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# Stabilization and Output Regulation of Nonlinear Systems in the Large

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## 1. Introduction

In the last decade, considerable research efforts aimed at the development of systematic design methodologies for nonlinear feedback systems. Interesting contributions to this research area were the studies of the nonlinear analog of the notion of zero of a transfer function, initiated by the authors in [1]. It is now apparent that the results of these studies have had a substantial impact of a large variety problems, which include asymptotic stabilization via smooth state-feedback [2][3], nonlinear adaptive control [21], design of (dynamic) state-feedback laws which render a nonlinear system locally diffeomorphic to a linear system [6], asymptotic tracking a prescribed set of reference trajectories [5][14], the feedback equivalence to a passive system [7].

In particular, in the paper [3], we were able to show that a certain class of systems having a globally asymptotically stable *zero dynamics,* namely those systems whose *relative degree* is uniformly equal to one and which possess a globally defined *normal* form, can be globally stabilized by smooth state feedback. In the papers [14][5], we addressed the problem of desiging a (locally defined) feeedback law yielding asymptotic tracking of prescribed trajectories and/or asymptotic attenuation of disturbances, along with internal exponential stability. We showed that the existence of these feedback laws corresponds the solvability of a certain partial differential equation - which expresses the existence of a *controlled invariant submanifold* - subject to a constraint expressed by a transcendental equation, and we also showed that the existence of solutions to these equations is in fact a property of the zero dynamics of an *augmented system* which incorporates the controlled plant and the dynamical system (the so-called *exosystem* ) which generates the desired output trajectories as well as the disturbances. Of course, the problems of feedback stabilization and output regulation are intimately related, and indeed the ability of globally stabilizing a nonlinear system is an obviuos prerequisite to the solvability of the problem of designing globally defined output regulators. As a matter of fact, as our inital work on (local) output regulation clearly shows, the existence global solutions to the problem of output regulation reposes on the existence of a globally defined invariant submanifold which eventually becomes a global attractor.

The content of this paper is twofold. In the first part, we wish to stress how, from our preliminary analysis of the *local* regulator problem (which is sketched in section 2), it is possible to isolate the main ingredients of a *global* solution to the output regulation problem (section 3). In the second part (sections 4-6), we summarize some very recent advances in the design of globally stabilizing feedback laws that were made possible by suitable use of the notion of passivity and, in particular, by the solution, that we recently obtained [7], to the problem of *when a given nonlinear system can be rendered passive via state feedback* .

## 2. Output Regulation of Nonlinear Systems

An important problem in control theory is that of controlling a fixed plant in order to have its output tracking (or rejecting) reference (or disturbance) signals produced by some external generator (the exosystem). In a general multivariable nonlinear setting, the problem in question can be formulated in the following terms. Consider a system modeled by equations of the form

$$\begin{aligned} \dot{x} &= f(x,w,u) \\ \dot{w} &= s(w) \\ e &= h(x,w) \end{aligned} \tag{2.1}$$

in which $x$ is the plant *state* vector, $u$ the *control input,* $w$ a vector of *exogenous signals,* $e$ the *output error,* the difference between the actual plant output and its desired reference behavior, and $f(\cdot,\cdot,\cdot)$, $s(\cdot)$, $h(\cdot,\cdot)$ are smooth functions, defined in a neighborhood of a reference (equilibrium) point, namely $(x,w,u) = (0,0,0)$. The control action to (2.1) is to be provided by a dynamic compensator, which processes the output error $e$, generates the appropriate control input $u$, and is modeled by equations of the form

$$\begin{aligned} \dot{z} &= \eta(z,e) \\ u &= \theta(z) \end{aligned} \tag{2.2}$$

in which $\eta(\cdot,\cdot)$, $\theta(\cdot)$ are $C^k$ functions (for some integer $k \geq 2$) defined in a neighborhood of the equilibrium $(z,e) = (0,0)$.

The purpose of the control is twofold: closed-loop stability and output regulation. More explicitly, we require

**(S) Closed-loop stability:** The interconnection of (2.1) and (2.2) with $w = 0$, i.e. the closed loop system

$$\begin{aligned} \dot{x} &= f(x,0,\theta(z)) \\ \dot{z} &= \eta(z,h(x,0)) \end{aligned} \tag{2.3}$$

has a (locally) exponentially stable equilibrium at $(x,z) = (0,0)$.

