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# REPRESENTATIONS OF SYMMETRIC LINEAR DYNAMICAL SYSTEMS

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## 1 Introduction.

Following the terminology explained in [1], we define a *dynamical system* $\Sigma$ to consist of a triple, $\Sigma = (\mathbf{T}, \mathbf{W}, \mathcal{B})$, with $\mathbf{T}$ a subset of $\mathbf{R}$, called the *time axis*, $\mathbf{W}$ a set called the *signal space*, and $\mathcal{B}$ a subset of $\mathbf{W}^{\mathbf{T}}$(:= all maps from $\mathbf{T}$ to $\mathbf{W}$), called the *behavior*. Thus the behavior consists of a given family of trajectories $w : \mathbf{T} \to \mathbf{R}$. In this talk we consider continuous-time dynamical systems with time axis $\mathbf{T} = \mathbf{R}$ and with signal space $\mathbf{W} = \mathbf{K}^q$, with $\mathbf{K} = \mathbf{R}$ (the real case) or $\mathbf{K} = \mathbf{C}$ (the complex case). The dynamical system $\Sigma = (\mathbf{R}, \mathbf{K}^q, \mathcal{B})$ will be said to be *linear* if $\mathcal{B}$ is a linear subspace of $(\mathbf{K}^q)^{\mathbf{R}}$ and *time-invariant* if $\sigma^t\mathcal{B} = \mathcal{B}$ for all $t \in \mathbf{R}$; $\sigma^t$ denotes the backwards $t$-shift.

We will assume that the behavior $\mathcal{B}$ is the solution set of a system of constant coefficient linear differential equations

$$R(\frac{d}{dt})w = 0 \tag{1}$$

defined in terms of a polynomial matrix $R \in \mathbf{K}^{\bullet \times q}[s]$ ; $\mathbf{K}^{\bullet \times q}[s]$ denotes the set of polynomial matrices over $\mathbf{K}$ with $q$ columns in the indeterminate $s$). The solution set of (1) is formally defined as follows

$$\mathcal{B} = \{w \in C^{\infty}(\mathbf{R}; \mathbf{K}^q) \mid (R(\frac{d}{dt})w)(t) = 0 \text{ for all } t \in \mathbf{R}\}$$

The assumption that $w$ is infinitely differentiable is used mainly for convenience. We will denote this class of dynamical systems as $\mathcal{L}^q$ and refer to its elements as *differential dynamical systems.* We will call the system of differential equations (1) or, equivalently, $R$, a *behavioral equation representation* of the corresponding dynamical system; (1) or, equivalently, $R$ is called (a) *minimal* (behavioral equation) representation if ($R_1 \in \mathbf{K}^{p_1 \times q}[s]$ and $R_1 \sim R$) implies ($p_1 \geq p$). The following characterization of minimal representations plays an important role. It turns out that (1) is minimal iff $R \in \mathbf{K}^{\bullet \times q}_{fr}[s]$ (that is, $R \in \mathbf{K}^{\bullet \times q}[s]$ is of full row rank. Moreover, if (1) is minimal and if $R_1 \in \mathbf{K}^{\bullet \times q}[s]$, then ($R_1 \sim R$) $\Leftrightarrow$ ($R_1$ and $R$ have the same number of columns and there exists a unimodular $U$ such that $R_1 = UR$). Finally, this $U$ is unique.

Let $\Sigma \in \mathcal{L}^q$, and let $R$ be a minimal behavioral equation of $\Sigma$. It follows from that the number of rows of $R \in \mathbf{K}^{\bullet \times q}[s]$ depends only on $\Sigma$ (and not on the particular minimal representation $R$).

We will denote the number of rows of $R$ by $p(\Sigma)$. Actually $p(\Sigma)$ is equal to the number of output variables in any input/output representation of $\Sigma$.

## 2 Symmetric systems.

The purpose of this talk is to study symmetries of dynamical systems in $\mathcal{L}^q$. A symmetry is induced by a transformation group, the basic idea being that we have a group of transformations mapping a dynamical system $\Sigma = (\mathbf{R}, \mathbf{K}^q, \mathcal{B}) \in \mathcal{L}^q$ into another such dynamical system. If this transformation leaves the behavior invariant, then we will call $\Sigma$ symmetric.

