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LQ-CONTROL: A BEHAVIORAL APPROACH

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Abstract

The behavioral approach to modelling dynamical system is characterized, among other things, by the fact that it does not require that the input/output structure of the system is displayed, nor does it require that a system is in state space form. In this presentation a formulation of the classical LQ-problem will be given which completely fits into the behavioral setting. The solution of the LQ-problem then involves quadratic polynomial matrix equations.

1. Introduction.

In the classical view of optimal control, the aim is choose the control input signal such that the cost is minimized when the system starts in a given initial state. In particular, in the case of LQ-control, this leads to the question of choosing the input $u : \mathbb{R} \to \mathbb{R}^m$ so as to minimize

$$\int_0^\infty (u^T(t)Ru(t) + x^T(t)Lx(t))dt \qquad (1)$$

subject to the constraints

$$\frac{dx}{dt} = Ax + Bu \qquad (2)$$

$$x(0) = x_0 \qquad (3)$$

This formulation has a number of obvious drawbacks, the most apparent one being that in most applications a feedback law is sought, while the formulation has an open-loop flavor. As we shall soon see, the LQ-problem allows a much more natural formulation in the behavioral context.

2. The behavioral approach.

Let $\mathcal{L}^q$ denote the set of linear time-invariant differential dynamical systems in q variables. Thus each element of $\mathcal{L}^q$ consists of a dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ (time set $\mathbb{R}$, signal space $\mathbb{R}^q$) whose *behavior* consists of the solution set of a system of differential equations

$$R\left(\frac{d}{dt}\right) w = 0 \qquad (4)$$

with $R \in \mathbb{R}^{\bullet \times q}[\xi]$ a polynomial matrix with q columns, and ξ the indeterminate. More explicitely, the behavior associated with (4) is formally defined by

$$\mathfrak{B} = \{w \in \mathcal{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q) \mid R\left(\frac{d}{dt}\right) w = 0 \text{ in the sense of distributions}\}$$

The decision to take w in $\mathcal{L}_1^{loc}$ is a somewhat arbitrary one. In fact, for the purposes of this paper, the reader is advised to assume w to be C^∞.

A dynamical system $\Sigma \in \mathcal{L}^q$ is said to be *controllable* if its behavior $\mathfrak{B}$ has the following property: $(w_1, w_2 \in \mathfrak{B}) \Rightarrow (\exists w \in \mathfrak{B}$ and $T \geq 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t+T) = w_2(t)$ for $t > T\}$. It is well-known from our earlier work [1] that (4) describes a controllable system if and only if the complex matrix $R(\lambda)$ has constant rank for $\lambda \in \mathbb{C}$.

An equivalent condition for controllability which will be useful in the sequel is the following. As argued in [1], models obtained from first principles will usually contain *latent variables* (ℓ) in addition to the *manifest variables* (w) which the model aims at describing, leading to a system of differential equations

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell \qquad (5)$$

with $R \in \mathbf{R}^{\bullet \times q}[\xi]$ and $M \in \mathbf{R}^{\bullet \times d}[\xi]$ polynomial matrices with the same number of rows. We define the *manifest behavior* of (5) to be the *closure* in the topology of $\mathcal{L}_1^{loc}$ of

$$\begin{gathered}\{w \in \mathcal{L}_1^{loc}(\mathbf{R}, \mathbf{R}^q) \mid \exists\, \ell \in \mathcal{L}_1^{loc}(\mathbf{R}, \mathbf{R}^d) \\ \text{such that}(w, \ell) \text{ satisfies (5)} \\ \text{in the sense of distributions}\}\end{gathered}$$

A special class of systems (5) are those with $R(\xi) = I$, yielding

$$w = M\left(\frac{d}{dt}\right) \ell \qquad (6)$$

It is easy to prove that the manifest behavior of such a system is always controllable. In fact, it can be shown that a system $\Sigma \in \mathcal{L}^q$ is controllable *if and only if* it is the manifest behavior of a system of the type (6).

Summaring, systems $\Sigma \in \mathcal{L}^q$ are those that admit a *kernel representation* (4), while the controllable systems $\Sigma \in \mathcal{L}^q$ are those that admit also an *image representation* (6).

