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H_∞ CONTROL IN A BEHAVIORAL CONTEXT

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1 Introduction

In the standard formulation of the H_∞ control problem the aim is to design a feedback loop around a given system in such a way that in the closed loop system the influence of the exogenous inputs on the exogenous outputs remains within a certain a priori given tolerance. The system under consideration typically has control inputs, exogenous inputs, measured outputs, and exogenous outputs. The controllers to be designed should take the measured outputs of the system as its inputs, and should, on the basis of these inputs, generate control inputs for the system. These controllers should be designed in such a way that the resulting closed loop operator (mapping exogenous inputs to exogenous outputs) has norm less than or equal to some a priori given upper bound.

Recently, it has been argued that in many cases it is more natural to view the problem of controller design as the problem of designing for a given system an additional set of 'laws' that the the signals appearing in the system should obey. More specifically, if a system is given in terms of a certain set of 'behavioral equations', then the problem of controller design is to invent an additional set of equations, involving the signals appearing in the system, in such a way that the 'controlled system' (i.e., the system consisting of those signals that are compatible with both set of equations) satisfies the a priori given control specifications, see e.g. [2], [4].

Of course, if we compare these to set-ups, the main difference is that in the behavioral set-up the controllers to be designed are no longer required to have the particular causality structure imposed by the requirement that the control inputs should be generated on the basis of measured outputs. This clearly has advantages if we want to control systems involving signals in which an a priori subdivision into control signals and measured output signals is unnatural.

In this note, we want to reformulate and study the H_∞ control problem in the behavioral framework, i.e., in the framework one gets by taking the alternative point of view to controller design as explained above.

2 Feedback in a behavioral setting

In this paper we consider linear time-invariant differential dynamical systems, i.e., the set of systems $\Sigma = \{\mathbb{R}, \mathbb{R}^q, \mathcal{B}\}$, with time set $\mathbb{R}$, signal space $\mathbb{R}^q$, and whose behavior is equal to the solution set of a system of differential equations of the form

$$R(\frac{d}{dt})w = 0,$$

where $R \in \mathbb{R}^{\bullet \times q}[\xi]$ is a real polynomial matrix in the indeterminate ξ with q columns. More precisely, the behavior of Σ is defined by

$$\mathcal{B} := \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0\}.$$

Here, $C^\infty(\mathbb{R}, \mathbb{R}^q)$ denotes the set of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^q$. The implied smoothness is impose for convenience only. The dynamical system Σ is called *controllable* if its behavior has the property that for any pair of trajectories $w_1, w_2 \in \mathcal{B}$, there exists a trajectory $w \in \mathcal{B}$ and $T \geq 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t+T) = w_2(t)$ for $t > 0$. In this paper, we will restrict ourselves to controllable systems. It can be shown (see [3]) that a system is controllable if and only if there exists a polynomial matrix $M \in \mathbb{R}^{q \times \bullet}[\xi]$, with q rows and l columns, such that $\mathcal{B}$ is given by

$$\mathcal{B} = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid w = M(\tfrac{d}{dt})\ell \text{ for some } \ell \in C^\infty(\mathbb{R}, \mathbb{R}^l)\}. \quad (2.1)$$

Such a representation of a linear dynamical system Σ is called an *image represenation* of Σ. It can be shown that if Σ has an image representation, or equivalently: if Σ is controllable, then the polynomial matrix M can be chosen such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. This property of M can be proven to be equivalent to *observability* of the representation 2.1, see [3].

We will now addres the issue of controlling the system Σ given by the image represenation 2.1. In this paper, a *controller* for Σ will be any system of equations

$$C(\frac{d}{dt})\ell = 0, \quad (2.2)$$

where $C \in \mathbb{R}^{\bullet \times l}[\xi]$ is a polynomial matrix with l columns. If we combine the original system equations $w = M(\frac{d}{dt})\ell$ with the equations 2.2 of the controller, we obtain the behavior $\mathcal{B}_C$ of the controlled system:

$$\mathcal{B}_C = \{w \in C^\infty(\mathbb{R}, \mathbb{R}^q) \mid w = M(\tfrac{d}{dt})\ell \text{ for some } \ell \in \ker C(\frac{d}{dt})\}, \quad (2.3)$$

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with $C(\frac{d}{dt}$ viewed as acting on $C^\infty(\mathbb{R},\mathbb{R}^l)$. In general, a controller design problem can now be formulated as: for the system Σ, design a set of equations 2.2 (equivalently: invent a polynomial matrix C) such that the behavior 2.3 of the controlled system satisfies certain a priori given design specifications. In this note, these design specifications will be 'H_∞-like' specifications.

