Proceedings of the 29th Conference
on Decision and Control
Honolulu, Hawaii • December 1990

CONTINUITY OF LATENT VARIABLE MODELS

by

JAN C. WILLEMS AND J.W. (HANS) NIEUWENHUIS

Dept. of Mathematics   Dept. of Econometrics
University of Groningen
P.O. Box 800
9700 AV Groningen
The Netherlands

ABSTRACT

We study the continuity of the behavior of dynamical systems as a function of the parameters in their behavioral equations. The problem is motivated by means of an *RLC*-circuit whose port behavior exhibits a surprising discontinuity as a function of the numerical values of the elements in the circuit. The main result states that a system described by means of difference equations involving manifest (external) and latent (internal) variables will have a continuous behavior in the limit if the limit system is observable.

1. INTRODUCTION. One of the important issues in the study of mathematical models is the continuity of their behavior as a function of the parameters describing the behavioral equations. Bifurcation theory and structural stability questions are examples of research areas which address such questions. However, also areas as system identification, robustness of control systems, and the performance of adaptive control schemes, are other areas where (implicitely) continuity questions are raised. The question of continuity is indeed an important issue in automatic control theory in particular, and in modelling in general.

A suitable mathematical formulation of the continuity property involves more than meets the eye. To begin with, it is not clear how to formalize it. Second, since a typical parametrization of a system's behavior involves a many-to-one map one cannot expect that system continuity will simply correspond to parameter continuity. In addition, in physical systems neither the input/output structure nor the (dimension of the) state space structure will be robust under parasitic perturbations and it is clear that the state space framework does not always provide a satisfactory general setting for studying the continuity issue. Finally, it is possible to give examples of innocent looking physical, lumped, linear, time-invariant systems whose behavior exhibits surprising discontinuities as a function of the numerical values of the physical elements. In Section 2 such an example is worked out in detail.

In [1,2] we have initiated a fundamental study of the continuity issue. Our setting follows the framework of [3,4,5]: a system is defined in terms of its behavior and continuity requires that this behavior is continuous in the limit. The main result of [1,2] may be formulated as follows. Consider a family of dynamical systems which can all be described by a system of difference or differential equations (σ denotes the shift)

$$R_\varepsilon(\sigma,\sigma^{-1})w = 0 \qquad \text{or} \qquad R_\varepsilon\left(\frac{d}{dt}\right)w = 0$$

in which the polynomial matrix $R_\varepsilon(s,s^{-1})$ or $R_\varepsilon(s)$ depends on a real parameter $\varepsilon \geq 0$. Then under suitable conditions (involving the rank and the maximal degree of the R_ε's) the behavior of these dynamical systems will be continuous as $\varepsilon \to 0$ if and only if the polynomials R_ε converge. The precise formulation of the 'only if' part of this claim is quite involved (see [1,2]). In any case, this is an appealing result which, roughly speaking, permits us to identify (for this class of systems and under suitable conditions!) system convergence with parameter convergence.

The present paper may be viewed as a continuation of this study. The problem raised stems from the observation (elaborated in [4,5]) that models obtained from first principles will invariably involve both manifest (external) and latent (internal) variables. For example, when describing the port behavior of an electrical circuit, one will need to introduce the voltages across and the currents through the internal branches in order to express the laws governing the circuit: the constitutive equations of the elements and Kirchhoff's voltage and current laws. In the context of the class of systems introduced above this leads to systems described by difference or differential equations of the type

$$R_\varepsilon(\sigma,\sigma^{-1})w = M_\varepsilon(\sigma,\sigma^{-1})a \qquad \text{or} \qquad R_\varepsilon\left(\frac{d}{dt}\right)w = M_\varepsilon\left(\frac{d}{dt}\right)a$$

with w the manifest and a the latent variables. The question arises whether, when the coefficients of the polynomial matrices $R_\varepsilon(s,s^{-1})$ and $M_\varepsilon(s,s^{-1})$ or $R_\varepsilon(s)$ and $M_\varepsilon(s)$ converge, this also holds for the manifest behavior (the port behavior in the case of electrical circuits). In Section 2 an example of a simple *RLC*-circuit is studied where this continuity does not hold. Our *Main Result* formulated in Section 5 states that *this continuity holds under the requirement that the limit system is observable!*