3. Quadratic differential functionals.

Let $\mathbf{R}^{q \times q}[\zeta, \eta]$ denote the set of 2-variable $q \times q$ polynomial matrices in the indeterminates ζ and η. Thus each $L \in \mathbf{R}^{q \times q}[\zeta, \eta]$ is a finite sum

$$L(\zeta, \eta) = \sum_{h,k} L_{hk} \zeta^h \eta^k \qquad (7)$$

with $L_{hk} \in \mathbf{R}^{q \times q}$. The *dual* of L is defined as

$$L^*(\zeta, \eta) := \sum_{h,k} L_{hk}^T \zeta^k \eta^h = L^T(\eta, \zeta) \qquad (8)$$

where ^{T} denotes transposition. L is said to be *symmetric* if $L \in L^*$, i.e., if $L_{hk} = L_{kh}^T$ for all k, h. The set of symmetric elements of $\mathbf{R}^{q \times q}[\zeta, \eta]$ will be denoted as $\mathbf{R}_s^{q \times q}[\zeta, \eta]$.

Associated with each $L \in \mathbf{R}_s^{q \times q}[\zeta, \eta]$ there is defined the *quadratic differential functional*, Q_L, defined for a (sufficiently smooth) map $w : \mathbf{R} \to \mathbf{R}^q$ by

$$Q_L(w) := \sum_{h,k} \left(\frac{d^h w}{dt^h}\right)^T L_{hk} \left(\frac{d^k w}{dt^k}\right) \qquad (9)$$

Obviously $Q_L : C^\infty(\mathbf{R}, \mathbf{R}^q) \to C^\infty(\mathbf{R}, \mathbf{R})$ but we can also let Q_L act on less smooth functions. In this outline we will, however, gloss over these smoothness issues.

4. LQ-control.

Our appraoch to optimal (feedback) control questions is as follows. We will first set up an optimality principle and determine the set of optimal trajectories. The very important issue of implementation, of synthesis (be it by memoryless or dynamic state feedback, or by regular or singular output feedback) will come later.

The LQ-control problem which we will consider is defined by a dynamical system $\Sigma \in \mathcal{L}^q$, the *plant*, and a quadratic differential functional Q_L, induced by $L \in \mathbf{R}_s^{q \times q}[\zeta, \eta]$, and called the *cost-functional*. We will consider uniquely the LQ-problem with imposed stability, because it is the most relevant one in applications. Let $\Sigma = (\mathbf{R}, \mathbf{R}^q, \mathfrak{B})$. Define $\mathfrak{B}_s$, the *stable part* of $\mathfrak{B}$, as

$$\mathfrak{B}_s := \{w \in \mathfrak{B} \mid \lim_{t \to \infty} w(t) = 0\}$$

and $\mathfrak{B}_c$, the *compact part* of $\mathfrak{B}$, as

$$\mathfrak{B}_c := \{w \in \mathfrak{B} \mid w \text{ has compact support}\}$$

Further, define for each (sufficiently smooth, say C^∞) $w \in \mathfrak{B}$ and $\Delta \in \mathfrak{B}_c$ the *cost-degradation* (by adding Δ to w), $J_w(\Delta)$, as

$$J_w(\Delta) := \int_{-\infty}^{+\infty} (Q_L(w + \Delta)(t) - Q_L(w)(t))dt \qquad (10)$$

Now define $\mathfrak{B}^*$, the *optimal behavior*, as

$$\mathfrak{B}^* := \{w \in \mathfrak{B}_s \mid J_w(\Delta) \geq 0 \text{ for all (sufficiently smooth) } \Delta \in \mathfrak{B}_c\}$$

The problem is to characterize the optimal dynamical system $\Sigma^* = (\mathbf{R}, \mathbf{R}^q, \mathfrak{B}^*)$ in terms of the dynamics of Σ and the cost-functional L. If Σ is given by a kernel representation (4) by $R \in \mathbf{R}^{\bullet \times q}[\zeta, \eta]$, then the problem reduces to finding $\mathfrak{B}^*$ from R and L; if it given by an image representation (6) then the problem reduces to finding $\mathfrak{B}^*$ from M and L.

Note the meaning of optimality as expressed by (10): an optimal trajectory is one which cannot be "improved" by adding a compact support trajectory. This formulation will, in fact, lead to the existence of many "optimal" trajectories.