3 H_∞ control in a behavioral setting

Consider the system Σ with behavior in image representation given by 2.1. Without loss of generality, assume that the representation is observable. We assume that the manifest variable w consists of two components,

$$w = \begin{pmatrix} z \\ d \end{pmatrix},$$

where z has the interpretation of a signal that we want to keep small in an appropriate sense, and where d is interpreted as an unknown disturbance signal that acts on the system. The signal d is assumed to take its values in $\mathbb{R}^r$, while the signal z takes its values in $\mathbb{R}^{q-r}$. Accordingly, we partition

$$M = \begin{pmatrix} N \\ D \end{pmatrix}$$

into polynomial matrices $N \in \mathbb{R}^{(q-r)\times l}[\xi]$ and $D \in \mathbb{R}^{r\times l}[\xi]$. Thus, we will be dealing with the system Σ given by

$$\begin{pmatrix} z \\ d \end{pmatrix} = \begin{pmatrix} N(\frac{d}{dt}) \\ D(\frac{d}{dt}) \end{pmatrix} \ell. \tag{3.1}$$

The interpretation of d being an unknown disturbance signal can be formalized by assuming that all (smooth) functions d can occur as the second component of the manifest variable w. The signal d is then called *free.* This is equivalent with the condition that the polynomial matrix D has full row rank as a matrix with entries in the field of real rational functions $\mathbb{R}(\xi)$.

Now, let $C \in \mathbb{R}^{\bullet\times l}[\xi]$ be a polynomial matrix with l columns, and consider the controller $C(\frac{d}{dt})\ell = 0$. In the sequel, we will simply call this 'the controller C'. Since d is interpreted as an unknown disturbance signal, we should assume that our controllers are not allowed to put restrictions on the signal d: in the controlled system, d should *remain a free variable.* It is easily seen that this requirement on C is equivalent to the condition that the polynomial matrix $\begin{pmatrix} D \\ C \end{pmatrix} \in \mathbb{R}^{\bullet\times l}[\xi]$ has full row rank as a matrix with entries in $\mathbb{R}(\xi)$. A controller C which has this property will be called *admissible.* In particular, an admissible controller has *at most* $l-r$ rows. For a given admissible controller C, the behavior of the controlled system is given by

$$\mathcal{B}_C = \{ \begin{pmatrix} z \\ d \end{pmatrix} \in C^\infty(\mathbb{R},\mathbb{R}^q) \mid \begin{pmatrix} z \\ d \end{pmatrix} = \begin{pmatrix} N(\frac{d}{dt}) \\ D(\frac{d}{dt}) \end{pmatrix}\ell \text{ for some } \ell \in \ker C(\frac{d}{dt}) \cap C^\infty(\mathbb{R},\mathbb{R}^l)\}.$$

In the sequel, let $\mathcal{L}_2^q(\mathbb{R})$ be the set of all $\mathbb{R}^q$ valued functions that are square integrable over $\mathbb{R}$. For a given vector valued function f, we denote its $\mathcal{L}_2$ norm by $\|f\|_2$. The idea of H_∞ control is to reduce the influence of the disturbance signals on the signals to be controlled. A controller C will be called a *contracting controller* if it is admissible and if the associated controlled behavior satisfies the following property:

$$\|z\|_2 \leq \|d\|_2 \text{ for all } \begin{pmatrix} z \\ d \end{pmatrix} \in \mathcal{B}_C \cap \mathcal{L}_2^q(\mathbb{R}). \tag{3.2}$$

The H_∞ control problem for the system Σ is the problem of finding a contracting controller. In addition, we will consider the H_∞ control problem with internal stability. A controller C will be called *internally stabilizing* if for the associated controlled behavior we have:

$$\begin{pmatrix} z \\ 0 \end{pmatrix} \in \mathcal{B}_C \implies \lim_{t\to\infty} z(t) = 0,$$

i.e., if from $d = 0$ we may concude that $z(t)$ tends to 0 as t runs off to infinity. For a given system Σ, the H_∞ control problem with internal stability is to find an internally stabilizing, contracting controller.

