \mathfrak{B}_f , and $x_1(0) = x_2(0)$, then the concatenation $(w, x) := (w_1, x_1) \wedge (w_2, x_2)$ will also satisfy $R' \left(\frac{d}{dt} \right) w = M \left(\frac{d}{dt} \right) x$. Since this concatenation need not be \mathcal{C}^∞ , it need only be a weak solution, that is a solution in the sense of distributions.

Let $\mathfrak{B} \in \mathcal{L}^q$. A latent variable representation of \mathfrak{B} is called a *state representation* of \mathfrak{B} if it is a state model. Given $w_1, w_2 \in \mathfrak{B}$, to decide whether $w_1 \wedge w_2 \in \mathfrak{B}$, we can look at the value of the state variables x_1 and x_2 at time $t = 0$. If $x_1(0) = x_2(0)$, then $w_1 \wedge w_2 \in \mathfrak{B}$. In other words, to decide whether a future continuation is possible within \mathfrak{B} , not the whole past needs to be remembered, but only the *present* value of the state is relevant. Thus x parametrizes the memory of the system.

An important role is played by latent variable models of the form

$$Gw + Fx + E \frac{dx}{dt} = 0 \quad (5.1)$$

Here, E , F , and G are real constant matrices. The important feature of (5.1) is that it is an (implicit) differential equation containing derivatives of order at most one in x and zero in w . It was shown in [6] that any latent variable model of the form (5.1) is a state space model. Conversely, every state model $R(\frac{d}{dt})w = M(\frac{d}{dt})x$ is equivalent to a representation of the form (5.1) in the sense that their full behaviors \mathfrak{B}_f coincide. This means that state representations of a given \mathfrak{B} of the form (5.1) are in fact *all* state representations of \mathfrak{B} : given a state representation \mathfrak{B}_f of \mathfrak{B} , it will have a kernel representation of the type (5.1) and hence, without loss of generality we can assume that it is of this form. In the case of state models, we call x the *state* or the vector of *state variables*. The number of state variables, i.e., the size of x , is called the *dynamic order* of the model. This number is denoted by n .

6 Main results

In this section we show that storage functions can always be represented as quadratic functions of a state variable, and that dissipation functions can always be represented as quadratic functions of a state variable, jointly with the manifest variable of a given system.

We will first treat the case that $\mathfrak{B} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Assume that $(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q), Q_\Phi)$ is dissipative. It turns out that every storage function is a quadratic function of any state variable of a particular system \mathfrak{B}_Φ obtained from the dynamics of Φ . Also, every dissipation function is a quadratic function of any state variable, jointly with the manifest variable of this system \mathfrak{B}_Φ . We now explain what we mean by \mathfrak{B}_Φ . The system \mathfrak{B}_Φ is defined as follows. Let $\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma_\Phi M(\eta)$ be a canonical factorization of Φ , with $\Sigma_\Phi \in \mathbb{R}^{r \times r}$. Now, consider the system $\mathfrak{B}_\Phi \in \mathcal{L}^r$ (with manifest variable $v \in \mathbb{R}^r$) with image representation

$$v = M\left(\frac{d}{dt}\right)w \quad (6.1)$$

Theorem 6.1 : *Let $Gv + Fx + E\frac{dx}{dt} = 0$ be a state representation of \mathfrak{B}_Φ , with full behavior \mathfrak{B}_f . Let Q_Ψ be a storage function for $(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q), Q_\Phi)$. Then there exists $K = K^\top \in \mathbb{R}^{n \times n}$ such that $\text{col}(M(\frac{d}{dt})w, x) \in \mathfrak{B}_f$ implies $Q_\Psi(w) = x^\top Kx$. Furthermore, if Q_Δ is a dissipation function for $(\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^q), Q_\Phi)$, then there exists $L = L^\top \in \mathbb{R}^{(n+q) \times (n+q)}$ such that $\text{col}(M(\frac{d}{dt})w, x) \in \mathfrak{B}_f$ implies*

$$Q_\Delta(w) = \begin{pmatrix} x \\ v \end{pmatrix}^\top L \begin{pmatrix} x \\ v \end{pmatrix}$$