5. The case $R = 0$.

In order to solve the LQ-problem as formulated above, we will first consider the case when the plant is free, i.e., when $R = 0$ in (4) (or, equivalently, $M = I$ in (6)). Intuitively speaking, then, the problem is to find the trajectories $w^* : \mathbf{R} \to \mathbf{R}^q$ such that $\lim_{t\to\infty} w^*(t) = 0$ and such that $\Delta = 0$ is the minimum of $J_w(\Delta)$ over all $\Delta : \mathbf{R} \to \mathbf{R}^q$ of compact support. In order to solve this problem we will first avoid the stability question. This yields

Proposition 1 Let $L \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$. Let $\mathfrak{B}^*$ denote the set of (sufficiently smooth) $w^* : \mathbf{R} \to \mathbf{R}^q$ is such that $J_{w^*}(\Delta) \geq 0$ for all (sufficiently smooth) $\Delta : \mathbf{R} \to \mathbf{R}^q$ of compact support. Then $\mathfrak{B}^*$ is non-empty if and only if

$$L(-i\omega, i\omega) \geq 0 \tag{11}$$

for all $\omega \in \mathbf{R}$. Note that $L(-i\omega, i\omega) \in \mathbf{C}^{q\times q}$ is an Hermitian matrix for all $\omega \in \mathbf{R}$. If (11) is satisfied for all $\omega \in \mathbf{R}$, then $w^* \in \mathfrak{B}^*$ if and only if

$$L\left(-\frac{d}{dt}, \frac{d}{dt}\right) w^* = 0 \tag{12}$$

In other words, $\Sigma^* = (\mathbf{R}, \mathbf{R}^q, \mathfrak{B}^*) \in \mathcal{L}^q$ and its kernel representation is parametrized by the polynomial matrix $\tilde{L} \in \mathbf{R}^{q\times q}[\xi]$ with $\tilde{L}(\xi) := L(-\xi, \xi)$.

Proof (outline): Note that $Q_L(w + \Delta) - Q_L(w) = 2B_L(w, \Delta) + Q_L(\Delta)$ where $B_L(w_1, w_2)$ denotes the *bilinear* differential functional $\sum_{h,k} \left(\frac{d^h w_1}{dt^h}\right)^T L_{hk} \left(\frac{d^k w_2}{dt^k}\right)$. From this it is easy to see that $J_{w^*}(\Delta) \geq 0$ for all Δ if and only if $\int_{-\infty}^{+\infty} (B_L(w^*, \Delta)(t))dt = 0$ and $\int_{-\infty}^{+\infty} (Q_L(\Delta)(t))dt \geq 0$ for all Δ. The first condition is equivalent to (12), while the second is equivalent to (11). Indeed,

$$\int_{-\infty}^{+\infty} (B_L(w^*, \Delta)(t))dt =$$

$$\int_{-\infty}^{+\infty} ((L\left(-\frac{d}{dt}, \frac{d}{dt}\right) w^*(t))^T \Delta(t))dt$$

This last expression is zero for all Δ of compact support if and only if (12) holds.

□

Note that if, in addition to (11), we require that $\det L(-i\omega, i\omega) \neq 0$ for some ω, then (12) defines an autonomous system in $\mathcal{L}^q$. *Autonomous* means that the behavior is a finite-dimensional subspace of $C^\infty(\mathbf{R}, \mathbf{R}^q)$ (various equivalent statements are given in [1]). It implies, in particular, that all elements of $\mathfrak{B}^*$ are C^∞ (in fact, analytic).

This proposition readily leads to our main result.

Theorem 2 Let $L \in \mathbf{R}_s^{q\times q}[\zeta, \eta]$ satisfy

$$L(-i\omega, i\omega) > 0$$

for all $\omega \in \mathbf{R}$. Then there exists a polynomial matrix $D \in \mathbf{R}^{q\times q}[\xi]$ with D Hurwitz (meaning that $\det D$ has all its roots in the open left half plane) such that

$$L(-\xi, \xi) = D^T(-\xi)D(\xi) \tag{13}$$

Moreover, a (sufficiently smooth) $w^* : \mathbf{R} \to \mathbf{R}^q$ is such that $\lim_{t\to\infty} w^*(t) = 0$ and $J_{w^*}(\Delta) \geq 0$ for all (sufficiently smooth) $\Delta : \mathbf{R} \to \mathbf{R}^q$ of compact support if and only if

$$D\left(\frac{d}{dt}\right) w^* = 0 \tag{14}$$

Proof (outline): Apply proposition 1, and observe that (14) extracts the elements of (12) which satisfy the stability requirement $\lim_{t\to\infty} w^*(t) = 0$.