**(R) Output regulation:** For each initial condition (($x(0)$, $z(0)$, $w(0)$) in a neighborhood of $(0,0,0)$, the response $x(t)$, $z(t)$, $w(t)$ of the closed loop system (2.1)-(2.2) satisfies

$$\lim_{t\to\infty} e(t) = 0,$$

As in the case of linear systems, in order to identify appropriate necessary and sufficient conditions for the solvability of the output regulator problem, it is very reasonable to assume that the exosystem does not contain any asymptotically stable "subsystem". If this were the case, in fact, at least for some subset of initial states $w(0)$, the output regulation property (R) would be a straightforward implication of the stability property (S). The property that $\dot{w} = s(w)$ does not contain any asymptotically stable subsystem, together with the already assumed stability of its equilibrium $w = 0$, can be, for instance, given the form of the following hypothesis.

**(H) Neutral Stability:** The exosystem has a Lyapunov stable equilibrium at $w = 0$ and, for some neighborhood $W$ of $w = 0$, the set $\Omega$ of all $\omega$-limit points of all trajectories which are initialized in $W$ is such that $\Omega \cap W$ is dense in $W$.

**Theorem 2.1** *[3] Assume (H). Assume the plant is locally exponentially stabilizable and the composition plant - exosystem is exponentially detectable. Then, the output regulation problem can be solved if and only if the following pair of equations*

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), w, c(w)) \tag{RE1}$$

$$0 = h(\pi(w), w) \tag{RE2}$$

*are solved by some $C^k$ mappings $\pi(w), c(w)$ (satisfying $\pi(0)$ 0, $c(0) = 0$).*

**Remark 2.2** In case the state $x$ of the plant and the state $w$ of the exosystem are available for feedback, then it is possible to prove that–under the assumption that the plant is locally exponentially stabilizable and the exosystem satisfies (H)–the solvability of the regulator equations (RE) is still a necessary and sufficient condition for the existence of a feedback law of the form $u = \alpha(x, w)$ yielding both properties (S) of exponential stability and (R) of output regulation.

## 3. Remarks on the General Regulator Problem

The results described in section 2 do give necessary and sufficient conditions for the local solution of the nonlinear regulator problem, provided the system is locally (exponentially) stabilizable and the exosystem is neutrally stable. However, in several rather interesting simulations involving the control of the "ball and beam" experiment and for lateral control of the Harrier II aircraft we have observed that these regulation schemes actually may continue to apply relatively far from the system equilibrium, leading to the possible improvement of those results in two important directions:

(i) a regulator theory valid on a perhaps a priori given open neighborhood $W$ of (0,0) in $\mathbb{R}^n \times \mathbb{R}^\ell$;

(ii) a regulator theory valid for exosystems which are not necessarily "neutrally stable", including for example exosystems which have stable limit cycles, invariant tori, etc.

In this direction we first note that the proof of the sufficiency in Theorem 2.1 is constructive and results–in case $x$ and $w$ are available for feedback–in the design of a control law of the form

$$u(x, w) = k(x - \pi(w)) + c(w) \tag{3.1}$$

where $u = k(x)$ is any (locally) exponentially stabilizing feedback law. Indeed the graph of the function $x = \pi(w)$ is a smooth submanifold which is complementary to the invariant submanifold $w = 0$. The first regulator equation asserts that the submanifold, $gr(\pi)$, is controlled invariant–in fact rendered invariant under (3.1). The second regulator equation asserts that the error vanishes on $gr(\pi)$. And, since $w = 0$ is in fact the stable manifold of the augmented system, the center manifold theorem implies that $gr(\pi)$ is locally attractive. In this light, one sees that there are three essential ingredients in the construction of a nonlinear regulator:

(R1) The existence of a controlled invariant submanifold, $gr(\pi)$;

(R2) $gr(\pi) \subset e^{-1}(0)$;

(R3) A principle of asymptotic phase for $gr(\pi)$; i.e. for all $(x_0, w_0) \in W$ there exists $(\pi(\bar{w}_0), \bar{w}_0) \in gr(\pi)$ such that

$$\|(x_t, w_t) - (\pi(\bar{w}_t), \bar{w}_t)\| \to 0 \text{ as } t \to \infty.$$

In this setting (R1) is equivalent to a solution of the first regulator equation (RE1) and (R2) is equivalent to (RE2). The principle of asymptotic phase, however, does not follow from the regulator equations. Locally, it will follow from the center manifold theory under weaker conditions than (H), which also plays a role in the necessity of solving the regulator equations. Indeed, this extension already allows for inclusion of more general exosystems, as in (ii).

