Proceedings of the 34th
Conference on Decision & Control
New Orleans, LA - December 1995

FM15 2:10

A SIMPLE PROOF THAT $n < m * p$ IMPLIES GENERIC EIGENVALUE ASSIGNABILITY BY REAL MEMORYLESS OUTPUT FEEDBACK*

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Abstract

In this talk we will give a simple proof of the remarkable result by Alex Wang [6] which states that if $n < m * p$, then generically the controlled eigenvalues can be assigned arbitrarily by real memoryless output feedback in a linear system with n states, m inputs, and p outputs.

Keywords: eigenvalue assignment, output feedback, behavioral systems, linear systems, memoryless controllers.

1 Introduction

Determining conditions for generic eigenvalue assignability by real memoryless output feedback has been one of the nagging puzzles in linear system theory over the last 25 years. We will not review its history here. Kimura [4] has given a nice account of this problem in the historical session at the 1994 CDC. For an in depth discussion of the status of this problem up to the time of publication, is given in Byrnes [2].

Let n be be number of states, m the number of inputs, and p the number of outputs of a linear time-invariant system. That $n \leq m * p$ is a necessary condition for generic eigenvalue assignability is easy to see by counting the number of equations and unknowns. That $n \leq m * p$ is sufficient over $\mathbb{C}$ has been shown in [3]. In [2] some very special cases, in addition to m or $p = 1$, are given for $n \leq m * p$ to be sufficient over $\mathbb{R}$: $m = 2$ and $p = 2^r - 1$, or $m = 2^r - 1$ and $p = 2$, for some $r \in \mathbb{N}$. That $n \leq m*p$ is not sufficient in general follows from [8], where it is shown that $m = p = 2$; $n = 4$ does not imply generic assignability over $\mathbb{R}$.

The main result about this problem undoubtedly the one by Alex Wang [6] where he proved, using rather unaccessible mathematics, that $n < m * p$ is a sufficient condition over $\mathbb{R}$: *one mere additional degree of freedom compared to* $\mathbb{C}$ *suffices* ! In [5] an elementary proof of this result has been given, exploiting a crucial idea provided again by Alex Wang [7]. By working completely in an (A,B,C)-setting and side-stepping behavioral thinking, the proof in [5], while elementary, turned out to be not particularly transparent. The purpose of the present note is to provide an elementary and simple proof, based on behavioral thinking, that $n < m * p$ is sufficient over $\mathbb{R}$.

Throughout the paper we will assume that the time functions under consideration are infinitely differentiable. A subset $S \subseteq \mathbb{R}^N$ will be called an *algebraic variety* if there exists a real polynomial in N variables such that S coincides with the zero set of this polynomial. If an algebraic variety S is a strict subset of $\mathbb{R}^N$, then it will be called *proper*. The complement of a proper algebraic variety is called *generic:* it is open, dense, and measure exhausting (meaning that its complement has Lebesgue measure zero).

2 The system

Consider the linear time-invariant system

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx \tag{2.1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$, and $(A,B,C) \in \mathbb{R}^{n^2+nm+pn}$. Let $\mu := ENT(\frac{n}{p})$, the smallest integer $\geq n/p$, $p_2 := \mu p - n$, and $p_1 := p - p_2$. When n is divisible by p, then $p_2 = 0$. In this case the vectors and matrices of size p_2 are assumed to be absent. It is well-

*Abbreviated version of a paper that will appear in *Communications, Computing, Control and Signal Processing 2000*, Festschrift on the Occasion of the Sixtieth Birthday of Thomas Kailath, Kluwer Publications.

known that the generic system (2.1) has observability indices

$$(\mu_1, \mu_2, \cdots, \mu_p) \tag{2.2}$$

with, if n is divisible by p, $\mu_1 = \cdots = \mu_p = \mu$, while, if n is not divisible by p, $\mu_1 = \cdots = \mu_{p_1} = \mu$ and $\mu_{p_1+1} = \cdots = \mu_p = \mu - 1$. It follows from [11] that the input/output behavior of the system (2.1) with the generic observability indices (2.2) will have the following *kernel representation*

$$R(\frac{d}{dt})w = 0, \qquad w := \begin{bmatrix} y \\ \hline u \end{bmatrix} \tag{2.3}$$