4 H_∞ control without internal stability

In this section we will study the problem of finding contracting controllers for a given system. We will first show that a necessary condition for a controller C to be contracting is, that the polynomial matrix $\begin{pmatrix} D \\ C \end{pmatrix}$ is *nonsingular* as a matrix with entries in $\mathbb{R}(\xi)$. In particular, this implies that a contracting controller C must have *exactly* $l-r$ rows. In order to show this, we need to show that $\begin{pmatrix} D \\ C \end{pmatrix}$ has full column rank. If this matrix fails to have full column rank, then there exists a nonzero $\ell \in C^\infty(\mathbb{R},\mathbb{R}^l)$ such that $C(\frac{d}{dt})\ell = 0$ and $d = D(\frac{d}{dt})\ell = 0$. By observability, we must have $z = N(\frac{d}{dt})\ell \neq 0$, which violates the contractiveness condition 3.2.

In the following, let $\mathcal{D}^q$ denote the subset of $C^\infty(\mathbb{R},\mathbb{R}^q)$ consisting of those functions that have compact support. By a density argument, we can show that 3.2 holds if and only if it holds for all $\begin{pmatrix} z \\ d \end{pmatrix} \in \mathcal{B}_C \cap \mathcal{D}^q$. Taking Fourier transforms, it follows that 3.2 is equivalent to the following condition:

$$\|N(i\omega)v\|^2 \leq \|D(i\omega)v\|^2 \text{ for all } v \in \ker C(i\omega), \text{ for all } \omega \in \mathbb{R}. \tag{4.1}$$

An important role in this paper is played by symmetric two-variable polynomial matrices (see [1], [5]). In particular, the following symmetric two-variable polynomial will play an central role:

$$\Phi(\zeta,\eta) := N^{\mathrm{T}}(\zeta)N(\eta) - D^{\mathrm{T}}(\zeta)D(\eta).$$

Then 4.1 is equivalent to

$$v^*\partial\Phi(i\omega)v \leq 0 \text{ for all } v \in \ker C(i\omega), \text{ for all } \omega \in \mathbb{R}, \quad (4.2)$$

where we define $\partial\Phi(\xi) := \Phi(-\xi, \xi)$. Thus we obtain the following result:

Lemma 4.1 *: Let $C \in \mathbb{R}^{\bullet \times l}[\xi]$. C is a contracting controller if and only if $\begin{pmatrix} D \\ C \end{pmatrix}$ is non-singular as a matrix with entries in $\mathbb{R}(\xi)$, and $v^*\partial\Phi(i\omega)v \leq 0$ for all $v \in \ker C(i\omega)$, for all $\omega \in \mathbb{R}$.*

Since a contracting controller C has exactly $l - r$ rows, we know that for all $\omega \in \mathbb{R}$ the dimension of $\ker C(i\omega)$ as a subspace of $\mathbb{C}^l$ satisfies $\dim\ker C(i\omega) \geq r$. For any given hermitian matrix M, let $\nu(M)$, $\zeta(M)$, and $\pi(M)$ denote the number (counting multiplicity) of negative, zero, and positive eigenvalues, respectively. The triple $(\nu(M), \zeta(M), \pi(M)$ is called the signature of M. It follows from 4.2 that if there exists a contracting controller, then we have

$$\nu(\partial\Phi(i\omega)) + \zeta(\partial\Phi(i\omega)) \geq r.$$

On the other hand, for all $v \in \ker D(i\omega)$ we have

$$v^*\partial\Phi(i\omega)v = \|N(i\omega)v\|^2 \geq 0.$$

Since $D(i\omega)$ has r rows, we have $\dim\ker D(i\omega) \geq l - r$, so

$$\pi(\partial\Phi(i\omega)) + \zeta(\partial\Phi(i\omega)) \geq l - r.$$

Now, as a standing assumption on the systems Σ under consideration, we are going to assume that $\partial\Phi(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$, equivalently that $\partial\Phi(\xi)$ has no zeros on the imaginary axis. Under this assumption we can conclude from the above that a necessary condition for the existence of a contracting controller is, that $\nu(\partial\Phi(i\omega) = r$ for all $\omega \in \mathbb{R}$, i.e., the number of negative eigenvalues of $\partial\Phi(i\omega)$ is constant and equal to the number of components of the disturbance signal. The following result states that this condition is also sufficient for the existence of a contracting controller:

Theorem 4.2 *: There exists a contracting controller if and only if $\nu(\partial\Phi(i\omega)) = r$ for all $\omega \in \mathbb{R}$.*

Proof : If $\nu(\partial\Phi(i\omega)) = r$ for all $\omega \in \mathbb{R}$, then $\partial\Phi(i\omega)$ has constant signature $(r, 0, l - r)$. Let J denote the signature matrix

$$J := \begin{pmatrix} I_{l-r} & 0 \\ 0 & -I_r \end{pmatrix}.$$

It is well known that the polynomial matrix $\partial\Phi(\xi)$ has a *J-spectral factorization*

$$\partial\Phi(\xi) = F^T(-\xi)JF(\xi),$$

with $F \in \mathbb{R}^{l \times l}[\xi]$. Now partition $F = \begin{pmatrix} C \\ D \end{pmatrix}$, with $C \in \mathbb{R}^{l-r \times l}[\xi]$ and $D \in \mathbb{R}^{r \times l}[\xi]$. Then obviously $\partial\Phi(i\omega) = C^T(-i\omega)C(i\omega) - Z^T(-i\omega)Z(i\omega)$. It can be verified that $\begin{pmatrix} D \\ C \end{pmatrix}$ is non-singular. Also, for all ω and for all $v \in \ker C(i\omega)$ we have $v^*\partial\Phi(i\omega)v = -\|Z(i\omega)v\|^2 \leq 0$, so C is contracting. ■

It follows from the above proof that any J spectral factorization of $\partial\Phi(\xi)$ yields a contracting controller. It is important to note however that, in fact, any polynomial matrix $C \in \mathbb{R}^{\bullet \times l}[\xi]$ such that $\begin{pmatrix} D \\ C \end{pmatrix}$ is non-singular and such that

$$\partial\Phi(i\omega) - C^T(-i\omega)C(i\omega) \leq 0, \text{ for all } \omega \in \mathbb{R}, \quad (4.3)$$

yields a contracting controller.

5 Dissipative systems

In this section we temporarily leave the H_∞-context, and consider general systems in image representation, given by

$$w = M(\frac{d}{dt})\ell, \quad (5.1)$$

with $M \in \mathbb{R}^{q \times l}[\xi]$. Denote by $\mathcal{B}$ the behavior of this system (see 2.1). In addition, we have a quadratic functional $Q_L : C^\infty(\mathbb{R}, \mathbb{R}^q) \to C^\infty(\mathbb{R}, \mathbb{R})$; $w \mapsto Q_L(w)$, associated with a given two-variable polynomial $L \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. For the precise definition of Q_L, see [5]. The functional Q_L will be called the *supply rate*. The system 5.1, together with the given supply rate, is called *cyclo-dissipative* if for all $w \in \mathcal{B}$ with compact support we have

$$\int_{-\infty}^{\infty} Q_L(w)(t)dt \geq 0. \quad (5.2)$$

For a given signal $w \in \mathcal{B}$, the quantity $Q_L(w)(t)$ is the rate at which supply (e.g. energy) flows into the system. The inequality 5.2 expresses the fact that, if the system happens to produce the compact support signal w, then supply flows net *into* the system over the interval $(-\infty, \infty)$. Define a two-variable polynomial $\tilde{L} \in \mathbb{R}_s^{l \times l}[\zeta, \eta]$ by

$$\tilde{L}(\zeta, \eta) := M^T(\zeta)L(\zeta, \eta)M(\eta).$$

It can be verified that if w and ℓ are related by 5.1, then $Q_L(w) = Q_{\tilde{L}}(\ell)$. Therefore, the system is cyclo-dissipative if and only if for all $\ell \in \mathcal{D}^l$

$$\int_{-\infty}^{\infty} Q_{\tilde{L}}(\ell)(t)dt \geq 0.$$

This condition is equivalent to

$$\partial \tilde{L}(i\omega) \geq 0, \text{ for all } \omega \in \mathbb{R}. \tag{5.3}$$