**Definition 3.1** The dynamical system

$$\dot{w} = s(w), \; w \in \mathbb{R}^\ell, \; s(0) = 0$$

is *weakly neutrally stable* provided it is Lyapunov stable and "purely center", in the sense that

$$Re(\lambda) = 0 \text{ for } \lambda \in \sigma(Jac(s)|_{w=0})$$

**Proposition 3.2** *Suppose the system to be controlled is locally exponentially stabilizable and the exosystem is weakly neutrally stable. If $\pi(w)$ and $c(w)$ are solutions to the regulator equation (RE1)-(RE2), then (3.1) is a feedback law which achieves local nonlinear regulation.*

**Example 3.3** (This example was suggested to us by J.W. Grizzle). Consider (2.1) where the exosystem is a van der Pol oscillator with a stable limit cycle

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 + \varepsilon(w_2 - w_2^3) \end{aligned} \tag{3.2}$$

with $\varepsilon > 0$.

This system cannot be immediately treated by means of the methods illustrated in section 2, because its Jacobian matrix

$$\left[\frac{\partial s}{\partial w}\right]_{w=0} = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix} \tag{3.3}$$

has eigenvalues in the right-half plane. However, one can consider the parameter $\varepsilon$ as an additional state variable satisfying

$$\dot{\varepsilon} = 0$$

thus obtaining an augmented exosystem which is weakly neutrally stable. In fact for the control law (3.1), the closed loop system (2.4) has a center manifold $M$ near $(x, w) = (0, 0)$, viz. the graph of the $C^k$ mapping $w \to \pi(w)$. Since this manifold is locally attractive, and the amplitude of the stable limit cycle goes to zero with $\varepsilon$, if the second regulator equation (RE2) is satisfied the output regulation requirement (R) is also achieved for some open set of initial data provided $\varepsilon$ is sufficiently small. In this sense, we can track sufficiently small amplitude, stable limit cycles.

## 4. Passive systems.

We shall henceforth consider nonlinear systems described by equations of the form

$$\dot{x} = f(x) + g(x)u \qquad (4.1a)$$

$$y = h(x) \qquad (4.1b)$$

with state space $X = \mathbb{R}^n$, set of input values $U = \mathbb{R}^m$ and set of output values $Y = \mathbb{R}^m$. The set $\mathcal{U}$ of admissible inputs consists of all $U$-valued piecewise continuous functions defined on $\mathbb{R}$. $f$ and the $m$ columns of $g$ are smooth (i.e. $C^\infty$) vector fields and $h$ is a smooth mapping. We suppose that the vector field $f$ has at least one equilibrium; thus, without loss of generality, after possibly a coordinates shift, we can assume $f(0) = 0$ and $h(0) = 0$.

We begin by reviewing a number of basic concepts related to the notions of dissipativity and passivity (see [24], [11] and [7] for additional details).

**Definition 4.1** A system $\Sigma$ of the form (4.1) is said to be *passive* if there exists a $C^0$ nonnegative function $V : X \to \mathbb{R}$, called the *storage function,* with $V(0) = 0$, such that for all $u \in \mathcal{U}, x^\circ \in X, t \geq 0$,

$$V(x) - V(x^\circ) \leq \int_0^t y^T(s)u(s)ds \qquad (4.2)$$

where $x = \phi(t, x^\circ, u)$.

**Remark 4.2.** Setting $u = 0$, we see from this definition that $V$ is decreasing along any unforced trajectory of (4.1); it follows then that passive systems having a positive definite storage function $V$ are Lyapunov stable. Reciprocally, we see also that $V$ is decreasing along any trajectory of (4.1) consistent with the constraint $y = 0$. Since all such trajectories define what are called the *zero dynamics* of a system [1], we can deduce that passive systems having a positive definite storage function $V$ have a Lyapunov stable zero dynamics. Sometimes, among the passive systems, it is convenient to identify those systems corresponding to the limiting situation in which the dissipation inequality (4.2) becomes a strict equality

**Definition 4.3** A passive system $\Sigma$ with storage function $V$ is said to be *lossless* if for all $u \in \mathcal{U}, x \in X, t \geq 0$,

$$V(x) - V(x^\circ) = \int_0^t y^T(s)u(s)ds$$

Passive systems are related to *positive real* systems. The latter can be defined as follows.

**Definition 4.4** A system $\Sigma$ is said to be *positive real* if for all $u \in \mathcal{U}, t \geq 0$,

$$0 \leq \int_0^t y^T(s)u(s)ds$$

whenever $x(0) = 0$.

