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Controllability for Delay-Differential Systems

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1. Introduction

The aim of this paper is to analyse the notion of controllability for delay-differential systems within the behavioral framework introduced in [5].

In this framework, a system is characterized by its behavior $\mathcal{B}$, which is a set consisting of all the admissible signals w.r.t. the system laws. The system variables correspond to the relevant attributes of the phenomenon which is described by the system. They are said to be external (or manifest) variables, as opposed to internal (or latent) variables. These latter are auxiliary variables that are often introduced in order to obtain a more suitable system description, but do not necessarily correspond to the relevant attributes of the phenomenon under consideration. As an example, for a classical i/s/o system the state variables will be internal, whereas both the inputs u and the outputs y are external variables. The system behavior will consist of all the joint input-output signals which are allowed to occur.

Since the behavior is the most intrinsic feature of a system, it is logical to define the system properties in terms of the set $\mathcal{B}$, i.e., at an external level. This applies in particular for the notion of controllability. Roughly speaking, we will say that a system (and/or its behavior) is controllable whenever it is possible to make compatible the 'past' and the 'future' of two arbitrary signals; so, the long term evolution is independent of the system history. In order to distinguish this property from the classical state controllability we will refer to it as behavioral controllability.

Behavioral controllability has been widely studied both for continuous- and for discrete-time systems, respectively described by differential and difference equations, see [5, 3]. In this paper we consider continuous-time systems described by differential equations with delays: i.e., delay-differential systems. More concretely, we will be concerned with systems whose behavior $\mathcal{B}$ can be described as the kernel of a delay-differential operator $R(D,\Delta)$ (where $R(z_1,z_2)$ is a 2D polynomial matrix in z_1 and z_2, D is the differential operator and Δ is the delay).

2. Controllability

Let Σ be a system in q real-valued variables evolving over R, with behavior $\mathcal{B} \subseteq \{w : R \rightarrow R^q\}$. Given two signals w_1 and w_2 in $\mathcal{B}$ we will say that w_1 is $\mathcal{B}$-compatible with w_2 if for all $t_1 \in R$ there exists $t_2 \geq t_1$ and $w \in \mathcal{B}$ such that $w^* := w_1 \wedge_{t_1} w \wedge_{t_2} w_2 \in \mathcal{B}$. Here $w^* := w_1 \wedge_{t_1} w \wedge_{t_2} w_2$ stands for the successive concatenation of w_1 with w at time t_1 and with w_2 at time t_2, and is defined as follows: $w^*(t) = w_1(t)$ for $t \leq t_1$, $w^*(t) = w(t)$ for $t_1 < t \leq t_2$ and $w^*(t) = w_2(t)$ for $t > t_2$.

Definition 1 *The system Σ and the behavior $\mathcal{B}$ are said to be behaviorally controllable if for all signals w_1 and w_2 in $\mathcal{B}$ w_1 is $\mathcal{B}$-compatible with w_2.*

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As mentioned before, behavioral controllability means then that any arbitrary past evolution w_1 in $(-\infty, t_1]$ can be associated in the long run (i.e., after a certain time t_2) to a desired future development w_2 in $(t_2, +\infty)$.
It turns out that the classical notion of state controllability is equivalent to behavioral controllability if the state is regarded as an external variable. However, if this is not the case, the behavioral controllability does not necessarily imply state controllability.

3. Delay-differential systems

In the sequel we will consider systems in q real-valued variables and with smooth signals, whose behavior $\mathcal{B} \subseteq C^\infty(R, R^q)$ can be given as follows. Let Δ be the delay operator defined by $\Delta : C^\infty(R, R^q) \to C^\infty(R, R^q)$, such that $(\Delta w)(t) = w(t-1)$ for all $t \in R$ and all $w \in C^\infty(R, R^q)$. Let further D be the differential operator. Then there exists a 2D polynomial matrix $R(z_1, z_2)$ with q columns such that $\mathcal{B} = \{w \in C^\infty(R, R^q) : R(D, \Delta)w = 0\}$. I.e., $\mathcal{B}$ is the kernel of a polynomial delay-differential (d-d) operator R.
Note that this kernel representation is more general than the polynomial input-output descriptions considered in [4] as well as than the pseudo-state descriptions of [1]. Indeed, any polynomial input-output d-d equation $P(D, \Delta)y = Q(D, \Delta)u$ can be regarded as a kernel representation with $R(D, \Delta) = [P(D, \Delta) \mid -Q(D, \Delta)]$ and with $w = col(y, u)$. On its turn, also the pseudo-state description $\{\dot{x} = A(\Delta)x + Bu \;\; y = Cx\}$ can be viewed as a kernel representation with $w = col(x, y, u)$ and $R(D, \Delta) = col([D - A(\Delta) \mid 0 \mid -B], [-C \mid I \mid 0])$.