Let $T$ be a transformation group acting on $\mathbf{K}^q$; $T$ induces a symmetry on $\mathcal{L}^q$ by defining for $\Sigma = (\mathbf{R}, \mathbf{K}^q, \mathcal{B}) \in \mathcal{L}^q, T_g\Sigma := (\mathbf{R}, \mathbf{K}^q, T_g\mathcal{B})$ with $T_g\mathcal{B} := \{w : \mathbf{R} \to \mathbf{R}^q \mid \exists w' : \mathbf{R} \to \mathbf{K}^q$ such that $w(t) = T_g w'(t)$ for all $t \in \mathbf{R}\}$. Note that, by a minor abuse of notation, we use the same symbol $T_g$ as acting on $\mathcal{L}^q$, on $\mathcal{B}$ and on $\mathbf{K}^q$. $\Sigma$ is symmetric in this sense if $w \in \mathcal{B}$ implies $T_g w \in \mathcal{B}$ for all $g \in \mathcal{G}$. Since $T_g$ transforms the trajectories $w$ in $\mathcal{B}$ by applying the memoryless map $T_g$, that is, in a non-dynamic way, we will call such a symmetry a *static symmetry.* In fact, we will be particularly interested in the case that $T_g$ is linear for all $g \in \mathcal{G}$. It is customary to denote $T$ by $\rho$ in that case.

Assume that $\rho : \mathcal{G} \to G\ell(\mathbf{K}^q)$ is a representation of the group $\mathcal{G}$ on $\mathbf{K}^q$ and assume that $\Sigma = (\mathbf{R}, \mathbf{K}^q, \mathcal{B}) \in \mathcal{L}^q$ is symmetric in the sense of the static symmetry induced by this representation. The problem considered is the following: *Can this symmetry be put into evidence by an appropriate behavioral equation representation of* $\Sigma$ *as (1), in which the polynomial matrix* $R$ *is such that this static symmetry becomes evident?*

Assume that $\rho : \mathcal{G} \to G\ell(\mathbf{K}^q)$ is a given representation of the *finite* group $\mathcal{G}$ on $\mathbf{K}^q$. Then $\rho$ defines a static symmetry on $\mathcal{L}^q$; $\Sigma = (\mathbf{R}, \mathbf{K}^q, \mathcal{B}) \in \mathcal{L}^q$ is thus *$\rho$-symmetric* iff $\rho_g\mathcal{B} = \mathcal{B}$ for all $g \in \mathcal{G}$. Let (1) be a minimal representation for such a $\rho$-symmetric $\Sigma \in \mathcal{L}^q$. It then follows that for each $g \in \mathcal{G}$, there will exist a unimodular polynomial matrix $U_g(s)$ such that $R(s)\rho_g = U_g(s)R(s)$. Our main result tells us that $R$ can be chosen such that $U_g(s)$ is a constant nonsingular matrix, thus independent of $s$!

**THEOREM 1:** *$\Sigma \in \mathcal{L}^q$ is $\rho$-symmetric iff there exists a minimal representation $R(\frac{d}{dt})w = 0$ of $\Sigma$ and a representation $\rho' : \mathcal{G} \to G\ell(\mathbf{K}^{p(\Sigma)})$ of the group $\mathcal{G}$ on $\mathbf{K}^{p(\Sigma)}$ such that $R(s)\rho_g = \rho'_g R(s)$ for all $g \in \mathcal{G}$. In fact $\rho'$ is isomorphic to a subrepresentation of $\rho$.*

In [2] we show that Theorem 2 leads to an appealing canonical form for symmetric systems. Here we will merely illustrate this by means of an example.

## 3 An Example

We will now show the implications of Theorem 2 on a system $\Sigma = (\mathbf{R}, \mathbf{R}^q, \mathcal{B}) \in \mathcal{L}^q$ with $\rho : S_q \to G\ell(\mathbf{R}^q)$ where $\rho_g col(w_1, w_2, \cdots, w_q) := col(w_{g(1)}, w_{g(2)}, \cdots, w_{g(q)})$. Here $S_q$ denotes the symmetric group of permutations of $q$ elements. It follows that $\Sigma \in \mathcal{L}^q$ will be $\rho$-symmetric in the case of permutations iff there exist (not-necessarily non-zero) polynomials $r_{av} \in \mathbf{R}[s]$, $r_\Delta \in \mathbf{R}[s]$ such that $\Sigma$ is described by

$$\begin{aligned} r_{av}(\frac{d}{dt})w_{av} &= 0 \\ r_\Delta(\frac{d}{dt})\Delta w_i &= 0 \quad i = 1, 2, \cdots, q \end{aligned}$$

with $w_{av} := \frac{1}{q}(w_1 + w_2 + \cdots + w_q)$ and $\Delta w_i := w_i - w_{av}$. One equation governs the dynamics of the average. The second equation governs the dynamics of the distance from the average and is identical for each of the components.

## References

[1] J.C. Willems, Paradigms and Puzzles in the Theory of Dynamical Systems, IEEE Transactions on Automatic Control, Vol AC-36, Number 3, 1991.

[2] F. Fagnani and J.C. Willems, Representations of Symmetric Linear Dynamical Systems, SIAM Journal on Control and Optimization, Submitted for publication.