The relation between passive and positive real systems depends on the property, of the state space realization, of being reachable form the equilibrium point $x = 0$. We recall that a state $x$ is reachable from 0 if there exists $t > 0$ and $u \in \mathcal{U}$ such that $x = \phi(t, 0, u)$. We also recall that the ***available storage,*** denoted $V_a$, of a system $\Sigma$ is the function $V_a : X \to \mathbb{R}$ defined by

$$V_a(x) = \sup_{x^\circ = x, u \in \mathcal{U}, t \geq 0} \; - \int_0^t y^T(s)u(s)ds$$

**Proposition 4.5***[26]. A passive system is positive real. Conversely, a positive real system in which any state is reachable from the origin and in which $V_a$ is $C^0$ is passive.*

We now turn to another fundamental property of passive systems which is one nonlinear enhancement of the ubiquitous Kalman- Yacubovitch-Popov Lemma for positive real linear systems.

**Definition 4.6** A system $\Sigma$ has the KYP property if there exists a $C^1$ nonnegative function $V : X \to \mathbb{R}$, with $V(0) = 0$, such that

$$L_f V(x) \leq 0 \qquad (4.3a)$$

$$L_g V(x) = h^T(x) \qquad (4.3b)$$

for each $x \in X$.

The two relations (4.3) can be interpreted as the *infinitesimal version* of the dissipation inequality (4.2) for a passive system (although one could, as in the papers [11][12], view the dissipation inequality itself as another nonlinear version of the Kalman-Yacubovitch-Popov Lemma). Concerning (4.3) it is possible to prove, as in [10], the following result.

**Proposition 4.7***[10]. A system $\Sigma$ which has the KYP property is passive, with storage function $V$. Conversely, a passive system having a $C^1$ storage function has the KYP property.*

**Remark 4.8.** Consider a system of the form (4.1a), i.e. with no specific output defined, and suppose there exists a $C^1$ nonnegative function $V : X \to \mathbb{R}$, with $V(0) = 0$, satisfying (4.3a), i.e. such that $L_f V(x) \leq 0$. Several authors (see e.g. [9][15][16][18]) have studied the problem of stabilizing such a system using a state feedback of the form $u = -[L_g V(x)]^T$. In view of (4.3) and of Proposition 4.8, we observe that this control law can be interpreted as a unit gain negative output feedback $u = -y$ imposed on the passive system defined choosing for (4.1a) the output map $y = [L_g V(x)]^T$. We will return later to this point.

We revisit now a certain number of known results about the possibility of asymptotically stabilizing a nonlinear passive system by means of memoryless output feedback. The asymptotic stability of interconnected passive system has been studied in depth in the literature by several authors, either from an operator theoretic point of view (as in [8][19] [20][22][23][25][28]) or in terms of the corresponding state space descriptions (as in [10]-[12], [26][27]). In particular, Hill and Moylan [10]- [12] have developed a synthesis of the techniques from the theory of passive systems and the Lyapunov stability theory which yields a number of important stability results under suitable observability hypotheses.

First of all, we will show how the observability condition used by Hill-Moylan can in fact be slightly weakened and brought to a form, that we call detectability, which is particularly suited to the analysis that will be presented later. In particular, we will derive a direct criterion for detectability for a passive system, stated in terms of Lie brackets of the vector fields which characterizes the input-state decription (4.1a), and is reminiscent of the well-known rank conditions for accessibility. This will enable us to show, in the last part of the section, that certain stabilization laws independently proposed in the literature on geometric nonlinear control, and some generalizations thereof, can all be derived from a basic stabilizability property of passive systems. Observability and detectability are defined as follows.

**Definition 4.9** A system $\Sigma$ is *locally zero-state detectable* if there exists a neighborhood $U$ of 0 such that, for all $x \in U$,

$$h(\phi(t,x,0)) = 0 \quad \text{for} \quad \text{all } t \geq 0 \;\Rightarrow\; \lim_{t\to\infty} \phi(t,x,0) = 0.$$

If $U = X$, the system is *zero-state detectable.* A system $\Sigma$ is *locally zero- state observable* if there exists a neighborhood $U$ of 0 such that , for all $x \in U$,

$$h(\phi(t,x,0)) = 0 \quad \text{for all } t \geq 0 \;\Rightarrow\; x = 0.$$

If $U = X$, the system is *zero-state observable.*

These two definitions are natural extensions of well estabilished concepts from linear system theory. Note however that in some of the literature on passive systems, the term ***detectability*** is used to mean what here is defined as *observability* (see e.g. [10]).

The following statement, whose proof is a natural adaptation of the proof of LaSalle's invariance principle, describes a basic stabilizability property of passive systems. For more general informations about stability of interconnected passive systems, we refer to the papers [10]-[12] by Hill and Moylan. For convenience, we recall that a nonnegative function $V : X \to R$ is said to be *proper* if for each $a > 0$, the set $V^{-1}([0,a]) = \{x \in X : 0 \leq V(x) \leq a\}$ is compact.

