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ON QUADRATIC DIFFERENTIAL FORMS

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Abstract

The purpose of this paper is to introduce some basic notation for quadratic differential forms and to provide a new result regarding the existence of a nonnegative storage function.

1 Introduction

In the behavioral approach to linear systems [1] it is customary to view a system as described by a high-order differential equation (for example as a kernel or as an image representation). This leads to an interesting interplay between polynomial matrices and linear dynamical systems. In control problems quadratic functionals are often also involved (witness LQ- and H_∞-control). Extending the behavioral point of view in this direction leads to the study of quadratic differential forms. These have been introduced earlier in the context of Lyapunov functions in [2] and for LQ-problems in [3]. The purpose of the present paper is to initiate a systematic study of quadratic differential forms. As a sequel to this note we will study H_∞-problems in [4].

2 Quadratic differential forms.

Let $\phi \in \mathbb{R}^{n_1 \times n_2}[\zeta,\eta]$ i.e., ϕ is a $n_1 \times n_2$ real polynomial matrix in the indeterminates ζ and η. Explicitely,

$$\phi(\zeta,\eta) = \sum_{k,\ell} \phi_{k\ell}\zeta^k\eta^\ell \tag{2.1}$$

The sum in (2.1) is a finite one and ranges over the nonnegative integers, and $\phi_{k\ell} \in \mathbb{R}^{n_1 \times n_2}$. Such a ϕ induces a *bilinear differential form*,

$$L_\phi : C^\infty(\mathbb{R},\mathbb{R}^{n_1}) \times C^\infty(\mathbb{R},\mathbb{R}^{n_2}) \to C^\infty(\mathbb{R},\mathbb{R}).$$

defined by

$$(L_\phi(v,w))(t) := \sum_{k,\ell} \left(\frac{d^k v}{dt^k}(t)\right)^T \phi_{k\ell} \left(\frac{d^\ell w}{dt^\ell}(t)\right) \tag{2.2}$$

If $n_1 = n_2$ $(=: n)$ then ϕ induces the *quadratic differential form*

$$Q_\phi : C^\infty(\mathbb{R},\mathbb{R}^n) \to C^\infty(\mathbb{R},\mathbb{R})$$

defined by

$$Q_\phi(w) := L_\phi(w,w). \tag{2.3}$$

Define the $^\bullet$ operator

$$^\bullet : \mathbb{R}^{n_1 \times n_2}[\zeta,\eta] \to \mathbb{R}^{n_2 \times n_1}[\zeta,\eta]$$

by

$$\phi^\bullet(\zeta,\eta) := \phi^T(\eta,\zeta) \tag{2.4}$$

where T denotes transposition. If $\phi \in \mathbb{R}^{n \times n}[\zeta,\eta]$ and $\phi = \phi^\bullet$, then ϕ will be called *symmetric*. The symmetric elements of $\mathbb{R}^{n \times n}[\zeta,\eta]$ will be denoted by $\mathbb{R}_s^{n \times n}[\zeta,\eta]$. Obviously

$$L_\phi(v,w) = L_{\phi^\bullet}(w,v) \text{ and } Q_\phi = Q_{\phi^\bullet} = Q_{\frac{1}{2}(\phi+\phi^\bullet)} \tag{2.5}$$

When considering quadratic differential forms we hence can in principle restrict attention to ϕ's in $\mathbb{R}_s^{n \times n}[\zeta,\eta]$.

Let $\partial : \mathbb{R}^{n_1 \times n_2}[\zeta,\eta] \to \mathbb{R}^{n_1 \times n_2}[\xi]$ be defined by

$$\partial(\phi)(\xi) := \phi(-\xi,\xi). \tag{2.6}$$

Denote (also) by $^\bullet$ the operator mapping $\mathbb{R}^{n_1 \times n_2}[\xi]$ into $\mathbb{R}^{n_2 \times n_1}[\xi]$ defined by

$$\Gamma^\bullet(\xi) := \Gamma^T(-\xi) \tag{2.7}$$

Call the element $\Gamma \in \mathbb{R}^{n \times n}[\xi]$ *symmetric* if $\Gamma = \Gamma^\bullet$. It follows trivially that

$$\partial(\phi^\bullet) = (\partial(\phi))^\bullet \tag{2.8}$$

Hence ∂ maps symmetric elements of $\mathbb{R}_s^{n \times n}[\zeta,\eta]$ into symmetric elements of $\mathbb{R}_s^n[\xi]$.