Thus, if our system is cyclo-dissipative, equivalently, if 5.3 holds, then we can factorize

$$\partial \tilde{L}(\xi) = R^{\mathrm{T}}(-\xi)R(\xi),$$

with $R \in \mathbb{R}^{l\times l}[\xi]$. Introduce now a two-variable polynomial Δ by

$$\Delta(\zeta,\eta) := \tilde{L}(\zeta,\eta) - R^{\mathrm{T}}(\zeta)R(\eta).$$

Since $\partial\Delta = 0$, the two-variable polynomial Δ must contain a factor $\zeta + \eta$, and therefore we can define a new two-variable polynomial by

$$P(\zeta,\eta) := (\zeta+\eta)^{-1}\Delta(\zeta,\eta). \tag{5.4}$$

Consider now the quadratic functionals Q_P and Q_Δ associated with P and Δ, respectively. We have

$$Q_\Delta(\ell) = Q_{\tilde{L}}(\ell) - \|R(\frac{d}{dt}\ell\|^2.$$

Furthermore, 5.4 is equivalent to:

$$\frac{dQ_P(\ell)}{dt} = Q_\Delta(\ell), \text{ for all } \ell \in C^\infty(\mathbb{R},\mathbb{R}^l).$$

Thus we obtain

$$\frac{dQ_P(\ell)}{dt}(t) \leq Q_{\tilde{L}}(\ell)(t), \tag{5.5}$$

for all $\ell \in C^\infty(\mathbb{R},\mathbb{R}^l)$, for all $t \in \mathbb{R}$. If we interpret $Q_P(\ell)(t)$ as the amount of supply (e.g., energy) stored inside the system at time t, then 5.5 expresses the fact that the rate at which the internal storage increases does not exceed the rate at which supply flows into the system. The inequality 5.5 is called the *dissipation inequality*. Any quadratic functional $Q_P : C^\infty(\mathbb{R},\mathbb{R}^l) \to C^\infty(\mathbb{R},\mathbb{R})$, associated with some two-variable polynomial P, that satisfies this inequality is called a *storage function*. A system is cyclo-dissipative if and only if it has a storage function Q_P. In general, storage functions are not unique. A storage function Q_P is called *positive semi-definite* if for all $\ell \in C^\infty(\mathbb{R},\mathbb{R}^l)$ we have $Q_P(\ell) \geq 0$, see also section 3.

6 H_∞ control with internal stability

We now return to the H_∞ control problem. In this section we will study the question under what conditions a contracting controller is internally stabilizing. Our main result will give necessary and sufficient conditions under which there exists a contracting, internally stabilizing controller for the system 3.1. We start with the following lemma:

Lemma 6.1 *:Let C be a contracting controller. Then C is internally stabilizing if and only if $\begin{pmatrix} D \\ C \end{pmatrix}$ is Hurwitz, i.e., has all its zeros in $C^- := \{\lambda \in \mathbb{C} \mid \Re e\ \lambda < 0\}$.*

Proof : Let λ be a zero of $\begin{pmatrix} D \\ C \end{pmatrix}$. Then there exists $v \neq 0$ such that $D(\lambda)v = 0$ and $C(\lambda)v = 0$. Define $\ell(t) := e^{\lambda t}v$. Then clearly $d = 0$ and $C(\frac{d}{dt})\ell = 0$. Furthermore $z(t) = e^{\lambda t}N(\lambda)v$. By observability, $N(\lambda)v \neq 0$. Thus $z(t) \to 0$ if and only if $\Re e\ \lambda < 0$. ■

Suppose now that $C \in \mathbb{R}^{\bullet\times l}[\xi]$ is such that $\begin{pmatrix} D \\ C \end{pmatrix}$ is nonsingular and such that 4.3 holds. Of course, 4.3 can be rewritten as

$$C^{\mathrm{T}}(-i\omega)C(i\omega) + D^{\mathrm{T}}(-i\omega)D(i\omega) - N^{\mathrm{T}}(-i\omega)N(i\omega) \geq 0, \quad \text{for all } \omega \in \mathbb{R}. \tag{6.1}$$

This inequality can be given an interpretation in the context of dissipative systems by considering the auxiliary system

$$\begin{pmatrix} z \\ d \\ v \end{pmatrix} = \begin{pmatrix} N(\frac{d}{dt}) \\ D(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{pmatrix} \ell, \tag{6.2}$$

with supply rate given by the (constant, i.e., of degree 0) two-variable polynomial

$$L(\zeta,\eta) := \begin{pmatrix} I_{l-r} & 0 & 0 \\ 0 & I_r & 0 \\ 0 & 0 & -I_{q-r} \end{pmatrix}.$$