**Theorem 4.10** *[7]. Suppose $\Sigma$ is passive with a storage function $V$ which is positive definite. Suppose $\Sigma$ is locally zero-state detectable. Let $\phi : Y \to U$ be any smooth function such that $\phi(0) = 0$ and $y^T\phi(y) > 0$ for each nonzero $y$. The control law*

$$u = -\phi(y) \tag{4.4}$$

*asymptotically stabilizes the equilibrium $x = 0$. If $\Sigma$ is zero-state detectable and $V$ is proper, the control law (4.4) globally asymptotically stabilizes the equilibrium $x = 0$.*

The previous Theorem shows that any passive system having a ***positive definite*** storage function $V$, if zero-state detectable, is (globally) asymptotically stabilized by pure gain output feedback. We will now describe how these assumptions can be tested and will use the conditions thus derived in order to state different criteria for stabilizability. For simplicity, we discuss only the conditions for global asymptotic stabilization. We first recall a result of Hill-Moylan (Lemma 1 of [10]) showing how the positive definiteness of V is implied by the property of zero-state observabilily.

**Proposition 4.11***[10]. Suppose $\Sigma$ is passive with storage function $V$. Suppose $\Sigma$ is zero-state observable. Then $V$ is positive definite.*

The next result, which is slightly more subtle, describes conditions which imply zero-state detectability. The result itself is, to the best of our knowledge, a new result, although its proof is substantially based on a clever argument proposed by Lee-Arapostatis (in the proof of Theorem 1 of [18]). In order to describe this result we need some preliminary material. With the vector fields $f, g_1, \ldots, g_m$ which characterize (4.1a) we associate the distribution

$$D = \text{span}\{ad_f^k g_i : 0 \leq k \leq n-1, 1 \leq i \leq m\}.$$

Moreover, we recall that for a passive system having a $C^1$ storage function $V$ which is positive definite and proper, for any initial condition $x \in X$ , the trajectory $\phi(.,x,0)$ is bounded, and the associated limit set is nonempty and compact. Set

$$\Omega = \cup_{x^\circ \in X}(\omega - \text{limit} \;\; \text{set of } \phi(.,x^\circ,0))$$

The objects thus introduced are useful in testing the zero-state detectability and/or observability of a passive system.

**Proposition 4.12** *[7]. Suppose $\Sigma$ is passive with a proper $C^r, r \geq 1$, storage function $V$. Let $S$ denote the set*

$$S = \{x \in X : L_f{}^m L_\tau V(x) = 0, \text{for all } \tau \in D, \text{ all } 0 \leq m < r\}$$

*If $S \cap \Omega = \{0\}$ and $V$ is positive definite, then $\Sigma$ is zero-state detectable. If $S = \{0\}$ and $\Sigma$ is lossless, then $\Sigma$ is zero-state observable.*

Using either one of the conditions described in Propositions 4.12 and 4.13 in order to check the assumptions required by the basic stabilization strategy expressed by Theorem 4.10, it is possible to recover a number of stabilization results independently proposed in the literature by various authors, thus showing that a number of apparently independent stabilization schemes (see [9][15][16][18]) reduce, in fact, to the one of a passive system subject to pure gain output feedback.

## 5. Feedback equivalence to a passive system.

In this section we address the problem of when a given nonlinear system is feedback equivalent to a passive system with positive definite storage function V. As we shall see in a moment, a role of major importance is played by property - for the system - of being minimum phase. We assume the reader familiar with the concepts of relative degree, normal form, and zero dynamics (see [1] or [13] for details). In particular, we recall that a system of the form (4.1) is said to have relative degree $\{1, \ldots, 1\}$ at $x = 0$ if the matrix $L_g h(0)$ is nonsingular. If this is the case and if the distribution spanned by the vector fields $g_1(x), \ldots, g_m(x)$ is involutive, it is possible to find $n-m$ real-valued functions $z_1(x), \ldots z_{n-m}(x)$, locally defined near $x = 0$ and vanishing at $x = 0$, which, together with the $m$ components of the output map $y = h(x)$, qualify as a new set of local coordinates. In the new coordinates $(z, y)$, the system is represented by equations having the following structure (normal form)

$$\dot{z} = q(z,y) \tag{5.1a}$$
$$\dot{y} = b(z,y) + a(z,y)u \tag{5.1b}$$

where the matrix $a(z,y)$ is nonsingular for all $(z,y)$ near $(0,0)$.

In the normal form (4.1) the zero dynamics of the system, which describe those internal dynamics which are consistent with the external constraint $y = 0$, are characterized by the equation $\dot{z} = q(z,0) := f^*(z)$. In what follows, we shall sometimes reexpress $q(z,y)$ in the form $q(z,y) = f^*(z) + p(z,y)y$ where $p(z,y)$ is a smooth function.