Let $\phi \in \mathbb{R}_s^{n\times n}[\zeta,\eta]$ and consider the associated quadratic differential form Q_ϕ. Let us call Q_ϕ *nonnegative* (denoted $Q_\phi \geq 0$ or simply as $\phi \geq 0$) if

$$Q_\phi(w) \geq 0 \qquad \forall w \in C^\infty(\mathbb{R},\mathbb{R}^n) \tag{2.9}$$

The purpose of this paper is to study the question when quadratic differential forms have certain positivity properties. We look for answers in terms of the defining two-variable polynomial matrices. In particular, we are interested in the following questions.

(*i*) When is $Q_\phi \geq 0$?

(*ii*) Given $\phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$, when does there exist a $\Psi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ such that
$\frac{d}{dt}Q_\Psi = Q_\phi$, or $\frac{d}{dt}Q_\Psi \leq Q_\phi$

(*iii*) When is Q_Ψ in (ii) itself ≥ 0?

As we shall see, these questions are very much related to another kind of positivity of quadratic differential forms called cyclo-positivity because it can be expressed in terms of periodic functions. Here we state it for elements of $\mathcal{D}^q$, the $C^\infty(\mathbb{R},\mathbb{R}^q)$ functions of compact support.

Let $\phi \in \mathbb{R}_s^{n\times n}[\zeta,\eta]$. We will call Q_ϕ *cyclo-nonnegative* (denoted $\oint Q_\Phi \geq 0$) if

$$\int_{-\infty}^{+\infty} Q_\phi(w)dt \geq 0 \qquad \text{for all } w \in \mathcal{D}^q \tag{2.10}$$

and *cyclic* (denoted $\oint Q_\phi = 0$) if (2.10) is zero.

Finally, we are also interested in the half line version of (2.10). Thus we will call Q_ϕ *half-line nonnegative* (denoted $\int_{\mathbb{R}+} Q_\phi \geq 0$) if

$$\int_{0}^{\infty} Q_\phi(w)dt \geq 0 \qquad \text{for all } w \in \mathcal{D}^q \tag{2.11}$$

3 Motivation.

The study of quadratic functions in the context of linear systems is a basic tool in the state space framework, as, for instance, in Lyapunov theory, LQ-control, H_∞-control. However, the positivity questions introduced in section 2 occured earlier in the context of electrical circuits, in particular in the theory of positive real functions. It turns out that the results of section 5 are new even in this case.

Let us explain how the positivity question occurs in electrical network analysis and synthesis. Consider the linear time-invariant input/output system with transfer function $G \in \mathbb{R}^{q\times q}(\xi)$. Let (N,D) be a right coprime polynomial factorization of G, i.e., $N, D \in \mathbb{R}^{q\times q}[\xi]$, $\begin{bmatrix} N \\ D \end{bmatrix}$ is right prime and $G = ND^{-1}$. In the language of behaviors [1] this means that

$$u = D(\frac{d}{dt})\ell,\ y = N(\frac{d}{dt})\ell \tag{3.1}$$

is an observable image representation of the (unique) controllable I/O-system with transfer function G. In (3.1), u denotes the input, y the output, and ℓ the (free) *latent* (driving) *variable.*