2. AN EXAMPLE. In order to illustrate the sort of phenomenon which will be studied in this paper, we will start with a simple concrete example. Let us model the *RLC*-circuit shown below:

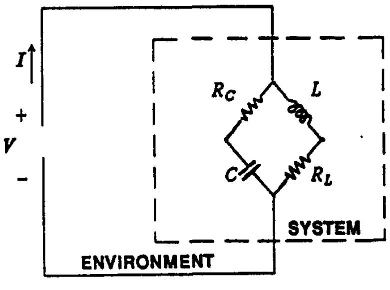

FIGURE 1

We assume that the values of the elements R_C, R_L, L, and C are all positive. The circuit interacts with its environment through the external port. The attributes which describe this interaction are the current I into the circuit and the voltage V across its external terminals. We will call these *manifest variables*. In order to specify the terminal behavior, we will introduce as auxiliary variables the currents through and the voltages across the internal branches of the circuit, as shown in Figure 2. We will call these auxiliary variables *latent variables*.

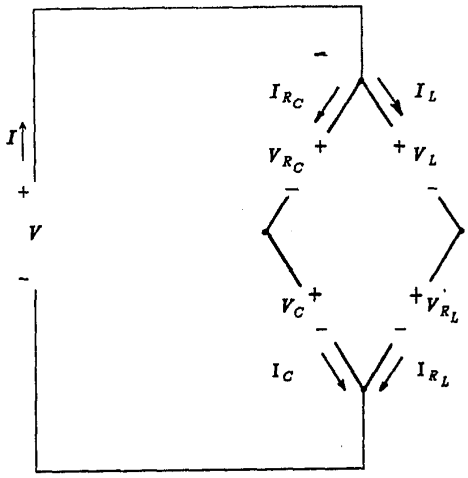

FIGURE 2

The following equations specify the laws governing the dynamics of this circuit. They define the relations between the port variables and the latent variables. We will call these equations the *full behavioral equations.*

Constitutive equations:

$$V_{R_C}=R_C I_{R_C} \; ; \; V_{R_L}=R_L I_{R_L} \; ;$$

$$C\frac{dV_C}{dt}=I_C \; ; \; L\frac{dI_L}{dt}=V_L \qquad (CE)$$

Kirchhoff's current laws:

$$I=I_{R_C}+I_L \; ; \; I_{R_C}=I_C \; ; \; I_L=I_{R_L} \; ;$$

$$I_C+I_{R_L}=I \qquad (KCL)$$

Kirchhoff's voltage laws:

$$V=V_{R_C}+V_C \; ; \; V=V_L+V_{R_L} \; ;$$

$$V_{R_C}+V_C=V_L+V_{R_L} \qquad (KVL)$$

This yields the *full* (or *internal*) *behavior*, defined as:

$$\mathfrak{B}_f=\{(I,V,I_{R_C},V_{R_C},I_{R_L},V_{R_L},I_C,V_C,I_L,V_L):\mathbb{R}\to\mathbb{R}^{10} \mid (CE),\ (KCL),\ \text{and } (KVL) \text{ are satisfied}\}$$

From this the (*manifest, external*) *port behavior*, follows. It is formally defined as:

$$\mathfrak{B}=\{(I,V):\mathbb{R}\to\mathbb{R}^2 \mid \exists (I_{R_C},V_{R_C},I_{R_L},V_{R_L},I_C,V_C,I_L,V_L):\mathbb{R}\to\mathbb{R}^8 \text{ such that } (I,V,I_{R_C},V_{R_C},I_{R_L},V_{R_L},I_C,V_C,I_L,V_L)\in\mathfrak{B}_f\}$$

Let us now carry out the elimination of the latent variables in order to come up with an explicit relation involving the port variables V and I only. Eliminating I_{R_C}, V_{R_C}, I_{R_L}, and V_{R_L} using (KCL) and (KVL) leads to:

$$V=V_C+R_C I_C \; ; \; V=V_L+R_L I_L \; ; \; C\frac{dV_C}{dt}=I_C \; ;$$

$$L\frac{dI_L}{dt}=V_L \; ; \; I=I_C+I_L$$

Next, eliminate I_C using the first of these equations and V_L using the second, to obtain the '*state equations*':

$$\begin{aligned} \frac{dV_C}{dt} &= -\frac{1}{CR_C}V_C+\frac{1}{CR_C}V \\ \frac{dI_L}{dt} &= -\frac{R_L}{L}I_L+\frac{1}{L}V \\ V-R_C I &= V_C-R_C I_L \end{aligned}$$

