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THE DISSIPATION INEQUALITY FOR SYSTEMS DESCRIBED BY HIGH-ORDER DIFFERENTIAL EQUATIONS

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1 Introduction

One of the important concepts in the analysis of physical systems is that of dissipativeness. Areas in system theory where this notion is applied are the synthesis of passive electrical circuits (circuits containing R, L, and C's), stability questions, H_∞-theory, etc. Classically, we have studied dissipativeness from the state space point of view [1], [2], [3]. The purpose of this paper is to study dissipativeness from a behavioral point of view for controllable linear time-invariant systems described by high-order differential equations.

We will consider linear time-invariant systems described by (a set of) high-order differential equations. Let $R(s)$ be a polynomial matrix in the indeterminate s, with q columns, $R \in \mathbb{R}^{\bullet \times q}[s]$, and consider the linear differential equation

$$R(\frac{d}{dt})\, w = 0. \tag{1}$$

In terms of the language developed in [4], equation (1) describes a continuous-time dynamical system

$$\Sigma_R := (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_R). \tag{2}$$

whose *behavior* $\mathfrak{B}_R$ is usually defined as the family of functions $w : \mathbb{R} \to \mathbb{R}^q$, which are locally integrable and satisfy the differential equation (1) in the sense of distributions. However, for the purposes of this note, where we do not want to get involved in smoothness considerations, we will define the behavior by

$$\mathfrak{B}_R := \{w \mid w \in C^\infty(\mathbb{R}, \mathbb{R}^q), R(\frac{d}{dt})\, w = 0\}. \tag{3}$$

We finally recall that systems of the form (2) which are *controllable* in the sense of [4] are characterized by a left-prime polynomial matrix $R(s)$, equivalently by a polynomial matrix $R(s)$ such that $R(\lambda)$ has constant rank for $\lambda \in \mathbb{C}$.

2 System dissipativeness

Let $M \in \mathbb{R}^{q_1 \times q_2}[\xi, \eta]$ be a $q_1 \times q_2$ polynomial matrix in the two indeterminates ξ and η.

$$M(\xi, \eta) = \sum_{k,\ell} M_{k\ell}\, \xi^k \eta^\ell. \tag{4}$$

and denote by $M^T \in \mathbb{R}^{q_2 \times q_1}[\xi, \eta]$ its transpose, $M^T(\xi, \eta) = \sum_{k,\ell} M_{k\ell}^T \xi^k \eta^\ell$. An element $M \in \mathbb{R}^{q \times q}[\xi, \eta]$ is said to be *symmetric* if $M(\xi, \eta) = M^T(\eta, \xi)$, i.e., if $M_{k\ell} = M_{\ell k}^T$, for all $k, \ell = 0, 1, \ldots$. We will denote these by $\mathbb{R}_S^{q \times q}[\xi, \eta]$. Every $M \in \mathbb{R}_S^{q \times q}[\xi, \eta]$ induces a quadratic mapping φ_M of $C^\infty(\mathbb{R}, \mathbb{R}^q)$ into $C(\mathbb{R}, \mathbb{R})$, defined by

$$\varphi_M(w) := \sum_{k,\ell} (\frac{d^k w}{dt^k})^T M_{k\ell} (\frac{d^\ell w}{dt^\ell}). \tag{5}$$

Conversely, every quadratic function of w and its derivatives of the form (5) determines a symmetric $q \times q$ polynomial matrix $M(\xi, \eta)$.