in which the polynomial matrix $R \in \mathbb{R}^{p\times(p+m)}[\xi]$ has the following structure

$$R_0 w + R_1 \frac{dw}{dt} + \cdots + R_{\mu-1}\frac{d^{\mu-1}w}{dt^{\mu-1}} + R_\mu \frac{d^\mu w}{dt^\mu} = 0 (2.4)$$

and $R_k \in \mathbb{R}^{p\times(p+m)}, k = 0, 1, \cdots, \mu$, and with $R_{\mu-1}$ and R_μ of the following special form

$$R_\mu = \begin{bmatrix} I_{p_1\times p_1} & O_{p_1\times p_2} & O_{p_1\times m} \\ O_{p_2\times p_1} & O_{p_2\times p_2} & O_{p_2\times m} \end{bmatrix} \tag{2.5}$$

$$R_{\mu-1} = \begin{bmatrix} * & O_{p_1\times p_2} & * \\ * & I_{p_2\times p_2} & O_{p_2\times m} \end{bmatrix} \tag{2.6}$$

The $*$'s in (2.6) and $R_0, \cdots, R_{\mu-2}$ in (2.4) are matrices without special structure. The total number of free parameters in $R_0, R_1, \ldots, R_{p-1}, R_p$, and hence in the associated differential equation (2.3), is thus $n(m+p)$. By a slight abuse of notation, we will therefore write $(R_0, R_1, \cdots, R_\mu) \in \mathbb{R}^{n(m+p)}$. We will denote the family of linear systems described by (2.3) with $R \in \mathbb{R}^{p\times(p+m)}[\xi]$ of the special form (2.4, 2.5, 2.6) by Σ. Hence $\Sigma \cong \mathbb{R}^{n(m+p)}$. This makes it clear what genericity in Σ signifies.

3 Feedback and interconnection

Now consider the linear memoryless output feedback law

$$u = Fy \tag{3.1}$$

Applied to (2.1), this yields the closed loop system given by

$$\frac{dx}{dt} = (A + BFC)x \tag{3.2}$$

The associated characteristic polynomial is

$$det(I\xi - A - BFC) \tag{3.3}$$

An easy calculation shows that the characteristic polynomial (3.3) equals

$$det\begin{bmatrix} R(\xi) \\ \hline -F \mid I_{m\times m} \end{bmatrix} \tag{3.4}$$

The eigenvalue assignability problem by memoryless output feedback is formulated as follows. Let (2.1) be given and let $d \in \mathbb{R}[\xi]$,

$$d(\xi) = d_0 + d_1\xi + \cdots + d_{n-1}\xi^{n-1} + \xi^n \tag{3.5}$$

be a given monic polynomial. The question is when there exits an $F \in \mathbb{R}^{p\times m}$ such that the characteristic polynomial associated with (3.2) equals d, i.e., such that

$$det(I\xi - A - BFC) = d(\xi) \tag{3.6}$$

Note that we have just shown that this is equivalent to

$$det\begin{bmatrix} R(\xi) \\ \hline -F \mid I_{m\times m} \end{bmatrix} = d(\xi) \tag{3.7}$$

Persons familiar with polynomial representations of linear systems could take (3.7) as the starting point of this paper: (2.4, 2.5, 2.6) is a canonical representation of a linear system with McMillan degree n and generic observability indices (2.2). The eigenvalue placement question is then simply formulated by (3.7), and we will prove that for generic elements in Σ, (3.7) is solvable for all d's of degree $\leq n$ if $n < m * p$.

4 Regular, singular, and dependent controllers

We will now consider control in a behavioral setting, in the spirit of [9,10]. Let (2.3) denote the *plant* and consider as *controller* the memoryless linear time-invariant dynamical system

$$Kw = 0 \tag{4.1}$$

with $K \in \mathbb{R}^{m\times(p+m)}$. This yields the *controlled system*

$$\begin{bmatrix} R(d/dt) \\ \hline K \end{bmatrix} w = 0 \tag{4.2}$$

Call the real monic polynomial whose roots coincide with those of

$$det\begin{bmatrix} R(\xi) \\ \hline K \end{bmatrix} \tag{4.3}$$

the *characteristic polynomial* of (4.2). Note that its degree does not exceed n, the McMillan degree of R. We will say that for a plant $R \in \Sigma$ *eigenvalue assignment* *by real memoryless feedback* is possible if for all monic polynomials $\pi \in \mathbb{R}[\xi]$, with *degree* $(\pi) \leq n$, there exists a K such that (4.2) has characteristic polynomial π.