Now, substitute V_C from the third of these equations in the first to obtain:

$$\frac{dI_L}{dt}+\frac{1}{CR_L}I_L=-\frac{1}{R_C}\frac{dV}{dt}+\frac{dI}{dt}+\frac{1}{CR_C}I$$

Using the second of the above equations, leads to:

$$\left(\frac{1}{CR_L}-\frac{R_L}{L}\right)I_L=\frac{dI}{dt}-\frac{1}{R_C}\frac{dV}{dt}+\frac{1}{CR_C}I+-\frac{1}{L}V$$

Now distinguish 2 cases:

CASE 1: $\frac{1}{CR_C}=\frac{R_L}{L}$. Then the above equation yields

$$\left(CR_C\frac{d}{dt}+\frac{R_C}{R_L}\right)V=\left(CR_C\frac{d}{dt}+1\right)R_C I \qquad (A)$$

as the desired port relation between the variables V and I.

CASE 2: $\frac{1}{CR_C}\neq\frac{R_L}{L}$. Then use the earlier equation to solve for I_L. Upon substitution in the previous equation this yields, after some reorganization:

$$\left(\frac{L}{R_L}CR_C\frac{d^2}{dt^2}+\left(1+\frac{R_C}{R_L}\right)CR_C\frac{d}{dt}+\frac{R_C}{R_L}\right)V = \left(CR_C\frac{d}{dt}+1\right)\left(\frac{L}{R_L}\frac{d}{dt}+1\right)R_C I \qquad (B)$$

as the desired port relation between the variables V and I.

Note that if we set $\frac{1}{CR_C}=\frac{R_L}{L}$ in the second equation we obtain:

$$\left(CR_C\frac{d}{dt}+1\right)\left(CR_C\frac{d}{dt}+\frac{R_C}{R_L}\right)V = \left(CR_C\frac{d}{dt}+1\right)\left(CR_C\frac{d}{dt}+1\right)R_C I \qquad (C)$$

Of further interest is the case in which in addition to $\frac{1}{CR_C}=\frac{R_L}{L}$, there also holds $\frac{R_C}{R_L}=1$.

Let us now state some results which follow from earlier work of ours. The first one concerns the presence of *common factors* in the polynomials p and q in a differential equation model

$$p\left(\frac{d}{dt}\right)y=q\left(\frac{d}{dt}\right)u \qquad p(s),q(s)\in\mathbb{R}[s]$$

describing the relation between $u(\cdot):\mathbb{R}\to\mathbb{R}$ and $y(\cdot):\mathbb{R}\to\mathbb{R}$. While a common factor can indeed be cancelled if we are only concerned with the transfer function, it cannot be cancelled when we are considering the behavior! Common factors do contribute to the behavior. The presence of common factors signifies *lack of controllability* in the dynamical system. *Cancelling common factors corresponds to taking the controllable part of the system* [4,5].

The second fact concerns the *limit behavior* of systems described by a set of differential equations whose coefficients depend on a parameter. Under reasonable conditions (which are satisfied for the full behavioral equations describing the above circuit and for equations (A), (B), and (C)) it can be shown that the limit behavior is, as may be expected, described by the equations obtained by substituting the limit of the parameters in the original equations.

The third fact concerns the observability of the latent variables in the full behavioral equations (CE), (KCL), and (KVL). These equations are observable if and only if $\frac{1}{CR_C} \neq \frac{R_L}{L}$.