In this note, we will pursue dissipation in linear systems described by (1), with supply rates of the quadratic type (5). Thus, assume that we have a dynamical system $\Sigma_R = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}_R)$ described by (1), parametrized by the polynomial matrix $R \in \mathbb{R}^{\bullet \times q}[s]$. Assume that this system exchanges a quantity (say energy) with its environment at a rate given by

$$\varphi_Q(w) := \sum_{k,\ell} \left(\frac{d^k w}{dt^k}\right)^T Q_{k\ell} \left(\frac{d^\ell w}{dt^\ell}\right). \qquad (6)$$

This function, called the *supply rate*, is parametrized by the symmetric two-variable polynomial matrix

$$Q(\xi,\eta) = \sum_{k,\ell} Q_{k\ell}\, \xi^k \eta^\ell \in \mathbf{R}_S^{q\times q}[\xi,\eta]. \qquad (7)$$

We will say that the pair (R, Q) defines a *dissipative system* if there exists a second symmetric two-variable polynomial matrix

$$P(\xi,\eta) = \sum_{k,\ell} P_{k\ell}\, \xi^k \eta^\ell \in \mathbf{R}_S^{q\times q}[\xi,\eta], \qquad (8)$$

defining the *storage function*

$$\varphi_P(w) := \sum_{k,\ell} \left(\frac{d^k w}{dt^k}\right)^T P_{k\ell} \left(\frac{d^\ell w}{dt^\ell}\right), \qquad (9)$$

such that $w \in \mathfrak{B}_R$ implies

$$\frac{d}{dt}\varphi_P(w) \leq \varphi_Q(w). \qquad (10)$$

This inequality is called the *dissipation inequality*. It means that the rate of increase of the storage cannot be larger than the supply. Related to this inequality is the *dissipation rate*. This is defined by a polynomial matrix $D \in \mathbf{R}^{\bullet\times q}[s]$ and must be such that for all $w \in \mathfrak{B}_R$ there holds

$$\frac{d}{dt}\varphi_P(w) = \varphi_Q(w) - \|D(\frac{d}{dt})\, w\|^2, \qquad (11)$$

with $\|\cdot\|$ the Euclidean norm. Thus, the square of the dissipation rate equals the difference between the supply and the rate of storage.

3 Result

The purpose of this note is to announce the following theorem.

Theorem 1 *Consider the system $\Sigma_R = (\mathbf{R}, \mathbf{R}^q, \mathfrak{B}_R)$ defined by (1), parametrized by $R \in \mathbf{R}^{\bullet\times q}[s]$, and assume that it is controllable. Consider the supply rate (6) defined by $Q \in \mathbf{R}_S^{q\times q}[\xi,\eta]$. Then the following statements are equivalent.*

1. *The pair (R, Q) defines a dissipative system, i.e., there exists a storage function $P \in \mathbf{R}_S^{q\times q}[\xi,\eta]$, such that for $w \in \mathfrak{B}_R$, (10) holds.*
2. *There exist a storage function $P \in \mathbf{R}_S^{q\times q}[\xi,\eta]$ and a dissipation rate $D \in \mathbf{R}^{\bullet\times q}[s]$, such that for $w \in \mathfrak{B}_R$, (11) holds.*
3. *For $w \in \mathfrak{B}_R$ with w periodic there holds*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi_Q(w)\, dt \geq 0. \qquad (12)$$

4. *For all $\omega \in \mathbf{R}$, the Hermitian matrix $Q(-i\omega, i\omega)$ is nonnegative definite on* $\ker R(i\omega)$.
5. *There exist $X \in \mathbf{R}^{\bullet\times q}[s]$ and $D \in \mathbf{R}^{\bullet\times q}[s]$, such that*

$$Q(-s,s) = X^T(-s)R(s) + R^T(-s)X(s) + D^T(-s)D(s). \qquad (13)$$

6. *There exist $X \in \mathbf{R}^{\bullet\times q}[s]$, $D \in \mathbf{R}^{\bullet\times q}[s]$ and $P \in \mathbf{R}_S^{q\times q}[\xi,\eta]$, such that*

$$Q(\xi,\eta) - (\xi+\eta)P(\xi,\eta) = X^T(\xi)R(\eta) + R^T(\xi)X(\eta) + D^T(\xi)D(\eta). \qquad (14)$$

References

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