Now consider (3.1) with the *supply rate*

$$s = u^T y \tag{3.2}$$

In electrical networks with u the port voltages and y the port currents, s will be the power (into the network when u and y are chosen with the appropriate sign convention). This supply rate (3.2) can be associated with the quadratic differential form

$$(N(\frac{d}{dt})\ell)^T(D(\frac{d}{dt})\ell)$$

The associated two-variable polynomial matrix is thus $N^T(\zeta)D(\eta)$ or, if preferred, after symmetrization

$$\frac{1}{2}(N^T(\zeta)D(\eta) + D^T(\zeta)N(\eta)). \tag{3.3}$$

A quadratic differential form Q_V is called a *storage function* w.r.t. the supply rate s if

$$\frac{d}{dt}Q_V \leq s \tag{3.4}$$

Since s is the power, V in (3.4) can be identified with the stored energy. In electrical circuit applications the stored energy is nonnegative. It is a well-known classical result that for the situation under consideration the following conditions are equivalent:

(*i*) There exists a $V \geq 0$ such that (3.4) is satisfied.

(*ii*) $\int_{-\infty}^{0} s\ dt \geq 0$ for all $\ell \in \mathcal{D}^q$. In other words, for all $\ell \in \mathcal{D}^q$ in (3.1), there holds

$$\int_{-\infty}^{0} u^T(t)y(t)dt \geq 0$$

(*iii*) ($\text{Re}\lambda \geq 0$ and λ not a singularity of G) $\Rightarrow$ ($G(\lambda) + G^T(\bar{\lambda})$ is nonnegative definite: i.e., G is positive real)

These conditions are moreover necessary and sufficient for the existence of a concrete realization of the transfer function $G(s)$ as the driving point impedance of a linear electrical circuit containing positive R, L, C's, transformers and gyrators. Many other equivalent conditions for positive realness are known. We will not enter into them. In section 5 we will obtain a (to the best of our knowledge) new equivalent condition for positive realness.

4 Results.

Let $\phi \in \mathbb{R}_s^{q\times q}[\zeta, \eta]$. Associate with ϕ the infinite matrix

$$\tilde{\phi} := \begin{bmatrix} \phi_{00} & \phi_{01} & \cdots & \phi_{0L} & \cdots \\ \phi_{10} & \phi_{11} & \cdots & \phi_{1L} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \phi_{L,0} & \phi_{L'1} & \cdots & \phi_{L'L} & \cdots \\ \vdots & \vdots & & \vdots & \end{bmatrix}$$

Obviously $\phi \geq 0$ if and only if $\tilde{\phi} \geq 0$ (with positivity of this infinite matrix defined in the obvious way — note that only a finite number of elements of $\tilde{\phi}$ are nonzero).

Observe that Q_ϕ is cyclic (defined as $\int_{-\infty}^{+\infty} Q_\phi(w)dt = 0$ for all $w \in \mathcal{D}^q$) if and only if $\phi(-i\omega, i\omega) = 0$, i.e. $\partial(\phi) = 0$. Now $\partial(\phi) = 0$, i.e., $\phi(-\xi, \xi) = 0$, means that $\phi(\zeta, \eta)$ contains a factor $(\zeta + \eta)$, implying that there exists $\Psi \in \mathbb{R}_s^{q\times q}[\zeta, \eta]$ such that $\phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta)$. In terms of quadratic differential forms this is equivalent to $\frac{d}{dt}Q_\Psi = Q_\phi$. It follows that the following conditions are equivalent:

(*i*) Q_ϕ is cyclic

(*ii*) $\partial(\phi) = 0$

(*iii*) $\exists \Psi \in \mathbb{R}_s^{q\times q}[\zeta, \eta]$ such that $\frac{d}{dt}Q_\Psi = Q_\phi$

(*iv*) $\exists \Psi$ such that $\phi(\zeta, \eta) = (\zeta + \eta)\Psi(\zeta, \eta)$

(*v*) for all $w_1 \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ and $w_2 \in D^q$ there holds: $\int_{-\infty}^{+\infty} (Q_\phi(w_1 + w_2) - Q_\phi(w_1))dt = 0$ (*path independence*)