This yields the following situation for the circuit at hand (remember that throughout we assume R_L, R_C, L, and $C>0$):

1. Assume $\frac{1}{CR_C} = \frac{R_L}{L}$. Then the port behavior is described by equation (A). The full behavioral equations are observable. If $\frac{R_C}{R_L} \neq 1$, then the port behavior is controllable.
2. Assume $\frac{1}{CR_C} \to \frac{R_L}{L}$. Then the limit port behavior is described by equation (C). This limit behavior differs from the behavior for $\frac{1}{CR_C} = \frac{R_L}{L}$ by the common factor $\left(CR_C \frac{d}{dt} + 1 \right)$, yielding a non-controllable mode. If $\frac{R_C}{R_L} \neq 1$, then the controllable part of this limit behavior equals the behavior at the limit.
3. Assume $\frac{1}{CR_C} = \frac{R_L}{L}$ *and* $\frac{R_C}{R_L} = 1$. Then the full behavioral equations are unobservable and the port behavior is described by
$$\left(CR_C \frac{d}{dt} + 1 \right) V = \left(CR_C \frac{d}{dt} + 1 \right) R_C I$$
This defines a non-controllable dynamical system. Its controllable part is described by $V = R_C I$.
4. Assume $\frac{1}{CR_C} \to \frac{R_L}{L}$ *and* $\frac{R_C}{R_L} \to 1$. Then the limit port behavior is described by
$$\left(CR_C \frac{d}{dt} + 1 \right)\left(CR_C \frac{d}{dt} + 1 \right) V = \left(CR_C \frac{d}{dt} + 1 \right)\left(CR_C \frac{d}{dt} + 1 \right) I$$

It is not controllable; its controllable part is described by $V = R_C I$. Thus in this case the limit behavior strictly includes the behavior at the limit which, in turn, strictly includes the controllable behavior.

Note that it follows from this example that the difference between the limit behavior and the behavior at the limit is more than merely a matter of cancelling common factors. It illustrates that taking *limit systems* (and *cancelling common factors*) is *'tricky business'*.

Note that careless calculation of the open circuit behavior of this circuit ($I = o$) would have given

$$\left(\frac{L}{R_L} CR_C \frac{d^2}{dt^2} + \left(1 + \frac{R_C}{R_L} \right) CR_C \frac{d}{dt} + \frac{R_C}{R_L} \right) V = o$$

with no warning bells (as the common factor phenomenon for the port behavior) that, when $CR_C = \frac{L}{R_L}$, these equations cease to be valid. Indeed taking $CR_C \to \frac{L}{R_L}$ yields

$$\left(CR_C \frac{d}{dt} + 1 \right)\left(CR_C \frac{d}{dt} + \frac{R_C}{R_L} \right) V = o$$

as the limit behavior, while the behavior when $CR_C = \frac{L}{R_L}$ is actually described by

$$\left(CR_C \frac{d}{dt} + \frac{R_C}{R_L} \right) V = o \, !$$

We hasten to correct two erroneous impressions which the above example may have given. First, the phenomenon that the limit behavior is unequal to the behavior at the limit for systems described in terms of latent variables can be seen from much simpler examples than the one given. For instance the full behavioral equations

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix} a$$

with latent variable a give $w_2 = w_1$ for the limit behavior, while the behavior at the limit is $w_1 = w_2 = o$. Second, as this last example also shows, the difference between the limit behavior and the behavior at the limit is not just a matter of uncontrollable modes (equivalently, of common factors) at the limit.

The point of this extensive example is to illustrate that the behavior of a linear, time-invariant dynamical system describing a simple electrical circuit need not be continuous in the physical parameters. This discontinuity is moreover not due to element values going to zero or to infinity. In particular, it is not a problem of singular perturbations. The purpose of this article is to formalize this phenomenon and to give a very natural system theoretic condition (observability) which yields the desired continuity of the behavior in terms of the system parameters. (The reason why the example gives the impression that controllability is an issue is due to the fact that for this electrical circuit observability and state point controllability simultaneously hold).

3. OUR MODELLING TRIPTICH. We start our development with a brief exposition of the modelling philosophy and language which we have been preaching during the last decade. The basic ingredients of this theory constitutes a tryptich consisting of the behavior, behavioral equations, and latent variables.