In [4], necessary and sufficient conditions for the existence of a globally defined normal form of the type (5.1) have been investigated. In addition to the nonsingularity of the matrix $L_gh(x)$, these conditions require further properties on set of $m$ vector fields $\bar{g}_1(x), \dots, \bar{g}_m(x)$ defined by $[\bar{g}_1(x) \dots \bar{g}_m(x)] = g(x)[L_gh(x)]^{-1}$. More precisely, there exists a globally defined diffeomorphism which tranforms the system (4.1) into a system having the normal form (5.1) if and only if:

(H1) the matrix $L_gh(x)$ is nonsingular for each $x \in X$,
(H2) the vector fields $\bar{g}_1(x), \dots, \bar{g}_m(x)$ are complete,
(H3) the vector fields $\bar{g}_1(x), \dots, \bar{g}_m(x)$ commute.

If this is the case, then globally defined zero dynamics exist for the system. Note that the condition (H3) is equivalent to the condition that the distribution spanned by $g_1(x), \dots, g_m(x)$ is involutive.

A system whose zero dynamics are asymptotically stable has been called a minimum phase system (see [1]-[3]). In the following definition, we specialize this concept in a more detailed manner.

**Definition 5.1** Suppose $L_gh(0)$ is nonsingular. Then $\Sigma$ is said to be:

(i) minimum phase if $z = 0$ is an asymptotically stable equilibrium of $f^*(z)$,
(ii) weakly minimum phase if there exists a $C^r, r \geq 2$, function $W^*(z)$, locally defined near $z = 0$ with $W^*(0) = 0$, which is positive definite and such that $L_{f^*}V(z) \leq 0$ for all $z$ near $z = 0$.

Suppose (H1) - (H2) - (H3) hold. Then $\Sigma$ is said to be:

(iii) globally minimum phase if $z = 0$ is a globally asymptotically stable equilibrium of $f^*(z)$,
(iv) globally weakly minimum phase if there exists a $C^r, r \geq 2$, function $W^*(z)$, defined for all $z$ with $W^*(0) = 0$, which is positive definite and proper and such that $L_{f^*}V(z) \leq 0$ for all $z$.

We proceed now to illustrate how the concepts of relative degree and zero dynamics arise naturally in the study of passive systems, playing in fact an important role. We begin by analyzing the relative degree of a passive system. In what follows, for convenience, we will say that a point $x$ is a regular point for a system $\Sigma$ of the form (4.1) if $\text{rank}\{L_gh(x)\}$ is constant in a neighborhood of $x$. We also assume throughout the section that $\text{rank}\{g(0)\} = \text{rank}\{dh(0)\} = m$.

**Theorem 5.2** *[7]. Suppose $\Sigma$ is passive with a $C^2$ storage function $V$ which is positive definite. Suppose $x = 0$ is a regular point for $\Sigma$. Then $L_gh(0)$ is nonsingular and $\Sigma$ has relative degree $\{1, \dots, 1\}$ at $x = 0$.*

The next result characterizes the asymptotic properties of the zero dynamics of a passive system.

**Theorem 5.3** *[7]. Suppose $\Sigma$ is passive with a $C^2$ storage function $V$ which is positive definite. Suppose that $x = 0$ is a regular point for $\Sigma$. Then the zero dynamics of $\Sigma$ locally exist at $x = 0$ and $\Sigma$ is weakly minimum phase.*

Theorems 5.2 and 5.3 show, in essence, that any passive system with a positive definite storage function, under mild regularity assumptions, necessarily has relative degree $\{1, \dots, 1\}$ at $x = 0$ and is weakly minimum phase. The next step of our investigation is to show that exactly these two conditions characterize the equivalence, via state feedback, to a passive system. We consider here regular static (i.e. memoryless) state feedback, i.e. a feedback of the form $u = \alpha(x) + \beta(x)v$ where $\alpha(x)$ and $\beta(x)$ are smooth functions defined either locally near $x = 0$ or globally, and $\beta(x)$ is invertible for all $x$.

**Theorem 5.4** *[7]. Suppose $x = 0$ is a regular point for $\Sigma$. Then $\Sigma$ is locally feedback equivalent to a passive system with a $C^2$ storage function $V$, which is positive definite, if and only if $\Sigma$ has relative degree $\{1, \dots, 1\}$ at $x = 0$ and is weakly minimum phase.*

**Remark 5.5.** In a linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

with $\text{rank}\{B\} = m$, $x = 0$ is always a regular point and a normal form - whenever it exists - is globally defined. Thus, from the previous result we immediately obtain that any linear system is feedback equivalent to a passive linear system with a storage function $V(x) = x^TQx$, which is positive definite, if and only if $CB$ is nonsingular and the system is weakly minimum phase. Since any controllable linear system is passive, with a storage function $V(x) = x^TQx$ which is positive definite, if and only if it is positive real (see e.g. [27]), we also see that any controllable linear system is feedback equivalent to a positive real system if and only if $CB$ is nonsingular and the system is weakly minimum phase.