□

6. The case $R \neq 0$.

In this presentation we will consider only controllable systems $\Sigma \in \mathcal{L}^q$. We can therefore assume that Σ is specified by an image representation (6), through the polynomial matrix $M \in \mathbb{R}^{q\times d}[\xi]$. In addition we will assume that this image representation is *observable.* This means that ℓ may be deduced from w. Observability for differential systems (5) or (6)is equivalent to asking that the complex matrix $M(\lambda) \in \mathbb{C}^{q\times d}$ is of rank d for all $\lambda \in \mathbb{C}$. A controllable differential system always admits an observable image representation. Then by defining

$$L'(\zeta,\eta) := M^T(\zeta)L(\zeta,\eta)M(\eta) \qquad (15)$$

we can reduce the minimization of w to that of ℓ. This leads to the following result, which solves the LQ-problem for controllable systems.

Assume that

$$L'(-i\omega, i\omega) > 0 \qquad (16)$$

for all $\omega \in \mathbb{R}$. Then there exists $D' \in \mathbb{R}^{d\times d}[\xi]$, Hurwitz, such that

$$L'(-\xi,\xi) = (D'(-\xi))^T(D'(\xi)) \qquad (17)$$

leading to

$$w^* = M\left(\frac{d}{dt}\right)\ell^* \qquad (18a)$$

$$D'\left(\frac{d}{dt}\right)\ell^* = 0 \qquad (18b)$$

as a specification of the optimal behavior $\mathfrak{B}^*$. Thus (18b) determines a finite-dimensional set of latent variable trajectories $\ell^* \in C^\infty(\mathbb{R},\mathbb{R}^d)$. Through (18a) this yields the (finite-dimensional) set of optimal manifest variable trajectories w^*. In other words, (18) is a latent variable representation of the optimal system, which is autonomous.

7. Feedback implementation.

It can be argued that the basic principle of applying control is interconnection. When this interconnection is compatible with a given signal flow structure, then we can speak of *feedback.* It is for good reasons that control theory has concentrated on feedback control. However, the omnipresence of feedback control (in contrast to interconnection, not to open-loop control) is not as compelling as the history of our subject leads us to believe.

Let $\Sigma = (\mathbb{R},\mathbb{R}^q,\mathfrak{B}) \in \mathcal{L}^q$. A system $\Sigma' = (\mathbb{R},\mathbb{R}^q,\mathfrak{B}') \in \mathcal{L}^q$ is said to be a *subsystem* of Σ' (denoted as $\Sigma' < \Sigma$) if $\mathfrak{B}' \subseteq \mathfrak{B}$. Let $\Sigma_k = (\mathbb{R},\mathbb{R}^q,\mathfrak{B}_k) \in \mathcal{L}^q$, $k = 1,2$. The *interconnection* of Σ_1 and Σ_2, denoted as $\Sigma_1 \wedge \Sigma_2$, is defined by $\Sigma_1 \wedge \Sigma_2 := (\mathbb{R},\mathbb{R}^q,\mathfrak{B}_1 \cap \mathfrak{B}_2)$.

A special type of interconnection is *full feedback interconnection* which can be defined as follows. As is well-known [1], any system $\Sigma_1 \in \mathcal{L}^q$ admits an *input/output representation*, i.e., a representation of the form

$$P_1(\frac{d}{dt})y = Q_1\left(\frac{d}{dt}\right)u \qquad (19a)$$

$$w = \Pi\begin{bmatrix} u \\ y \end{bmatrix} \qquad (19b)$$

with $\Pi \in \mathbb{R}^{q\times q}$ a permutation matrix, $P_1 \in \mathbb{R}^{p\times p}[\xi]$, $Q_1 \in \mathbb{R}^{p\times m}[\xi]$, $\det P_1 \neq 0$, and transfer function $P_1^{-1}Q_1 \in \mathbb{R}^{p\times m}(\xi)$ proper.