We define a *(mathematical) model* as a pair $(\mathfrak{U},\mathfrak{B})$ with $\mathfrak{U}$ a *universum* and $\mathfrak{B}$ a subset of $\mathfrak{U}$ called the ***behavior***. Thus a mathematical model is an exclusion law: from all the a priori possibilities in $\mathfrak{U}$, it recognizes only those in $\mathfrak{B}$ as being in principle possible. In most examples the behavior $\mathfrak{B}$ will be specified by equations $b_1(u) = b_2(u)$. Formally, we assume that we are given two maps $b_1, b_2: \mathfrak{U} \to \mathfrak{E}$ (with $\mathfrak{E}$ a set called the *equating space*) specifying $\mathfrak{B}$ as $\mathfrak{B} = \{u \in \mathfrak{U} \mid b_1(u) = b_2(u)\}$. The 'equilibrium' equations $b_1(u) = b_2(u)$ are called ***behavioral equations***. Clearly these behavioral equations specify $\mathfrak{B}$, but the converse is not true: there are obviously very many equivalent behavioral equations (equivalent in the sense that they specify the same behavior). This illustrates the ancillary role which equations play in mathematical modelling.

In addition to the behavior and behavioral equations, there is a third element in our general language for mathematical modelling: ***latent variables***. The introduction of latent variables stems from the realization that in most modelling exercises, it will be necessary to introduce other variables in addition to those which we are trying to model. We will call these auxiliary variables *latent variables* and the basic variables which our models aims at describing ***manifest variables***.

Formally, a ***latent variable model*** is a triple $(\mathfrak{U},\mathfrak{L},\mathfrak{B}_f)$ with $\mathfrak{U}$ the universum of manifest variables, $\mathfrak{L}$ the universum of latent variables, and $\mathfrak{B}_f$ a subset of $\mathfrak{U} \times \mathfrak{L}$, called the *full* (or *internal*) *behavior*. It induces the *manifest mathematical model* $(\mathbb{U},\mathfrak{B})$ with $\mathfrak{B}=\{u\in\mathbb{U}\mid\exists\,\ell\in\mathfrak{L}$ such that $(u,\ell)\in\mathfrak{B}_f\}$ the *manifest* (or *internal*, or *intrinsic*) *behavior*. In practice, $\mathfrak{B}_f$ will often be described by *full behavioral equations* $b_{f,1}(u,\ell)=b_{f,2}(u,\ell)$ leading, after elimination of ℓ (if this proves to be possible), to the behavioral equations $b_1(u)=b_2(u)$ for the manifest behavior.

We view a dynamical system merely as a mathematical model in which the behavior consists of a family of time functions. Thus a ***dynamical system*** Σ is defined as a triple $\Sigma=(\mathbb{T},\mathbb{W},\mathfrak{B})$ with $\mathbb{T}\subseteq\mathbb{R}$ the *time axis*, $\mathbb{W}$ the *signal space*, and $\mathfrak{B}\subseteq\mathbb{W}^{\mathbb{T}}$ the ***behavior***. Thus $\mathfrak{B}$ consists of those maps from $\mathbb{T}$ to $\mathbb{W}$ satisfying the laws governing the dynamical system. A dynamical system is said to be *linear* if $\mathbb{W}$ is a vector space and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$. We will consider only the case with time axis $\mathbb{T}=\mathbb{Z}$ (*discrete-time* systems) or $\mathbb{T}=\mathbb{R}$ (*continuous-time* systems). We will call $\Sigma=(\mathbb{T},\mathbb{W},\mathfrak{B})$ with $\mathbb{T}=\mathbb{Z}$ or $\mathbb{R}$ *time-invariant* if $\sigma^t\mathfrak{B}=\mathfrak{B}$ for all $t\in\mathbb{T}$. Here σ^t denotes the *t-shift*: $(\sigma^t\boldsymbol{w})(t'):=\boldsymbol{w}(t'+t)$. Thus in a linear system the behavior satisfies the *superposition principle*: $\boldsymbol{w}_1,\boldsymbol{w}_2\in\mathfrak{B}$ and α,β scalar imply $\alpha\boldsymbol{w}_1+\beta\boldsymbol{w}_2\in\mathfrak{B}$. Time-invariant systems on the other hand are governed by laws which do not depend explicitely on time, $\boldsymbol{w}\in\mathfrak{B}$ implying $\sigma^t\boldsymbol{w}\in\mathfrak{B}$.