A global version of Theorem 5.4 indeed exists if the system in question has a global form.

**Theorem 5.6** *[7]. Assume (H1) - (H3). Then $\Sigma$ is globally feedback equivalent to a passive system with a $C^2$ storage function $V$, which is positive definite, if and only if $\Sigma$ is globally weakly minimum phase.*

So far, we have investigated the feedback equivalence of a given system to a passive system with positive definite storage function $V$. In the next statement, we analyze the particular configuration in which the system in question can be expressed in the form

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y \tag{5.3a}$$
$$\dot{x} = f(x) + g(x)u \tag{5.3b}$$
$$y = h(x) \tag{5.3c}$$

which we assume to be globally valid (of course, corresponding local results also hold). The analysis of configurations of this type was considered in a number of previous papers (see e.g. [3][17][24]). In view of the particular structure of (5.3), we will call (5.3b)-(5.3c) the *driving system*, while (5.3a) will be called the *driven system*.

First of all, note that if the point $(\zeta, x) = (0,0)$ were a point of regularity for the full system (5.3), then its local feedback equivalence to a passive system would be of course determined by the conditions described in Theorem 5.5, namely the properties of having relative degree $\{1, \dots, 1\}$ at $(\zeta, x) = (0,0)$ and of being weakly minimum phase. Note also that, in view of the special structure of (5.3) the point

$(\zeta, x) = (0,0)$ is a point of regularity for the full system if and only if the point $x = 0$ is a point of regularity for the driving system and that, in particular, the full system has relative degree $\{1, \dots, 1\}$ at $(\zeta, x) = (0,0)$ if and only if the driving system has relative degree $\{1, \dots, 1\}$ at $x = 0$. Finally, note that, in this case (that is, if $L_g h(0)$ is nonsingular) the zero dynamics of the full system have the form

$$\dot{\zeta} = f_0(\zeta) \tag{5.4a}$$
$$\dot{z} = f^*(z) \tag{5.4b}$$

where $f^*(z)$ is exactly the zero dynamics vector field of the driving system. Thus, the full system is weakly minimum phase if and only if the driving system is, and there exists a positive definite function $U(\zeta)$, locally defined near $\zeta = 0$ with $U(\zeta) = 0$, such that $L_{f_0}U(\zeta) \leq 0$ for all $\zeta$. Similar considerations can be repeated in a global setting, and we can therefore deduce, as an immediate application of our previous discussion, the following result.

**Corollary 5.7.** *Suppose the triplet $\{f, g, h\}$ satisfies the assumptions (H1) - (H3) (or, what is the same, suppose a normal form of the type (5.1) globally exists for the driving system of (5.3)). Then, the full system (5.3) is feedback equivalent to a passive system with a $C^2$ storage function $V$, which is positive definite, if and only if the driving system is weakly minimum phase and there exists a positive definite function $U(\zeta)$, defined for all $\zeta = 0$ with $U(\zeta) = 0$, such that $L_{f_0}U(\zeta) \leq 0$ for all $\zeta$.*

In the next statement, we show how feedback equivalence of (5.3) to a passive system can be determined without assuming the existence of a normal form for the driving system.

**Theorem 5.8** *[7]. Consider the system (5.3). Suppose (5.4a) is globally asymptotically stable and $\{f, g, h\}$ is passive with a $C^r, r \geq 1$, storage function $V$, which is positive definite. The system is feedback equivalent to a passive system with a $C^r$ storage function which is positive definite.*

## 6. Global stablization of weakly minimum phase nonlinear systems.

We now apply some of the results illustrated so far to the problem of deriving globally asymptotically stabilizing feedback laws for certain classes of nonlinear systems. In particular, we give a fairly general theorem which incorporates and extends a number of interesting results which recently appeared in the literature.