Next observe that $\oint Q_\phi \geq 0$ if and only if $\phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. This can be expressed in terms of the existence of a factorization of $\partial(\phi)$ as

$$\partial(\phi)(\xi) = \phi(-\xi, \xi) = D^T(-\xi)D(\xi) \tag{4.1}$$

Defining

$$\Psi(\zeta, \eta) := \frac{\phi(\zeta, \eta) - D^T(\zeta)D(\eta)}{\zeta + \eta} \tag{4.2}$$

shows that (4.1) implies that

$$\frac{d}{dt}Q_\Psi(w) = Q_\phi(w) - \|D(\frac{d}{dt})w\|^2 \tag{4.3}$$

It follows that the following conditions are equivalent

(*i*) $\oint Q_\phi \geq 0$

(*ii*) $\partial(\phi)(i\omega) = \phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$

(*iii*) $\exists \Psi$ such that $\frac{d}{dt}Q_\Psi \leq Q_\phi$

This result can be refined in an important direction. When $\partial(\phi)(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, then $\int_{-\infty}^{+\infty} Q_\phi(w)dt \geq 0$ for all $w \in \mathcal{D}^q$ which shows that $w = 0$ is a minimum (over $\mathcal{D}^q$) of this integral expression. We are also interested (with H_∞-applications in mind) in the case that $w = 0$ is a saddle.

In [1] (and elsewhere) we have studied differential systems as subspaces of $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$. In order to avoid smoothness questions, we will study in this paper linear shift-invariant subspaces $\mathcal{B}$ of $\mathcal{D}^q$. Call $\mathcal{B} \subseteq \mathcal{D}^q$ a *differential system* if there exists a $R \in \mathbb{R}^{\bullet\times q}[\xi]$ such that

$$R(\frac{d}{dt})w = 0 \tag{4.4}$$

is a kernel representation of $\mathcal{B}$, i.e., $\mathcal{B} = \ker R(\frac{d}{dt})$. Note that here we only consider solutions in $\mathcal{D}^q$. Let $\mathcal{L}^q$ denote the set of all such differential systems. Now, call $w = 0$ a *saddle* with respect to the quadratic form $\int_{-\infty}^{+\infty} Q_\phi(w)dt$ induced by ϕ if there exists $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}^q$ with $\mathcal{B}_1 \oplus \mathcal{B}_2 = \mathcal{D}^q$ such that

$$\int_{-\infty}^{+\infty} Q_\phi(w_1, 0)dt \geq 0 \tag{4.5}$$

and

$$\int_{-\infty}^{+\infty} Q_\phi(0, w_2)dt \leq 0 \tag{4.6}$$

and with (4.5) ((4.6)) zero only if $w_1 = 0$ ($w_2 = 0$). Here $w = (w_1, w_2)$ denotes the partition of $w \in \mathcal{D}^q$ into $(w_1, 0) \in \mathcal{B}_1$ and $(0, w_2) \in \mathcal{B}_2$. If $\mathcal{B}_1, \mathcal{B}_2$ exist we will say that $\int_{-\infty}^{+\infty} Q_\phi$ has a saddle structure.

Now assume that $\det\phi(-i\omega, i\omega) \neq 0$ for $\omega \in \mathbb{R}$. Then $\partial(\phi)(i\omega)$ has constant signature and ϕ may be then factored as

$$\phi(-\xi, \xi) = D^T(-\xi)\Sigma D(\xi) \tag{4.7}$$

with $D \in \mathbb{R}^{q\times q}[\xi]$ $\det D \neq 0$, and Σ a signature matrix, say $\Sigma = \begin{bmatrix} I_1 & 0 \\ 0 & -I_2 \end{bmatrix}$ with I_1 and I_2 identity matrices of appropriate dimensions. Defining

$$\Psi(\zeta, \eta) = \frac{\phi(\zeta, \eta) - D^T(\zeta)\Sigma D(\eta)}{\zeta + \eta} \tag{4.8}$$

and partitioning D as $D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ shows that

$$\frac{d}{dt}Q_\Psi(w) = Q_\phi(w) - \|D_1(\frac{d}{dt})w\|^2 + \|D_2(\frac{d}{dt})w\|^2 \tag{4.9}$$

This shows how D and Ψ can be defined also when we have a saddle rather than a minimum. Note that (4.9) shows the saddle structure of $\int_{-\infty}^{+\infty} Q_\phi$: let $\mathcal{B}_1$ be defined by the kernel representation $D_1(\frac{d}{dt})w = 0$ and $\mathcal{B}_2$ by $D_2(\frac{d}{dt})w = 0$. Thus $\int_{-\infty}^{+\infty} Q_\phi$ has a saddle structure if $\det\phi(-i\omega, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$.