We will be especially interested in dynamical systems described by difference or (ordinary) differential equations. Let us consider the discrete-time case first. Let $\mathbb{R}^{\bullet\times q}[s,s^{-1}]$ denote the family of polynomial matrices with q columns and any (-finite-) number of rows. Specifically, we will consider dynamical systems $\Sigma=(\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ in which $\mathfrak{B}$ is described in terms of a polynomial matrix $R(s,s^{-1})\in\mathbb{R}^{\bullet\times q}[s,s^{-1}]$ by the difference equations

$$\boxed{R(\sigma,\sigma^{-1})\boldsymbol{w}=\boldsymbol{o}} \qquad \textbf{(AR)}$$

yielding as behavior $\ker R(\sigma,\sigma^{-1})$ with $R(\sigma,\sigma^{-1})$ viewed as a linear map from $(\mathbb{R}^q)^{\mathbb{Z}}$ into $(\mathbb{R}^g)^{\mathbb{Z}}$ (g equals the number of rows of R). Clearly this defines a linear time-invariant system $(\mathbb{Z},\mathbb{R}^q,\ker R(\sigma,\sigma^{-1}))$. It is easy to see that $\ker R(\sigma,\sigma^{-1})$ defines a linear shift-invariant closed subset of $(\mathbb{R}^q)^{\mathbb{Z}}$ (equipped with the topology of pointwise convergence). In fact, it can be shown that every such subspace can be expressed as the kernel of a polynomial operator in the shift. We will call the behavioral equations (*AR*) ***AutoRegressive*** or, briefly, ***(AR) equations***, and we will denote the family of dynamical systems induced by them by $\mathfrak{L}^q$. In [1,4] it has been shown that $\mathfrak{L}^q$ consists of all dynamical systems $(\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ which are *linear*, *time-invariant*, and *complete* (see [1,4] for a formal definition and a discussion of completeness).

If we introduce latent variables in this setting then we obtain a ***latent variable dynamical*** system $\Sigma_f=(\mathbb{T},\mathbb{W},\mathbb{L},\mathfrak{B}_f)$ with $\mathbb{T}$ the *time axis*, $\mathbb{W}$ the *signal space*, $\mathbb{L}$ the *latent variable space*, and $\mathfrak{B}_f=(\mathbb{W}\times\mathbb{L})^{\mathbb{T}}$ the *full behavior*. We will consider latent variable dynamical systems $\Sigma_f=(\mathbb{Z},\mathbb{R}^q,\mathbb{R}^d,\mathfrak{B}_f)$ with $\mathfrak{B}_f$ described by behavioral equations

$$\boxed{R(\sigma,\sigma^{-1})\boldsymbol{w}=M(\sigma,\sigma^{-1})\boldsymbol{a}} \qquad \textbf{(ARMA)}$$

with $R(s,s^{-1})\in\mathbb{R}^{\bullet\times q}[s,s^{-1}]$, $M(s,s^{-1})\in\mathbb{R}^{\bullet\times d}[s,s^{-1}]$, both having the same number of rows; $\boldsymbol{w}:\mathbb{Z}\to\mathbb{R}^q$ is the *manifest* time-series, and $\boldsymbol{a}:\mathbb{Z}\to\mathbb{R}^d$ is the *latent variable* time-series, yielding the *full behavior* $\mathfrak{B}_f=\ker\,[R(\sigma,\sigma^{-1})\,\vdots\,-M(\sigma,\sigma^{-1})]$. We will call the system of equations (ARMA) ***AutoRegressive-Moving-Average*** or, briefly, ***(ARMA) equations***, with the left hand side the *autoregressive* part, and the right hand side the *moving-average* part.

The *manifest behavior* is defined as $\mathfrak{B}=\{\boldsymbol{w}\mid\exists\,\boldsymbol{a}$ such that $R(\sigma,\sigma^{-1})\boldsymbol{w}=M(\sigma,\sigma^{-1})\boldsymbol{a}\}$. Equivalently, $\mathfrak{B}=(R(\sigma,\sigma^{-1}))^{-1}\mathrm{im}\,M(\sigma,\sigma^{-1})$. It turns out that $\mathfrak{B}$ itself can also be expressed as the kernel of a polynomial operator in the shift. That is, there exists a polynomial $R'(s,s^{-1})$ such that $\mathfrak{B}=\ker R'(\sigma,\sigma^{-1})$. The polynomial matrix R' can be obtained by premultiplying $M(s,s^{-1})$ by an unimodular polynomial matrix $U(s,s^{-1})$ such that $U(s,s^{-1})M(s,s^{-1})=\begin{bmatrix}0\\ \hdashline M''(s,s^{-1})\end{bmatrix}$ with $M''(s,s^{-1})$ of full row rank (implying that $M''(\sigma,\sigma^{-1})$ is surjective). Then $R'(s,s^{-1})$ is obtained by taking the conformable partition of $U(s,s^{-1})R(s,s^{-1})=\begin{bmatrix}R'(s,s^{-1})\\ \hdashline R''(s,s^{-1})\end{bmatrix}$.