**Theorem 6.1** *[7]. Consider a system $\Sigma$ described by*

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y \tag{6.1a}$$
$$\dot{x} = f(x) + g(x)u \tag{6.1b}$$
$$y = h(x) \tag{6.1c}$$

*Suppose $\dot{\zeta} = f_0(\zeta)$ is globally asymptotically stable. Suppose $\{f, g, h\}$ is passive with a $C^r, r \geq 1$, storage function $V$, which is positive definite and proper, and suppose $S = \{0\}$ with $S$ defined as in section 4. Then $\Sigma$ is globally asymptotically stabilizable by smooth state feedback.*

*As an immediate application of this result we obtain.*

**Corollary 6.2.** *Consider a system $\Sigma$ described by*

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y$$
$$\dot{y} = f(\zeta, y) + g(\zeta, y)u$$

*Suppose $g(\zeta, y)$ is invertible for all $\zeta, y$. Suppose $\dot{\zeta} = f_0(\zeta)$ is globally asymptotically stable. Then $\Sigma$ is globally asymptotically stabilizable by smooth state feedback.*

The system considered in this statement is just a globally minimum phase system with relative degree $\{1, \dots, 1\}$ represented in its global normal form. Thus, Corollary 6.2 coincides with Theorem 2.1 of [3]. One of our next applications consists in showing that the minimum phase assumption of Theorem 2.1 of [3] can in fact be weakened, in the sense that also weakly minimum phase systems may be globally asymptotically stabilized by smooth feedback. We will prove this result after having shown how Theorem 6.1 specializes in case the driving system has a globally defined normal form.

To this end, recall that if a system

$$\dot{z} = f^*(z) + p(z, y)y \tag{6.3a}$$
$$\dot{y} = b(z, y) + a(z, y)v \tag{6.3b}$$

is globally weakly minimum phase, there exists a $C^r, r \geq 1$, function $W^*(z)$, defined for all $z$ with $W^*(0) = 0$, which is positive definite and proper, and such that $L_{f^*}W^*(z) \leq 0$ for all $z$. Set $g^*(z) = p(z, 0)$ and define

$$D^* = \text{span}\{ad_{f^*}{}^k g_i{}^*, 0 \leq k \leq n - m - 1, 1 \leq i \leq m\} \tag{6.4a}$$
$$S^* = \{z \in Z^* : L_{f^*}{}^m L_\tau W^*(z) = 0, \text{ for all } \tau \in D^*, \text{ all } 0 \leq m < r\} \tag{6.4b}$$

Then, the following statement expresses the form to which Theorem 6.1 reduces in case the driving system has a globally defined normal form.

**Theorem 6.3.** *Consider a system $\Sigma$ described by*

$$\dot{\zeta} = f_0(\zeta) + f_1(\zeta, y)y \tag{6.5}$$
$$\dot{z} = f*(z) + p(z, y)y \tag{6.6a}$$
$$\dot{y} = b(z, y) + a(z, y)u \tag{6.6b}$$

*Suppose the unforced dynamics of the driven system (6.5) is globally asymptotically stable and suppose the driving system (6.6) has relative degree $\{1, \dots, 1\}$ at each point and is globally weakly minimum phase. Suppose $S^* = \{0\}$ (where $S*$ is defined as in (6.4)). Then $\Sigma$ is globally asymptotically stabilizable by smooth state feedback.*

Taking the driven system to be trivial in Theorem 6.4, we obtain the following Corollary, an extension of Theorem 2.1 of [3], which describes conditions under which a globally weakly minimum phase system can be globally asymptotically stabilized by smooth feedback.

**Corollary 6.4.** *Suppose the system $\Sigma$ described by*

$$\dot{z} = f^*(z) + p(z, y)y$$
$$\dot{y} = b(z, y) + a(z, y)u$$

*has relative degree $\{1, \dots, 1\}$ at each point and is globally weakly minimum phase. Suppose $S^* = \{0\}$. Then $\Sigma$ is globally asymptotically stabilizable by smooth state feedback.*

Finally, we consider the special situation in which the driving system is linear and controllable. We will also assume that the driving system satisfies either of the following conditions:

(i) CB is nonsingular and the system is weakly minimum phase; or
(ii) the system is feedback equivalent to a passive system, with positive definite storage function $V(x) = x^T Q x$; or

(iii) the system is feedback equivalent to a positive real system.

As a matter of fact, the characterization we obtained for feedback equivalence of a nonlinear system to a passive system proves, in the case of linear systems, the equivalence of (i), (ii) and (iii). Accordingly, we obtain a Corollary, which contains as a particular case Theorem 2.1 of [17].

**Corollary 6.5.** *Consider a system $\Sigma$ described by*

$$\dot{z} = f_0(\zeta) + f_1(\zeta, y)y$$
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

*Suppose the unforced dynamics of the driven system is globally asymptotically stable. Suppose $(A, B)$ is controllable and suppose the driving system $\{A, B, C\}$ satisfies either one of the three equivalent conditions (i), (ii) or (iii). Then $\Sigma$ is globally asymptotically stabilizable by smooth state feedback.*

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