5 Positivity of the storage function.

In this section, we will consider the existence of a positive storage function in the cyclo-dissipative case, i.e., when $\phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. For simplicity we will consider only the case when $\partial(\phi)(i\omega) > 0$ for all $\omega \in \mathbb{R}$.

Consider the equations

$$\phi(-\xi, \xi) = D^T(-\xi)D(\xi) \tag{5.1}$$

and

$$\Psi(\zeta, \eta) = \frac{\phi(\zeta, \eta) - D^T(\zeta)D(\eta)}{\zeta + \eta} \tag{5.2}$$

Since (5.1) and (5.2) are equivalent to $\frac{d}{dt}Q_\Psi(w) = Q_\phi(w) - \|D(\frac{d}{dt})(w)\|^2$ we think of Q_ϕ as the *supply rate*, $\|D(\frac{d}{dt})w\|^2$ the *dissipation rate*, and Q_Ψ the *storage function*. Note that D in (5.1) is not necessarily square.

The question which we will now study is what conditions on ϕ ensure the existence of a $\Psi \geq 0$. We think of Ψ as a *storage function*. The question is thus when there exists a nonnegative storage function.

A polynomial matrix $P \in \mathbb{R}^{q\times q}[\xi]$ is said to be *Hurwitz* if $\det P \neq 0$ and if all roots of $\det P$ have negative real part. If P^* is Hurwitz, then P is called *anti-Hurwitz*. It may be shown that if $\det\phi(-i\omega, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$, then (5.1) admits a Hurwitz solution D_+ and a anti-Hurwitz solution D_-. Let Ψ_+ defined by (5.2) through D_+ and Ψ_- by D_-.

It can be shown that every Ψ satisfies

$$Q_{\Psi_+} \leq Q_\Psi \leq Q_{\Psi_-} \tag{5.3}$$

We give the idea of the proof. Let D and Ψ be any other solutions of (5.1,5.2). Let $\Delta = \Psi - \Psi_+$. Then Δ satisfies

$$\Delta(\zeta, \eta) = \frac{D_+^T(\zeta)D_+(\eta) - D^T(\zeta)D(\eta)}{\zeta + \eta} \tag{5.4}$$

Whence

$$\frac{d}{dt}Q_\Delta(w) = \|D_+(\frac{d}{dt})w\|^2 - \|D(\frac{d}{dt})w\|^2$$

Let $w \in C^\infty(\mathbb{R}, \mathbb{R}^q)$ be such that $\lim_{t\to\infty} \frac{d^k w}{dt^k}(t) = 0$ for all $k \in \mathbb{Z}_+$. Then obviously

$$Q_\Delta(w)(0) = \int_0^\infty \|D(\frac{d}{dt})w\|^2 dt - \int_0^\infty \|D_+(\frac{d}{dt})w\|^2 dt \tag{5.5}$$

Now taking w to be a solution of $D_+(\frac{d}{Dt})w = 0$ showes, after some further analysis to treat initial conditions other than those of $D_+(\frac{d}{dt})w = 0$, that $Q_\Delta(w)(0) \geq 0$.

Assume now that $\det\phi(-i\omega, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Then $\partial(\phi)(i\omega)$ has constant signature, as a function of ω, say, (n_+, n_-) with n_+ the number of positive and n_- the number of negative eigenvalues of $\phi(-i\omega, i\omega)$. Now consider the following generalization of (5.1)

$$\phi(-\xi, \xi) = D^T(-\xi)\Sigma D(\xi) \tag{5.6}$$

with Σ given by,

$$\Sigma = \begin{bmatrix} I_{n_+} & 0 \\ 0 & -I_{n_-} \end{bmatrix} \tag{5.7}$$

(5.6) should now be regarded as an equation in D with ϕ given. Each solution leads to a $\Psi \in \mathbb{R}_s^{n\times n}[\zeta,\eta]$ defined by

$$\Psi(\zeta,\eta) = \frac{\phi(\zeta,\eta) - D^T(\zeta)\Sigma D(\eta)}{\zeta+\eta} \tag{5.8}$$