The above is easily generalized to differential equations. However, in this case there are difficulties with the required smoothness of $\boldsymbol{w}$ and the sense in which we want the differential equation to be satisfied. However, for the purpose of this paper we will use a C^∞ setting (even though in other context this has serious disadvantages). The behavioral equations

$$R\left(\frac{d}{dt}\right)\boldsymbol{w}=\boldsymbol{o} \qquad \textbf{(DE)}$$

with $R(s)\in\mathbb{R}^{\bullet\times q}[s]$ defines the continuous-time linear time-invariant dynamical system $(\mathbb{R},\mathbb{R}^q,\mathfrak{B})$ with $\mathfrak{B}=\ker R\left(\frac{d}{dt}\right)$ and $R\left(\frac{d}{dt}\right)$ viewed as a map from $C^\infty(\mathbb{R};\mathbb{R}^q)$ into $C^\infty(\mathbb{R};\mathbb{R}^g)$. Introducing latent variables leads to

$$R\left(\frac{d}{dt}\right)\boldsymbol{w}=M\left(\frac{d}{dt}\right)\boldsymbol{a} \qquad \textbf{(LVDE)}$$

with $R(s)\in\mathbb{R}^{\bullet\times q}[s]$ and $M(s)\in\mathbb{R}^{\bullet\times d}[s]$, both having the same number of rows. Elimination of the latent variables is carried out by pre-multiplying by a unimodular polynomial matrix $U(s)\in\mathbb{R}^{g\times g}[s]$ such that $U(s)M(s)=\begin{bmatrix}0\\ \hdashline M''(s)\end{bmatrix}$ with $M''(s)$ of full row rank (implying again that $M''\left(\frac{d}{dt}\right)$ is surjective). The behavioral equations specifying the manifest behavior are then given by $R'\left(\frac{d}{dt}\right)\boldsymbol{w}=\boldsymbol{o}$, where $\begin{bmatrix}R'(s)\\ \hdashline R''(s)\end{bmatrix}$ is a conformable partition of $U(s)R(s)$.

Let us illustrate these notions at the hand of the circuit example of Section 2. In this electrical circuit we consider as manifest variables the port variables $(\boldsymbol{V},\boldsymbol{I}):\mathbb{R}\to\mathbb{R}^2$, while $\mathfrak{B}$ consists of all $\boldsymbol{V},\boldsymbol{I}$-pairs which the circuit can conceivably produce. Our mathematical model involves also the latent variables $(\boldsymbol{V}_{R_C},\boldsymbol{I}_{R_C},\boldsymbol{V}_{R_L},\boldsymbol{I}_{R_L},\boldsymbol{V}_C,\boldsymbol{I}_C,\boldsymbol{V}_L,\boldsymbol{I}_L):\mathbb{R}\to\mathbb{R}^8$. The full behavior equations consist of (*CE*), (*KCL*), and (*KVL*). In this case the manipulations leading to equations (*A*) when $\frac{1}{CR_C}=\frac{L}{R_L}$ and (*B*) when $\frac{1}{CR_C}\neq\frac{L}{R_L}$, correspond to carrying out explicitely the unimodular pre-multiplication alluded to above.

4. PARAMETRIZATIONS AND THEIR CONTINUITY. In our framework a mathematical model is a very abstract object. When we represent it by behavioral equations it becomes a bit more concrete. Often, however, it is possible to view a model as being induced by some concrete parameters.