Examine the characteristic polynomial (4.3) and partition K as

$$K = [K_1 \mid K_2] \tag{4.4}$$

with $K_1 \in \mathbb{R}^{m\times p}$ and $K_2 \in \mathbb{R}^{m\times m}$. Observe that (4.3) has degree exactly n if and only if K_2 is invertible. We call a controller (4.1) *regular* if (4.3) has degree n; *singular* if it has degree $< n$; and *dependent* if it is zero. In [1] dependent controllers are called *non-admissible.* It may be surprising that dependent controllers will play a crucial role in the sequel.

A regular control law (4.4) is obviously equivalent to the feedback control law of the form (3.1) given by

$$[K_2^{-1}K_1 \mid I\,]w = 0 \tag{4.5}$$

i.e.,

$$u = -K_2^{-1}K_1 y \tag{4.6}$$

Regular control laws lead to polynomials (4.3) which are monic and of degree n.

5 The main result

The main result of this paper is the

Theorem 1: *Assume that $n < m * p$. Then generically for $R \in \Sigma \cong \mathbb{R}^{n(m+p)}$ eigenvalue assignment by real memoryless feedback is possible.*

Proof: (i) In the first step of the proof we will examine the expansion of (4.3) as a power series in K around a point K_0. Let $R \in \Sigma$ be given. Consider the map h which assigns to K the polynomial (4.3). Thus h is a map from $\mathbb{R}^{m(m+p)}$ to $\mathbb{R}^{n+1}$ (associate the coefficients of the polynomial (4.3) – a polynomial of degree at most n – with a vector in $\mathbb{R}^{n+1}$); h is, in fact, a polynomial map.

Consider the power series expansion of h at the point $K_0 \in \mathbb{R}^{m\times(p+m)}$. There holds

$$h(K_0+\Delta)(\xi) = det\begin{bmatrix} R(\xi) \\ \hline K_0\end{bmatrix} + \sum_{k=1}^{m}\sum_{\ell=1}^{m+p} \Delta_{k\ell}M_{k\ell}(\xi) + h.o.t. \tag{5.1}$$

where $\Delta_{k\ell}$ denotes the (k,ℓ)-th element of Δ, $M_{k\ell}$ equals $(-1)^{p+k+\ell}$ times the minor obtained by crossing out the $(p+k)$-th row and ℓ-th column of $\begin{bmatrix} R(\xi) \\ \hline K_0\end{bmatrix}$, and $h.o.t.$ means quadratic or higher order terms in the elements of Δ.

We will now show that R has the eigenvalue assignability property if K_0 has the following properties

$$\text{(a)} \quad det\begin{bmatrix} R(\xi) \\ \hline K_0\end{bmatrix} = 0 \tag{5.2}$$

i.e., K_0 is a dependent controller, and

(b) the polynomials $M_{k\ell}(\xi)$, $k = 1,\ldots,m; \ell = 1,\ldots,m+p$, span the $(n+1)$-dimensional vector space of real polynomials of degree $\leq n$.

Indeed, by the implicit function theorem, (a) and (b) combined imply (since the linear part of h is surjective at K_0) that the image of h contains an open neighborhood of the origin. However, if the image of h contains the polynomial χ, then it also contains $\alpha\chi$ (simply premultiply K by the diagonal matrix diag$(\alpha, 1, \cdots, 1)$ and examine (4.2)). Hence (a) and (b) imply that h is surjective.

(ii) In the second step of the proof, we will the existence of a dependent controller K_0, with K_0 of rank m. The row degrees of $R \in \Sigma$ are equal to the observability indices (2.2). Note that $n < m * p$ implies $\mu_p < m$. We will prove that $\mu_p < m$ in turn implies the existence of a rank m $K_0 \in \mathbb{R}^{m\times(p+m)}$ such that (5.2) holds. Indeed, we can associate with each $R \in \Sigma$ such a dependent controller K_0 in a canonical way, as follows.

