Proceedings of the 29th Conference
on Decision and Control
Honolulu, Hawaii • December 1990

The solution of the rational interpolation problem: a summary

A. C. Antoulas
Rice University & E.T.H. Zürich

J. A. Ball, J. Kang
Virginia Polytechnic Institute and State University

and

J. C. Willems
University of Groningen

Abstract. *A new method for dealing with the rational interpolation problem, first proposed in Antoulas and Willems [1990] and extended in Antoulas, Ball, Kang and Willems [1990], is summarized. It is based on the reachability of an appropriately defined pair of matrices.*

1. Introduction

Given the array of scalar pairs of points

$$(x_i, y_i),\ i \in \underline{N}, x_i \neq x_j, i \neq j, \tag{1}$$

we seek all *rational interpolants*, that is rational functions

$$y(x) = \frac{n(x)}{d(x)},\ n, d\text{: coprime},$$

which *interpolate* the array (1), i.e.

$$y(x_i) = y_i,\ i \in \underline{N}.$$

Moreover, we wish to keep track of the complexity of interpolants, which is defined to be their McMillan degree:

$$deg\ y := \max\{deg\ n, deg\ d\}.$$

The following problems arise: **(a)** Find all *admissible* degrees of complexity, i.e. those positive integers κ for which there exist interpolants $y(x)$ with $deg\ y = \kappa$. **(b)** Given an admissible degree κ, construct *all* corresponding solutions.

The first complete answer to the above questions was provided by Antoulas and Anderson [1986] using the so-called Löwner matrix as the main tool. For further developments, see Antoulas and Anderson [1989], [1990] as well as Anderson and Antoulas [1990]. In the present paper we will discuss a novel approach to answering the above questions.

2. The main result

Define the following pair of matrices:

$$F := \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_N \end{bmatrix},\quad G := \begin{bmatrix} 1 & -y_1 \\ 1 & -y_2 \\ \vdots & \vdots \\ 1 & -y_N \end{bmatrix}. \tag{2}$$

Since $x_i \neq x_j, i \neq j$, this pair is *reachable*. Furthermore, there exist right coprime polynomial matrices $W(x), \Theta(x)$ such that

$$(xI - F)W(x) = G\Theta(x),\ \det\Theta(x) \neq 0. \tag{3}$$

The 2×2 matrix $\Theta := (\theta_{ij})$, in the above equation can be assumed, wihtout loss of generality, *column reduced*. Let the corresponding column indices be

$$\kappa_1 := deg \begin{bmatrix} \theta_{11}(x) \\ \theta_{21}(x) \end{bmatrix} \leq \kappa_2 := deg \begin{bmatrix} \theta_{12}(x) \\ \theta_{22}(x) \end{bmatrix},\ \kappa_1 + \kappa_2 = N.$$

We are now ready to state the

Main Theorem. *Given the array (1), define the pair of matrices (2), and let $\Theta(x)$ be a column reduced polynomial matrix satisying (3).*
(i) If $\kappa_1 < \kappa_2$ and θ_{11}, θ_{21} are coprime,

$$y^{\min}(x) = \frac{\theta_{11}(x)}{\theta_{21}(x)},\ \delta(y^{\min}) = \kappa_1,$$

is the unique minimal interpolant. Furthermore, there are no interpolants of complexity between κ_1 and κ_2.
(ii) Otherwise, there is a family of interpolating functions of minimal complexity which can be parametrized as follows:

$$y^{\min}(x) = \frac{\theta_{12}(x) + p(x)\theta_{11}(x)}{\theta_{22}(x) + p(x)\theta_{21}(x)},\ \delta(y^{\min}) = \kappa_2 = N - \kappa_1,$$

where the polynomial $p(x)$ satisfies

$$deg\ p = \kappa_2 - \kappa_1,\ \theta_{22}(x_i) + p(x_i)\theta_{21}(x_i) \neq 0,\ i \in \underline{N}. \tag{14}$$

(iii) In both cases (i) and (ii), there are families of interpolants $y = n/d$ of every degree $\kappa \geq \kappa_2$, satisfying (10), where $deg\ p = \kappa - \kappa_1$ and $deg\ q = \kappa - \kappa_2$.

Questions (a) and (b) posed in the introduction have thus been answered.

3. Generalizations

The above result can be generalized with little additional effort to the various matrix interpolation problems, like the tangential or directional problems, as well as to the multiple point versions thereof. For details, the reader is referred to the two references given in the abstract.

4. References

B. D. O. Anderson and A. C. Antoulas [1990], "Rational interpolation and state variable realization", *Linear Algebra and Its Applications*, Special Issue on Matrix Problems, in press.

A. C. Antoulas and B. D. O. Anderson [1986], "On the scalar rational interpolation problem", *IMA J. of Mathematical Control and Information*, **3**: 61-88.

A. C. Antoulas and B. D. O. Anderson [1989], "On the problem of stable rational interpolation", *Linear Algebra and Its Applications*, **122-124**: 301-329.

A. C. Antoulas and B. D. O. Anderson [1990], "State-space and polynomial approaches to rational interpolation", Proc. MTNS '89, Birkhäuser, Boston.

A. C. Antoulas, J. A. Ball, J. Kang, and J. C. Willems [1990], "On the solution of the minimal rational interpolation problem", *Linear Algebra and Its Applications*, Special Issue on Matrix Problems, in press.

A. C. Antoulas and J. C. Willems [1990], "Minimal rational interpolation and Prony's method", in Analysis and Optimization of Systems, A. Bensoussan and J. L. Lions Editors, Lecture Notes in Control and Information Sciences, **144**: 297-306, Springer Verlag.