In fact, 6.1 says that the above system with supply rate $Q_L(v,d,z) := \|v\|^2 + \|d\|^2 - \|z\|^2$ is cyclo-dissipative. As explained in the previous section, there exists at least one storage function Q_P for this cyclo-dissipative system. We have the following theorem:

Theorem 6.2 *:Let $C \in \mathbb{R}^{\bullet\times l}[\xi]$ be such that $\begin{pmatrix} D \\ C \end{pmatrix}$ is nonsingular and such that 6.1 holds. C is internally stabilizing if and only if there exists a positive semi-definite storage function Q_P. Under this condition, every storage function is positive semi-definite.*

Proof : Suppose that Q_P (associated with the two-variable polynomial P) is a positive semi-definite storage function. We want to show that $\begin{pmatrix} D \\ C \end{pmatrix}$ is Hurwitz. We will first explain the idea of the proof. The dissipation inequality states that

$$\frac{dQ_P(\ell)}{dt} \leq \|C(\frac{d}{dt})\ell\|^2 + \|D(\frac{d}{dt})\ell\|^2 - \|N(\frac{d}{dt})\ell\|^2,$$

for all ℓ. Now let ℓ be a solution of $\begin{pmatrix} D(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{pmatrix}\ell = 0$. Then we get

$$\frac{dQ_P(\ell)}{dt} \leq -\|N(\frac{d}{dt})\ell\|^2.$$

This means that $\frac{dQ_P(\ell)}{dt} \leq 0$ along solutions of the (autonomous) system $\begin{pmatrix} D(\frac{d}{dt}) \\ C(\frac{d}{dt}) \end{pmatrix}\ell = 0$. Since $Q_P(\ell) \geq 0$, we expect Q_P to act as a Lyapunov function and to be able to conclude that $\ell(t) \to 0$ as $t \to \infty$.

We will now make this idea precise. let λ be a zero of $\begin{pmatrix} D \\ C \end{pmatrix}$. Then there is a non-zero vector v such that $D(\lambda)v = 0$ and $C(\lambda)v = 0$. Define $\ell(t) := e^{\lambda t}v$. We calculate that

$$Q_P(\ell)(t) = e^{2(\Re e\ \lambda)t} v^* P(\overline{\lambda}, \lambda)v.$$

Then according to the dissipation inequality, we have

$$2\Re e\ \lambda e^{2(\Re e\ \lambda)t} v^* P(\overline{\lambda}, \lambda)v \leq -\|N(\lambda)v\|^2 e^{2(\Re e\ \lambda)t}$$

for all t. In particular, for $t = 0$ this yields

$$2\Re e\ \lambda v^* P(\overline{\lambda}, \lambda)v \leq -\|N(\lambda)v\|^2.$$

Since $v^* P(\overline{\lambda}, \lambda)v \geq 0$ by positive semi-definiteness of Q_P, we find that $\Re e\ \lambda \leq 0$. From $\Re e\ \lambda = 0$ it would follow that $N(\lambda)v = 0$, which would contradict observability. This proves that $\Re e\ \lambda < 0$. The rest of the proof is omitted here. ■

If the polynomial matrix $C \in \mathbb{R}^{(l-r)\times r}[\xi]$ is obtained from a J-spectral factorization of $\partial\Phi$, i.e., if we take C to be a polynomial matrix obtained by factorizing

$$\partial\Phi(\xi) = F^{\mathrm{T}}(-\xi) \begin{pmatrix} I_{l-r} & 0 \\ 0 & I_r \end{pmatrix} F(\xi), \tag{6.3}$$

with $F \in \mathbb{R}^{l\times l}[\xi]$, and by partitioning

$$F = \begin{pmatrix} C \\ Z \end{pmatrix},$$

then C satisfies

$$C^{\mathrm{T}}(-i\omega)C(i\omega) - \partial\Phi(i\omega) - Z^{\mathrm{T}}(-i\omega)Z(i\omega) = 0,$$

for all $\omega \in \mathbb{R}$. The auxiliary system 6.2 corresponding to this choice of C is cyclo-dissipative, and we can obtain a particular storage function Q_Ψ by defining

$$\Psi(\zeta, \eta) := (\zeta + \eta)^{-1}(C^{\mathrm{T}}(\zeta)C(\eta) - \Phi(\zeta, \eta) - Z^{\mathrm{T}}(\zeta)Z(\eta)).$$

According to the previous theorem, the contracting controller C is internally stabilizing if and only if Q_Ψ is positive semi-definite.

