Proceedings of the 36th
Conference on Decision & Control
San Diego, California USA • December 1997

A behavioral framework for periodically time-varying systems

Margreet Kuijper
Dept. of EE Engineering
University of Melbourne
Parkville, Vic., 3052
Australia
m.kuijper@ee.mu.oz.au

Jan C. Willems
Dept. of Mathematics
University of Groningen
P.O. Box 800, 9700 AV Groningen
The Netherlands
j.c.willems@math.rug.nl

Abstract

We set out to develop a framework for the analysis and synthesis of discrete-time periodically time-varying systems. Adopting a behavioral approach, we define the concept of periodicity in terms of the trajectories of the system. We subsequently use this framework to investigate several basic notions, such as controllability, on the level of trajectories and also present several techniques for associating time-invariant systems in a behavioral way.

1 Introduction and basic definitions

Recall that in the behavioral framework a dynamical system Σ is defined as a triple $\Sigma = (\mathbf{T}, W, \mathcal{B})$, where $\mathbf{T}$ is the *time set*, W is the *space of external variables*, and $\mathcal{B}$ is a subset of $W^{\mathbf{T}}$ called the *behavior* of the system. In this paper we focus on $\mathbf{T} = \mathbf{Z}$ and $W = \mathbf{R}^q$. We define a system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ to be *P-periodic* if its behavior $\mathcal{B}$ satisfies

$$\sigma^P \mathcal{B} = \mathcal{B}.$$

Here $\sigma : (\mathbf{R}^q)^{\mathbf{Z}} \mapsto (\mathbf{R}^q)^{\mathbf{Z}}$ is the *backward shift*:

$$(\sigma w)(t) = w(t+1).$$

Of course, this also defines σ^k for $k \in \mathbf{Z}$. In particular, we call σ^{-1} the *forward shift*. As usual, we refer to a 1-periodic system as *time-invariant*.

Note that our definition of periodic systems does not involve a system representation. Our set-up is to first define concepts on a very general level in terms of the trajectories of the system and then derive specific results in terms of representations. The type of systems that we will focus on in section 3, is described by periodically time-varying difference equations

$$(R_t(\sigma, \sigma^{-1})w)(kP + t) = 0 \qquad t = 1, \ldots, P. \qquad (1)$$

Here $R_1, \ldots, R_p$ are polynomial matrices. More specifically, each of the R_t's belong to $\mathbf{R}^{\bullet \times q}[\xi, \xi^{-1}]$, the set of polynomial matrices with q columns. It is easy to see that a system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ for which $\mathcal{B}$ is represented by (1) defines a P-periodic system. We will call (1) a *kernel representation* of $\mathcal{B}$ and use the short notation $(R_1, \ldots, R_p)$ for (1). Note that we do not require the R_t's to have the same number of rows. In particular, (1) is a more general type of representation than the type of difference equations studied in [4].

In [6] the concept of controllability was defined as an intrinsic property of a dynamical system. The definition of [6] carries through to P-periodic systems:

Definition 1 The P-periodic system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ is said to be *controllable* if for all $t_0 \in \mathbf{Z}$ and $w_1, w_2 \in \mathcal{B}$, there exists $t_1 \in \mathbf{Z}, t_1 \geq 0$ and $w \in \mathcal{B}$, such that

$$\begin{aligned} w(t) &= w_1(t) \ \text{ for } t < t_0 \\ &= w_2(t - t_1) \ \text{ for } t \geq t_0 + t_1. \end{aligned}$$

$\diamond$

As explained in [6], the counterpart of a controllable system is an autonomous system. In an autonomous system the past of a trajectory completely determines its future. The following definitions from [6] carry over to P-periodic systems:

Definition 2 The P-periodic system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ is said to be *autonomous* if for all $t_0 \in \mathbf{Z}$

$$\{w_1, w_2 \in \mathcal{B} \text{ and } w_1(t) = w_2(t) \ \text{ for } t < t_0\} \implies \{w_1 = w_2\}.$$