4. Controllability of D-D Systems

Let $\mathcal{B}$ be the behavior with kernel representation $\mathcal{B} = kerR(D, \Delta)$, where $R(z_1, z_2)$ is a full row rank r x q polynomial matrix. It can be shown that:

Lemma 1 *With the previous notation, if $rankR(\lambda, e^{-\lambda}) = r$ for all $\lambda \in C$ then there exists a 2D polynomial matrix $M(z_1, z_2)$ such that $\mathcal{B} := kerR(D, \Delta) = imM(D, \Delta)$, with the operator $M(D, \Delta)$ acting on $C^\infty(R, R^l)$ for a certain integer l.*

This means that, in case $R(\lambda, e^{-\lambda})$ has no rank drops, $kerR(D, \Delta)$ can be alternatively described as the image of a polynomial delay-differential operator.
Based on this result it is not difficult to conclude that $\mathcal{B}$ is then controllable. Indeed, suppose that $\mathcal{B}$ has an image representation $\mathcal{B} = imM(D, \Delta)$, and let w_1 and w_2 be two arbitrary signals in $\mathcal{B}$. Then, there exist a_1 and a_2 in $C^\infty(R, R^l)$ such that $w_i = Ma_i, (i = 1, 2)$. Now, it is possible to construct a smooth signal a^* which coincides with a_1 in the past and with a_2 in the (sufficiently far) future. Such signal yields an element $w^* = Ma^*$ in $\mathcal{B}$ which coincides with w_1 in the past and with w_2 in the future. Therefore w_1 is $\mathcal{B}$-compatible with w_2 showing that $\mathcal{B}$ is controllable.
The converse implication also holds true. If $rankR(\lambda, e^{-\lambda}) < r$ for some $\lambda_0 \in C$ we can show that there exists a signal associated with the frequency λ_0 which is not $\mathcal{B}$-compatible with the identically zero signal and hence $\mathcal{B}$ is not controllable.
The foregoing considerations lead to the following characterization of behavioral controllability for delay-differential systems with kernel representations.

Theorem 1 *With the previous notation, $\mathcal{B} = kerR(D, \Delta)$ is (behaviorally) controllable if and only if $rankR(\lambda, e^{-\lambda}) = r$ for all $\lambda \in C$.*

Note that in case the system is a pure differential one, i.e., $R(D, \Delta) = S(D)$, this characterization of behavioral controllability reduces to the condition $rankS(\lambda) = r$ for all $\lambda \in C$, which has been derived in [5].

In order to give a further insight, it is useful to compare this result with results on state controllability for d-d systems of [1].

We will focus on the class of d-d systems Σ considered in [1] which have a pseudo-state description of the form

$$\begin{cases} \dot{x} &= A(\Delta)x + Bu \\ y &= Cx \end{cases}$$

where x is the (n dimensional) pseudo-state, u is the input, y is the output and $A(z) = A_N z^N + \ldots + A_1 z + A_0$ is a polynomial matrix in z.
For the system Σ, the state at time t is defined in [1] as being $z(t) = col(x(t), x_t)$, where $x_t \in L_2[(-N,0], R^n]$ is given by $x_t(\tau) = x(t+\tau)$ for all $\tau \in (-N, 0]$. This yields the infinite dimensional state space $Z = R^n x L_2[(-N,0), R^n]$. Define, in this state space, the set K_t of all attainable states in time t, and let $K_\infty := \cup_{t>0} K_t$. Then Σ is said to be approximately controllable if K_∞ is dense in Z. The next theorem, providing a characterization of approximate controllability, has been derived in [1].

Theorem 2 *Σ is approximately controllable if an only if:*

(1) $rank[(\lambda I - A(e^{-\lambda}) \mid B] = n \quad \forall \lambda \in C$ and
(2) $rank[A_N \mid B] = n$.

The first condition of the theorem is known as spectral controllability.

Note that the pseudo-state description that we have considered here can be regarded as a kernel representation with $R(D, \Delta) = col([D - A(\Delta) \mid 0 \mid -B], [-C \mid I \mid 0])$ if Σ is viewed as a system with external variable vector $w = col(x, y, u)$ and with smooth signals. It turns out from Theorem 1 that Σ is behaviorally controllable iff $rank[(\lambda I - A(e^{-\lambda}) \mid B] = n$ for all $\lambda \in C$. So behavioral controllability seems to correspond to spectral rather than to approximate controllability.
The situation can be illustrated by the following example.

Example 1 *Let $A(z) = A_0 + A_1 z$ with $A_0 = col([0 \mid 1], [0 \mid 0])$, $A_1 = col([0 \mid 0], [-1 \mid 0])$ and $B = col(0, -1)$. Then the corresponding system Σ is not approximately controllable since $rank[A_1 \mid B] = 1 < 2$. However,it is easy to check that $[\lambda I - A(e^{-\lambda}) \mid B]$ has rank two for all $\lambda \in C$ and hence Σ is behaviorally controllable.*
What happens in this case is that the pseusostate components x_1 and x_2 are related by $\dot{x_1} = x_2$. This holds in particular in the interval [-1, 0) ; therefore, not all the elements in the state-space $R^2 x L_2([-1,0), R^2)$ are feasible, which prevents approximate controllability. This obstacle does not arise for behavioral controllability since this property exclusively regards admissible system signals (and hence one does not take into account the signals which do not satisfy $\dot{x_1} = x_2$).

5. Concluding Remarks

We have presented a necessary and sufficient condition for the behavioral controllability of delay-differential systems with kernel representations. Moreover, we have compared the notion of behavioral controllability with the notions of approximate and spectral controllability considered in [1]. Besides these latter, other controllability properties (namely, weak and strong comtrollability) have been introduced within an algebraic approach to delay-differential systems ,see [2]. However, it turns out that these properties do not clearly relate to the behavior of the system signals.
Finally, we would like to remark that our results hold for all types of delay-differential systems, and not only for retarded ones as is the case of the results of [1] and [2].

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