Now, we would like to obtain conditions, *in terms of the two-variable polynomial Φ only (so in terms of the system matrices N and D only)*, under which Q_Ψ is indeed positive semi-definite. The surprising fact is, that such conditions can indeed be given, if instead of working with an arbitrary J-spectral factorization we work with a *Hurwitz* J-spectral factorization, i.e., a J-spectral factorization

$$\partial\Phi(\xi) = F_H^{\mathrm{T}}(-\xi) \begin{pmatrix} I_{l-r} & 0 \\ 0 & I_r \end{pmatrix} F_H(\xi) = C_H^{\mathrm{T}}(-\xi)C_H(\xi) - Z_H^{\mathrm{T}}(-\xi)Z_H(\xi) \tag{6.4}$$

with $F_H = \begin{pmatrix} C_H \\ Z_H \end{pmatrix}$ Hurwitz. The corresponding quotient two-variable polynomial will be denoted by Ψ_H, the corresponding storage function by Q_{Ψ_H}.

Definition 6.3 : *The J-spectral factorization 6.3 is called a* regular factorization *(see also [6]) if* $\begin{pmatrix} N \\ D \end{pmatrix} F^{-1}$ *is* proper. *In that case, F is called a* regular factor. *The two-variable polynomial $\Phi(\zeta, \eta)$ will be called* regular *if there exists a regular factorization of $\partial\Phi$.*

In the following, let $\lambda_1, \dots \lambda_k$ be the distinct zeros of $\partial\Phi$ in $\mathbb{C}^-$. For simplicity, assume that every zero has multiplicity one. Furthermore, let v_i be a singular vector associated with the zero λ_i, i.e., assume that $v_i \neq 0$ and $\partial\Phi(\lambda_i)v_i = 0$.

Define the constant Hermitian matrix $k \times k$ matrix $\mathcal{T}$ by

$$\mathcal{T}_{i,j} := -\frac{(v_i)^* \Phi(\overline{\lambda}_i, \lambda_j)v_j}{\overline{\lambda}_i + \lambda_j}.$$

The next lemma states that if there exists a regular Hurwitz J spectral factorization, then the storage function Q_{Ψ_H} associated with that factorization is positive semi-definite if and only if the Hermitian matrix $\mathcal{T}$ is positive semi-definite:

Lemma 6.4 : *Assume that F_H is a regular Hurwitz J-spectral factor. Then we have: $Q_{\Psi_H} \geq 0$ if and only if $\mathcal{T} \geq 0$.*

Now let us return to the H_∞ control problem with internal stability. Recall that the polynomial matrix C_H obtained from the Hurwitz J-spectral factorization 6.4 is contracting, and that it is internally stabilizing if and only if its associated storage function Q_{Ψ_H} is positive semi-definite. By applying the previous lemma we can therefore conclude that if C_H is obtained from a regular Hurwitz factorization, then it is internally stabilizing if and only if $\mathcal{T} \geq 0$. At this point, note that we have obtained a sufficient condition for the existence of a contracting, internally stabilizing controller for our system: under the assumption that the two-variable polynomial $\Phi(\zeta, \eta)$ is regular, if the negative signature $\nu(\partial\Phi(i\omega)$ is equal to r (the dimension of the disturbance) for all ω, and if $\mathcal{T} \geq 0$, then there exists a contracting, internally stabilizing controller (take C_H). We state this formally:

Theorem 6.5 :*Assume that Φ is regular. There exists a contracting, interally stabilizing controller if the following two conditions are satisfied:*

(i) $\nu(\partial\Phi(i\omega) = r$ *for all* $\omega \in \mathbb{R}$,

(ii) $\mathcal{M} \geq 0$.

If these two conditions hold, then a contracting, internally stabilizing controller is given by C_H, where C_H is obtained from a regular Hurwitz J-spectral factorization 6.4 of $\partial\Phi$.

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