$\diamond$

Definition 3 The P-periodic system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ is said to be *stable* if $w \in \mathcal{B}$ implies that

$$w(t) \longrightarrow 0 \ \text{ for } t \to \infty.$$

$\diamond$

0-7803-3970-8/97 $10.00 © 1997 IEEE

2 Lifting and twisting

With a P-periodic system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ we can associate the time-invariant system $\Sigma^L = (\mathbf{Z}, \mathbf{R}^{Pq}, L\mathcal{B})$ through the mapping

$$L : (\mathbf{R}^q)^{\mathbf{Z}} \mapsto (\mathbf{R}^{Pq})^{\mathbf{Z}},$$

defined by

$$(Lw)(t) = \begin{pmatrix} w(Pt+1) \\ \vdots \\ w(Pt+P) \end{pmatrix}.$$

The idea is that trajectories of a σ^P-invariant subspace $\mathcal{B}$ are "folded" into trajectories of the σ-invariant subspace $L\mathcal{B}$. The mapping L captures the idea of the so-called lifting technique, as presented in e.g. [1]. We denote the ***lifted system*** by $\Sigma^L := (\mathbf{Z}, \mathbf{R}^{Pq}, L\mathcal{B})$.

Below the concepts of continuity and closedness are taken as usual, that is, w.r.t. the topology of pointwise convergence. Recall (see e.g.[5]) that a *homeomorphism* is a bijective function h for which both h and h^{-1} are continuous. Recall also that a function h is called *closed* if the image under h of a closed subspace is again closed. Any homeomorphism is closed.

The following lemma is obvious.

Lemma 4 The mapping $L : (\mathbf{R}^q)^{\mathbf{Z}} \mapsto (\mathbf{R}^{Pq})^{\mathbf{Z}}$ defined above has the following properties:

(i) L is linear

(ii) $L\sigma = \begin{bmatrix} 0 & I & 0 & \cdots \\ \vdots & & \ddots & \\ & & & I \\ \sigma I & 0 & \cdots & \end{bmatrix} L$; consequently $L\sigma^P = \sigma L$

(iii) L is a homeomorphism; consequently L is closed

Theorem 5 Let $L : (\mathbf{R}^q)^{\mathbf{Z}} \mapsto (\mathbf{R}^{Pq})^{\mathbf{Z}}$ be defined as above. Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be a dynamical system. Then

(i) Σ is P-periodic $\Leftrightarrow$ Σ^L is time-invariant

(ii) $\mathcal{B}$ is linear $\Leftrightarrow$ $L\mathcal{B}$ is linear

(iii) $\mathcal{B}$ is closed $\Leftrightarrow$ $L\mathcal{B}$ is closed

(iv) $\mathcal{B}$ is autonomous $\Leftrightarrow$ $L\mathcal{B}$ is autonomous

(v) $\mathcal{B}$ is stable $\Leftrightarrow$ $L\mathcal{B}$ is stable

Proof: Properties (i)-(iii) follow immediately from the previous lemma. Properties (iv)-(v) can be easily verified from the definitions. ∎

Let us now present the alternative technique of [2, 3] for associating a time-invariant system with a periodically time-varying system. Define the mapping

$$H : ((\mathbf{R}^q)^{\mathbf{Z}})^P \mapsto (\mathbf{R}^{Pq})^{\mathbf{Z}},$$

by

$$H(w_1, \ldots, w_P)(Pk+t) := Q^t \begin{bmatrix} w_1(Pk+t) \\ w_2(Pk+t+1) \\ \vdots \\ w_P(Pk+t+P-1) \end{bmatrix},$$

where $t = 1, \ldots, P$ and

$$Q = \begin{bmatrix} 0 & \cdots & 0 & I \\ I & 0 & \cdots & \cdots \\ & \ddots & \ddots & \\ 0 & \cdots & I & 0 \end{bmatrix}. \tag{2}$$

The idea is that trajectories of $\mathcal{B}$ are "twisted" into trajectories of $H(\mathcal{B}^P)$. We denote the *twisted system* by $\Sigma^H := (\mathbf{Z}, \mathbf{R}^{Pq}, H(\mathcal{B}^P))$.