Next, we treat the general case. Let $\mathfrak{B} \in \mathcal{L}^q$ be an arbitrary controllable system. Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta]$. Assume that (\mathfrak{B}, Q_Φ) is dissipative. Also in this case, every storage function turns out to be a quadratic function of any state variable, and every dissipation function a quadratic function of any state variable, jointly with the manifest variable of a system \mathfrak{B}_Φ . This time, however, the system \mathfrak{B}_Φ is obtained by combining the dynamics of \mathfrak{B} and Φ . Again, let $\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma_\Phi M(\eta)$ be a canonical factorization of Φ , with $\Sigma_\Phi \in \mathbb{R}^{r \times r}$. Now, consider the system $\mathfrak{B}_\Phi \in \mathcal{L}^r$ (with manifest variable $v \in \mathbb{R}^r$) represented by

$$v = M\left(\frac{d}{dt}\right)w, \quad w \in \mathfrak{B} \quad (6.2)$$

Theorem 6.2 : *Let $Gv + Fx + E\frac{dx}{dt} = 0$ be a state representation of \mathfrak{B}_Φ , with full behavior \mathfrak{B}_f . Let Q_Ψ be a storage function for (\mathfrak{B}, Q_Φ) . Then there exists $K = K^\top \in \mathbb{R}^{n \times n}$ such that $w \in \mathfrak{B}$ and $\text{col}(M(\frac{d}{dt})w, x) \in \mathfrak{B}_f$ implies $Q_\Psi(w) = x^\top Kx$. If Q_Δ is a dissipation function for (\mathfrak{B}, Q_Φ) , then there exists $L = L^\top \in \mathbb{R}^{(n+d) \times (n+d)}$ such that $w \in \mathfrak{B}$ and $\text{col}(M(\frac{d}{dt})w, x) \in \mathfrak{B}_f$ implies*

$$Q_\Delta(w) = \begin{pmatrix} x \\ v \end{pmatrix}^\top L \begin{pmatrix} x \\ v \end{pmatrix}$$

Finally, we discuss the special case that the supply rate Q_Φ is first order in w , i.e., $Q_\Phi(w) = w^\top Pw$, with $P = P^\top \in \mathbb{R}^{q \times q}$. Let $\mathfrak{B} \in \mathcal{L}^q$ be controllable, and assume that (\mathfrak{B}, Q_Φ) is dissipative. In this case every storage function is simply a quadratic function of any state variable of \mathfrak{B} , and every dissipation function is a quadratic function of any state variable of \mathfrak{B} , jointly with the manifest variable of \mathfrak{B} .

Corollary 6.3 : *Let $Gw + Fx + E\frac{dx}{dt} = 0$ be a state representation of \mathfrak{B} , with full behavior \mathfrak{B}_f . Let Q_Ψ be a storage function for (\mathfrak{B}, Q_Φ) . Then there exists $K = K^\top \in \mathbb{R}^{n \times n}$ such that $\text{col}(w, x) \in \mathfrak{B}_f$ implies $Q_\Psi(w) = x^\top Kx$. If Q_Δ is a dissipation function for (\mathfrak{B}, Q_Φ) , then there there exists $L = L^\top \in \mathbb{R}^{(n+d) \times (n+d)}$ such that $\text{col}(w, x) \in \mathfrak{B}_f$ implies*

$$Q_\Delta(w) = \begin{pmatrix} x \\ w \end{pmatrix}^\top L \begin{pmatrix} x \\ w \end{pmatrix}$$

Example 6.4 : Consider the mechanical system (4.3) together with the supply rate Q_Φ . A canonical factorization of $\Phi(\zeta, \eta)$ is given by

$$\Phi(\zeta, \eta) = \frac{1}{2} \begin{pmatrix} \zeta & -\zeta \\ I & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \frac{1}{2} \begin{pmatrix} \eta & I \\ -\eta & I \end{pmatrix}$$