Let the vector polynomial

$$r_p^0 + r_p^1\xi + \cdots + r_p^{\mu_p-1}\xi^{\mu_p-1} + r_p^{\mu_p}\xi^{\mu_p} \tag{5.3}$$

denote the p-th (i.e., the last) row of R. Take

$$K_0 = \begin{bmatrix} r_p^0 \\ r_p^1 \\ \vdots \\ r_p^{\mu_p-1} \\ r_p^{\mu_p} \\ e_{p+\mu_p+1} \\ \vdots \\ e_{m+p-1}\end{bmatrix} \tag{5.4}$$

where e_k equals the row vector $[0,\cdots,0,1,0,\cdots,0]$ with the 1 in the k-th entry.

It is obvious, by examining (5.3) and (5.4), that (5.2) will indeed hold for the K_0 given by (5.4).

(iii) The third part of the proof consists in showing that condition (b) of part (i) is generically satisfied for the K_0 given by (5.4). In other words, we will prove that $n < m * p$ implies that the K_0 given by (5.4) will yield polynomials $M_{k\ell}(\xi)$ which, generically for Σ, span the polynomials of degree $\leq n$.

In order to see this, observe first that the set of R's in Σ such that the $M_{k\ell}$'s do not span, forms an algebraic variety in $\Sigma \cong \mathbb{R}^{n(p+m)}$. Indeed, the K_0 given by (5.4) is a linear function of R; the coefficients of the $M_{k\ell}$'s are polynomial functions of the coefficients of R; non-spanning thus requires a certain polynomial in the coefficients of R to be zero. The algebraic variety in question is a proper one, for the following polynomial matrix R (suggested to me by Joachim Rosenthal) is an exception

$$\begin{bmatrix} \xi^{\mu_1} & 1 & 0 \cdots 0 & 0 & 0 & \cdots 0\,0 \cdots 0 \\ 0 & \xi^{\mu_2} & 1 \cdots 0 & 0 & 0 & \cdots 0\,0 \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots \xi^{\mu_{p-1}} & 0 & 0 & \cdots 0\,0 \cdots 1 \\ 0 & 0 & 0 \cdots 0 & \xi^{\mu_p} & \xi^{\mu_p - 1} & \cdots 1\,0 \cdots 0 \end{bmatrix} \tag{5.5}$$

The corresponding K_0 is

$$[O_{m\times(p-1)} \; I_{m\times m} \; O_{m\times 1}] \tag{5.6}$$

and the set of $M_{k\ell}(\xi)$'s contains the monomials

$$\begin{array}{l} 1, \xi, \cdots, \xi^{\mu_p} \\ \xi^{\mu_1}, \xi^{\mu_1+1}, \ldots, \xi^{\mu_1+\mu_p} \\ \xi^{\mu_1+\mu_2}, \xi^{\mu_1+\mu_2+1}, \ldots, \xi^{\mu_1+\mu_2+\mu_p} \\ \quad\vdots \\ \xi^{\mu_1+\cdots+\mu_{p-1}}, \xi^{\mu_1+\cdots+\mu_{p-1}+1}, \ldots, \xi^{\mu_1+\cdots+\mu_{p-1}+\mu_p} \end{array} \tag{5.7}$$

These obviously span the polynomials of degree $\leq n$. This completes the proof of Theorem 1. ∎

Note the surprisingly crucial role played in the proof by the dependent controller (5.4), whose existence was guaranteed by the fact that μ_p is less than m!

6 Memoryless output feedback in state models

The generic set of R's from Σ for which h is surjective will also have the property that for each real *monic* polynomial π of degree n, there exists a control law of the type

$$[-F \mid I\,]w = 0 \tag{6.1}$$

i.e.,

$$u = Fy \tag{6.2}$$

achieving π as characteristic polynomial. By (3.4) this π will also be equal to (3.3). Together with the observation that the map that associates with $(A, B, C) \in \mathbb{R}^{n^2+nm+pn}$ the system $(R_0, R_1, \cdots, R_\mu) \in \Sigma \cong \mathbb{R}^{n(m+p)}$ is rational, we thus obtain Wang's result [6] as a direct consequence of Theorem 1.

Theorem 2: *Assume that $n < m*p$. Then generically for $(A, B, C) \in \mathbb{R}^{n^2+nm+pn}$ there exists, for each real monic polynomial π of degree n, and $F \in \mathbb{R}^{m\times p}$ such that*

$$\pi(\xi) = det(I\xi - A - BFC) \tag{6.3}$$

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