It has been shown in [2, 3] that P-periodicity of Σ implies that the twisted system Σ^H is time-invariant.

We now present the following theorem:

Theorem 6 Let $H : ((\mathbf{R}^q)^{\mathbf{Z}})^P \mapsto (\mathbf{R}^{Pq})^{\mathbf{Z}}$ be defined as above. Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be a dynamical system. Then $\mathcal{B}$ is controllable if and only if $H(\mathcal{B}^P)$ is controllable.

Proof: The "if" part is proved by twisting trajectories in $\mathcal{B}$ with, for example, zero trajectories. To prove the "only if" part, we note that two trajectories w_1 and $\tilde{w}_1$ in $\mathcal{B}$ can always be concatenated with two desired trajectories w_2 and $\tilde{w}_2$, in such a way that the corresponding time lapses t_1 and $\tilde{t}_1$ (see Definition 1) are equal. Indeed, this is achieved by steering to the zero trajectory as an intermediate step. The "only if" part now follows in a straightforward way. ∎

3 Representations

In this section we first concentrate on the question

> What requirements should a P-periodic system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ satisfy in order to have a representation of the type (1)?

Before stating the result, we investigate how the lifting L transforms the representation (1).

With every sequence of polynomial matrices $(R_1, \ldots, R_P)$, where the R_t's belong to $\mathbf{R}^{\bullet \times q}[\xi, \xi^{-1}]$, we can associate a polynomial matrix $R^L \in \mathbf{R}^{\bullet \times Pq}[\xi, \xi^{-1}]$, as follows. First decompose each of the R_t's $(t = 1, \ldots P)$ in terms of their powers ξ^k with k taken modulo P, i.e. write

$$R_t(\xi, \xi^{-1}) = R_t^0(\xi^P, \xi^{-P}) + \xi^{-1} R_t^1(\xi^P, \xi^{-P}) + \cdots + \xi^{-(P-1)} R_t^{P-1}(\xi^P, \xi^{-P}). \tag{3}$$

Now define R^L as

$$R^L(\xi, \xi^{-1}) := \begin{bmatrix} R_1^0(\xi, \xi^{-1}) & \xi^{-1} R_1^{P-1}(\xi, \xi^{-1}) & \cdots & \xi^{-1} R_1^1(\xi, \xi^{-1}) \\ R_2^1(\xi, \xi^{-1}) & R_2^0(\xi, \xi^{-1}) & \cdots & \xi^{-1} R_2^2(\xi, \xi^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ R_P^{P-1}(\xi, \xi^{-1}) & R_P^{P-2}(\xi, \xi^{-1}) & \cdots & R_P^0(\xi, \xi^{-1}) \end{bmatrix} \tag{4}$$

The following lemma is easily verified.

Lemma 7 A behavior $\mathcal{B} \subset (\mathbf{R}^q)^{\mathbf{Z}}$ is given by the kernel representation (1), abbreviated as $(R_1, \ldots, R_P)$, if and only if $L\mathcal{B}$ is given by the kernel representation

$$R^L(\sigma, \sigma^{-1}) w = 0.$$

$\diamond$

We are now ready for the main result of this section:

Theorem 8 The following are equivalent:

(i) $\mathcal{B}$ is a σ^P-invariant linear closed subspace of $(\mathbf{R}^q)^{\mathbf{Z}}$

(ii) there exist polynomial matrices $R_1, \ldots R_P$, such that (1) is a kernel representation for $\mathcal{B}$