It can again be shown that (5.6) admits a Hurwitz solution D_+ and a anti-Hurwitz solution D_-. Denote the corresponding Ψ's again by Ψ_+ and Ψ_-, respectively. Thus in this case the condition $\Psi_- \geq 0$ is still a sufficient condition for the existence of a nonnegative storage function (while in the cyclo-dissipative case it was a necessary and sufficient condition).

Let $M \in \mathbb{C}^{q\times q}[\xi]$ and assume $\det M \neq 0$. Call $\lambda \in \mathbb{C}$ a *singularity* of M if $\det M(\lambda) = 0$. The order of λ as a root of $\det M$ is called the *order* of λ as a singularity of M. $M \in \mathbb{C}^{q\times q}[\xi]$ is said to be *semi-simple* if all its singularities are semi-simple. A singularity $\lambda \in \mathbb{C}$ is *semi-simple* if dim $\ker M(\lambda)$ equals the order of λ as a singularity of M.

Let $\phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ and assume that $\det\phi(-i\omega, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Consider $\partial(\phi)$ and assume that it is semi-simple. Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be the distinct singularities of $\partial(\phi)$ is the right half complex plane. Define $\mathcal{N}_k := \ker\partial(\phi)(\lambda_k)$. Let, $\{a_1^k, a_2^k, \cdots, a_{n_k}^k\}$ be a basis for $\mathcal{N}_k$. Now consider the matrix

$$\mathcal{T} := \begin{bmatrix} T_{11} & \cdots & T_{1N} \\ \vdots & \ddots & \vdots \\ T_{N1} & \cdots & T_{NN} \end{bmatrix} \tag{5.9}$$

with $T_{k\ell}$ the $n_k \times n_\ell$ matrix formed by the elements

$$\frac{(a_r^\ell)^* \phi(\bar\lambda_k, \lambda_\ell) a_s^k}{\bar\lambda_k + \lambda_\ell}$$

Note that (by diagonalizing the symmetric matrix $\tilde\phi$) it is always possible to write ϕ as

$$\phi(\zeta,\eta) = M^T(\zeta)\Sigma_M M(\eta) \tag{5.10}$$

with $M \in \mathbb{R}^{\bullet\times q}[\xi]$ of full column rank and Σ_M a signature matrix $\Sigma_M = \begin{bmatrix} I_M^+ & 0 \\ 0 & -I_M^- \end{bmatrix}$. We will say that (5.6) defines a *regular* factorization [5] if the McMillan degree of M equals that of $\begin{bmatrix} M \\ D \end{bmatrix}$. The regularity of a factorization is immediately related to the condition that the McMillan degree of $\partial(\phi)$ should equal 2 times that of M.

The main purpose of this paper is to announce the following results.

Theorem 5.1 :*Let $\phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ and assume that $\phi(-i\omega, i\omega) > 0$ for all $\omega \in \mathbb{R}$. Assume, for notational simplicity, that $\partial(\phi)$ is semi-simple, and that D_- defines a regular factor of 5.6. Then the following conditions are equivalent:*

(i) $\exists\, D$ satisfying (5.1) such that the corresponding $\Psi \geq 0$

(ii) $\Psi_- \geq 0$

(iii) $\mathcal{T} \geq 0$

This theorem can to some extent be generalized to the case that $\det\phi(-i\omega, i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Note in fact, that $\mathcal{T}$ is well-defined also in this case.

Theorem 5.2 :*Let $\phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$ and assume that $\det\phi(-i\omega, i\omega) \neq 0$ for all $w \in \mathbb{R}$. Assume also that D_- defines a regular factorization of (5.6). Then $\Psi_- \geq 0$ if and only if $\mathcal{T} \geq 0$.*

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