Now assume that $\Sigma_2 \in \mathcal{L}^q$ admits a similar representation

$$P_2\left(\frac{d}{dt}\right)u = Q_2\left(\frac{d}{dt}\right)y \qquad (20)$$

with $P_2 \in \mathbb{R}^{m\times m}[\xi]$, $Q_2 \in \mathbb{R}^{m\times p}[\xi]$, $\det P_2 \neq 0$, and $I - P_1^{-1}Q_1P_2^{-1}Q_2 \in \mathbb{R}^{p\times p}[\xi]$ non-singular. Then $\Sigma_1 \wedge \Sigma_2$ will be called a *full feedback* interconnection. If in addition $P_2^{-1}Q_2$ is also proper, then we call the full feedback interconnection *regular.* Otherwise it is called *singular.*

As we have defined things here, regular and singular full feedback interconnections seem to depend on the input/output structure. But this is not the case. Define, for $\Sigma \in \mathcal{L}^q$ described by (4), $p(\Sigma)$ as rank (R). Equivalently, $p(\Sigma) = q - m(\Sigma)$ with $m(\Sigma)$ the number of free variables in Σ. In other words, $p(\Sigma)$ = the number of outputs and $m(\Sigma) = q - p(\Sigma)$ = the number of inputs. Define further $n(\Sigma) :=$ McMillan degree (R); $n(\Sigma)$ equals the number of states in a minimal representation of Σ [1]. Now for $\Sigma_1, \Sigma_2 \in \mathcal{L}^q$, $\Sigma_1 \wedge \Sigma_2$ is a full feedback interconnection (in the sense that *any* input/output representation (19) of Σ, will yield a corresponding input/output representation on Σ_2 satisfying the required conditions) if and only if

$$p(\Sigma_1 \wedge \Sigma_2) = p(\Sigma_1) + p(\Sigma_2) = q \qquad (21)$$

and a regular one of in addition

$$n(\Sigma_1 \wedge \Sigma_2) = n(\Sigma_1) + n(\Sigma_2) \qquad (22)$$

More details on this will be given in a future extensive paper on this subject. See, however, [2].

Note that if $\Sigma_1 \wedge \Sigma_2$ is a full feedback interconnection, then $\Sigma_1 \wedge \Sigma_2$ is an autonomous subsystem of Σ_1. It can be shown that if $\Sigma_1 \in \mathcal{L}^q$ is controllable, then *every* subsystem $\Sigma' \in \mathcal{L}^q$ may be realized this way, i.e., there will exist a $\Sigma'' \in \mathcal{L}^q$ such that $\Sigma \wedge \Sigma''$ equals Σ' and, such that this interconnection is a is a full feedback interconnection.

We may conclude that the optimal system (18) can always be implemented by means of a singular output feedback controller. In fact, it can be implemented by means of a memoryless state feedback law. In this case, however, the definition of state involves the plant as well as the cost-functional.

8. Concluding remarks.

8.1 Optimal control can be defined in the behavioral context and merely requires a suitable definition of optimal trajectories. In the LQ-context we have taken these to be the stable trajectories which minimize the cost-functional against compact support variations. This notion does not require explicit mention of a fixed initial state, not does it require displaying the control input.

8.2 The solution of the LQ-problem requires the solution of a quadratic polynomial matrix equation (13) or (17), in many ways reminiscent of the Algebraic Riccati Equation on the one hand, and of spectral factorization on the other. It is, in fact, possible to obtain a full generalization of the (ARE) provided that we stick to kernel representations. Details will be given elsewhere.

8.3 Our approach completely dissociates the problem of finding the optimal behavior from finding a feedback implementation of it. In general this has drawbacks as well, but it appears to be a quite essential feature of the LQ-case.

8.4 The solution of the LQ-problem as given hare can be generalized to situations where stability is not imposed and where a larger class of trajectory variations Δ is considered. This yields in particular the behavioral analogue of what is usually called the *free-end point problem* in LQ-control.

8.5 We are in the process of extending these results to the H_∞-problem. Hopefully, this extension will lead to an integration of H_∞-robustness with system perturbations defined by means of the type of parametrizations introduced in the behavioral descriptions of dynamical systems.

References

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