Proof: To prove that (i) implies (ii), we note that it follows from Theorem 5 that (i) implies that $L\mathcal{B}$ is a σ-invariant linear closed subspace of $(\mathbf{R}^{Pq})^{\mathbf{Z}}$. By Theorem III-1 of [6] this implies that there exists a representation

$$R(\sigma, \sigma^{-1}) w = 0$$

for $L\mathcal{B}$, where R is a polynomial matrix with Pq columns. Now partition R as in (4) to obtain a kernel representation $(R_1, \ldots, R_P)$ for $\mathcal{B}$ (use Lemma 7). Vice versa, we have, by Lemma 7, that $L\mathcal{B}$ has a kernel representation if (ii) holds. By Th. III-1 of [6], $L\mathcal{B}$ then has to be σ-invariant, linear and closed, which implies (i) because of Theorem 5. $\blacksquare$

Note that a partitioning of a polynomial matrix with Pq columns into $(R_1, \ldots, R_P)$ is not necessarily unique, as the R_t's need not have the same number of rows.

Next, we investigate how the twisting operator H transforms a representation (1). Using the notation of (3), we define

$$R^H(\xi, \xi^{-1}) := \begin{bmatrix} R_1^0(\xi^P, \xi^{-P}) & \xi^{-(P-1)} R_1^{P-1}(\xi^P, \xi^{-P}) & \cdots & \xi^{-1} R_1^1(\xi^P, \xi^{-P}) \\ \xi^{-1} R_2^1(\xi^P, \xi^{-P}) & R_2^0(\xi^P, \xi^{-P}) & \cdots & \xi^{-2} R_2^2(\xi^P, \xi^{-P}) \\ \vdots & \vdots & \ddots & \vdots \\ \xi^{-(P-1)} R_P^{P-1}(\xi^P, \xi^{-P}) & \xi^{-(P-2)} R_P^{P-2}(\xi^P, \xi^{-P}) & \cdots & R_P^0(\xi^P, \xi^{-P}) \end{bmatrix} \tag{5}$$

and have the following lemma from [2, 3].

Lemma 9 A behavior $\mathcal{B} \subset (\mathbf{R}^q)^{\mathbf{Z}}$ is given by the kernel representation (1), abbreviated as $(R_1, \ldots, R_P)$, if and only if $H(\mathcal{B}^P)$ is given by the kernel representation

$$R^H(\sigma, \sigma^{-1}) w = 0.$$

$\diamond$

4 Characteristic polynomial

Our next aim is to formulate a concept of "characteristic polynomial" for an autonomous P-periodic system. For this, we first need the following lemma.

Lemma 10 Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be a P-periodic system. Then Σ is autonomous if and only if $\mathcal{B}$ is finite dimensional. Furthermore, if (1), abbreviated as $(R_1, \ldots, R_P)$, is a kernel representation for $\mathcal{B}$ then Σ is autonomous if and only if the matrix R^L defined in (4) satisfies

$$\text{rank } R^L = Pq.$$

Proof: Use Theorem 5 (iv) and Prop. V-7 from [6]. $\blacksquare$

As a result of the above lemma, for a linear autonomous P-periodic system $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$, the map $\sigma^P : \mathcal{B} \to \mathcal{B}$ is a linear map on a finite-dimensional space. We now have the following definition:

Definition 11 For a linear autonomous P-periodic system Σ we define the *characteristic polynomial* of Σ, denoted as χ_Σ, as the characteristic polynomial of the map $\sigma^P_{|\mathcal{B}}$.

Theorem 12 Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be an autonomous P-periodic system with $\mathcal{B}$ linear and closed. Then there exists a kernel representation (1), abbreviated as

$(R_1, \ldots, R_P)$, for which the matrix R^L defined in (1) is nonsingular. For any such representation we have

$$\chi_\Sigma = \det R^L. \tag{6}$$

Proof: The existence of a kernel representation (1) follows from Lemma 10 and Prop. V-7 from [6]. It is not difficult to see that the characteristic polynomial of the map $\sigma^P_{|\mathcal{B}}$ equals the characteristic polynomial of the map $\sigma_{|L\mathcal{B}}$ and thus (6) holds. ∎

Recall that a polynomial $p \in \mathbf{R}[\xi, \xi^{-1}]$ is called a *Schur polynomial* if for $\lambda \in \mathbf{C} \setminus \{0\}$ the identity $p(\lambda, \lambda^{-1}) = 0$ implies that $|\lambda| < 1$.