The corresponding system \mathfrak{B}_Φ (with manifest variable $v = \text{col}(v_1, v_2)$) is represented by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{dq}{dt} + F \\ -\frac{dq}{dt} + F \end{pmatrix}, M \frac{d^2q}{dt^2} + D \frac{dq}{dt} + Kq = F$$

It is easily seen that $\text{col}(\frac{dq}{dt}, q)$ is a state variable for \mathfrak{B}_Φ . It was indeed shown that a storage function is given by $Q_\Psi(q, F) = \frac{1}{2}(\frac{dq}{dt})^T M \frac{dq}{dt} + \frac{1}{2}q^T Kq$ and that a dissipation function is given by $Q_\Delta(q, F) = \frac{1}{2}(\frac{dq}{dt})^T (D + D^T) \frac{dq}{dt}$.

Example 6.5 : The relation between force F and position q due to a potential field $V(q)$ is given by $F = (\nabla V)(q)$. This defines a (in general nonlinear) system \mathfrak{B} with manifest variable $w = \text{col}(q, F)$. This system is dissipative (even lossless) with respect to the supply rate $Q_\Phi(q, F) = F^T \frac{dq}{dt}$, and $V(q)$ defines a storage function: $\frac{d}{dt} V(q) = (\nabla V)(q)^T \frac{dq}{dt} = F^T \frac{dq}{dt}$. The storage function $V(q)$ is a function of the position q . The question is now: in what sense is $V(q)$ a function of the state? For the case that \mathfrak{B} is linear, equivalently $V(q) = \frac{1}{2}q^T Kq$, $(\nabla V)(q) = Kq$, with $K = K^T$, the answer is provided by theorem 6.2: storage functions of \mathfrak{B} are quadratic functions of state variables of the system \mathfrak{B}_Φ represented by

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{dq}{dt} + F \\ -\frac{dq}{dt} + F \end{pmatrix}, F = Kq$$

It is easily seen that q is indeed a state variable for \mathfrak{B}_Φ .

References

- [1] F.M. Callier, On polynomial spectral factorization by symmetric extraction, *IEEE Transactions on Automatic Control*, volume 30, pages 453-464, 1985.
- [2] W.A. Coppel, *Linear Systems*, Notes in Pure Mathematics, Australian National University, volume 6, 1972.
- [3] C.A. Desoer and E.S. Kuh, *Basic Circuit Theory* McGraw-Hill, Tokyo, 1969.
- [4] D.J. Hill and P.J. Moylan, Stability of nonlinear dissipative systems, *IEEE Transactions on Automatic Control*, volume 21, pages 708-711, 1976.
- [5] M. Kuijper, *First-order Representations of Linear Systems*, Birkhäuser, Boston, 1994.
- [6] P. Rapisarda and J.C. Willems, State maps for linear systems, *SIAM Journal on Control and Optimization*, to appear.
- [7] H.L. Trentelman and J.C. Willems, The dissipation inequality and the algebraic Riccati equation *The Riccati Equation* (Eds. S. Bittanti, A.J. Laub, and J.C. Willems) Springer Verlag, pages 197-242, 1991.

[8] J.C. Willems, Dissipative dynamical systems - Part I: General theory, Part II: Linear systems with quadratic supply rates, *Archive for Rational Mechanics and Analysis*, volume 45, pages 321-351, 352-393, 1972.

[9] J.C. Willems, From time series to linear system. Part I: Finite dimensional linear time invariant systems ; Part II: Exact modelling, Part III: Approximate modelling, *Automatica*, volume 22, pages 561-580, 675-694, 1986, volume 23, pages 87-115, 1987.

[10] J.C. Willems, Models for dynamics, *Dynamics Reported*, volume 2, pages 171-269, 1989.

[11] J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Transactions on Automatic Control*, volume 36, pages 259-294, 1991.

[12] J.C. Willems and H.L. Trentelman, On quadratic differential forms, *preprint, submitted for publication*, 1996.