The following theorem is an immediate corollary of Theorem 5 (v) and Theorem 12.

Theorem 13 Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be an autonomous P-periodic system, represented by the kernel representation (1), abbreviated as $(R_1, \ldots, R_P)$. Let R^L be defined as in (4). Then Σ is stable if and only if det R^L is a Schur polynomial.

5 Controllability

In section 3 we investigated how the twisting operator H, defined in section 2, transforms a kernel representation. In this section we use these results to obtain a way to determine the controllability of a P-periodic system from its representation. The next theorem follows immediately from Theorem 6 and Theorem V-2 of [6].

Theorem 14 Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be a P-periodic system, represented by the kernel representation (1), abbreviated as $(R_1, \ldots, R_P)$. Let R^H be defined as in (5). Then Σ is controllable if and only if $R^H(\lambda, \lambda^{-1})$ has constant rank for all $\lambda \in \mathbf{C} \setminus \{0\}$.

In the following theorem we generalize the pole-placement result of [7] to P-periodic systems.

Theorem 15 Let $\Sigma = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B})$ be a P-periodic system that is not autonomous (but, for example, controllable). Let $\mathcal{B}_{des}$ be an autonomous σ^P-invariant desired behavior. Then there exists a controller $\Sigma_c = (\mathbf{Z}, \mathbf{R}^q, \mathcal{B}_c)$ such that

$$\mathcal{B}_{des} = \mathcal{B} \cap \mathcal{B}_c.$$

Proof: Since $L\mathcal{B}$ and $L\mathcal{B}_{des}$ are σ-invariant spaces, we can apply the theory on time-invariant systems of [6] to conclude that there exists a σ-invariant $\bar{\mathcal{B}}_c$ such that

$$L\mathcal{B} \cap \bar{\mathcal{B}}_c = L\mathcal{B}_{des}.$$

Now $L^{-1}\bar{\mathcal{B}}_c$ defines a σ^P-invariant controller. ∎

6 Conclusions

In this paper we have introduced and investigated several system theoretic notions for periodically time-varying systems on the level of the system's trajectories. We have also addressed the question: how do these notions express themselves in terms of a representation of the system? Here the type of representation used is more general than usually considered in the periodically time-varying literature. The type studied is the natural one that comes up in a behavioral framework. It is a topic of future research to investigate this type of representation in more detail as well as exploit the presented "lifting" and "twisting" techniques further.

7 Acknowledgement

The first author acknowledges the support of the Australian Research Council.

References

[1] P.P. Khargonekar, K. Poolla and A. Tannenbaum, "Robust control of linear time-invariant plants using periodic compensation", *IEEE Trans. Automat. Control*, **30** (1985), 1088-1096.

[2] M. Kuijper, "How to obtain a periodically time-varying difference equation", *Proceedings of the 2nd Asian Control Conference* (ASCC, Seoul, Korea, 1997) II-143-146.

[3] M. Kuijper, "How to construct a periodically time-varying system of minimal lag: an algorithm based on twisting", submitted (1997).

[4] R.A. Meyer and C.S. Burrus, "A unified analysis of multirate and periodically time-varying digital filters", *IEEE Trans. Circuits and Systems*, **22** (1975) 162-168.

[5] B.T. Sims, *Fundamentals of Topology*, (MacMillan Publishing, 1976).

[6] J.C. Willems, "Paradigms and puzzles in the theory of dynamical systems", *IEEE Trans. Automat. Control*, **36** (1991) 259-294.

[7] J.C. Willems, "On interconnections, control, and feedback", *IEEE Trans. Automat. Control* **42** (1997) 326-339.