Let $\mathbb{M}$ be a set, for example a family of mathematical models. In this case each element $M\in\mathbb{M}$ defines a mathematical model $(\mathbb{U},\mathfrak{B})$. A ***parametrization*** $(\mathbb{P},\pi)$ of $\mathbb{M}$ consists of a set $\mathbb{P}$ and a *surjective* map $\pi:\mathbb{P}\to\mathbb{M}$. The set $\mathbb{P}$ is called the *parameter* *space.* We think of $\mathbb{P}$ as a concrete space and of an element $p \in \mathbb{P}$ as a parameter, for example a set of real or complex numbers, or vectors, or polynomials, or polynomial matrices. Typically $p \in \mathbb{P}$ determines a behavioral equation and in this way induces a mathematical model. If $\mathbb{P}$ is a set of matrices we speak of $(\mathbb{P},\pi)$ as a *matrix parametrization* of $\mathbb{M}$. A similar nomenclature is used for *polynomial parametrizations*, *polynomial matrix parametrizations*, etc. When $\mathbb{P}$ is an abstract space related to $\mathbb{M}$ in the same way as above, then we will call $(\mathbb{P},\pi)$ also a ***representation*** of $\mathbb{M}$. Hence *representation* (abstract) is essentially synonymous to *parametrization* (concrete). For representations, think of a behavioral equation representation of a mathematical model, or of a latent variable representation, etc.

Let $(\mathbb{P},\pi)$ be a parametrization of $\mathbb{M}$, and assume that $\mathbb{P}$ and $\mathbb{M}$ are endowed with a topology. We will call $(\mathbb{P},\pi)$ a ***continuous parametrization*** of $\mathbb{M}$ if whenever $p_\varepsilon \in \mathbb{P}$, $\varepsilon \geq 0$ satisfies $\lim_{\varepsilon \to 0} p_\varepsilon = p_0$, then $\lim_{\varepsilon \to 0} \pi(p_\varepsilon) = \pi(p_0)$ and, conversely, whenever $M_\varepsilon \in \mathbb{M}$, $\varepsilon \geq 0$, satisfies $\lim_{\varepsilon \to 0} M_\varepsilon = M_0$, then there exists $p_\varepsilon \in \mathbb{P}$ such that $\pi(p_\varepsilon) = M_\varepsilon$ and such that $\lim_{\varepsilon \to 0} p_\varepsilon = p_0$. In other words, in a continuous parametrization convergence of the 'abstract' objects in $\mathbb{M}$ can be put into evidence by convergence of the parameters representing these objects, that is by the 'concrete' objects in $\mathbb{P}$ with the corresponding specific convergence with which typical parameter spaces are endowed. Actually, for the problem which we will study, all the results remain valid when ε is a vector in some $\mathbb{R}^N$. However, for the sake of concreteness, we will look at the case of a real parameter $\varepsilon \geq 0$ with $\varepsilon \to 0$.

Now consider the family of discrete-time linear time-invariant complete dynamical systems $(\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ denoted, as mentioned earlier, by $\mathfrak{L}^q$. Each element $R(s,s^{-1}) \in \mathbb{R}^{\bullet \times q}[s,s^{-1}]$ induces the dynamical system $(\mathbb{Z},\mathbb{R}^q, \ker R(\sigma,\sigma^{-1}))$. Let π denote the map which associates this dynamical system with the polynomial matrix R. A basic result from [3] reads:

THEOREM: $(\mathbb{R}^{\bullet\times q}[s,s^{-1}],\pi)$ ***defines a parametrization of*** $\mathfrak{L}^q$.

The question whether this is a continuous parametrization is a more delicate one. For one thing, it requires specifying a topology on $\mathfrak{L}^q$ and on $\mathbb{R}^{\bullet\times q}[s,s^{-1}]$. For $\mathfrak{L}^q$ we will use the following topology. In [2] it has been shown that $(\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ belongs to $\mathfrak{L}^q$ if and only if $\mathfrak{B}$ is a linear shift-invariant closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$, equipped with the topology of pointwise convergence. This leads to a topology on $\mathfrak{L}^q$ with the following notion of convergence. A family $\Sigma_\varepsilon = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B}_\varepsilon) \in \mathfrak{L}^q$ with $\varepsilon > 0$ a real number is defined to converge to $\Sigma_0 = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B}_0) \in \mathfrak{L}^q$

if (i) whenever $w_{\varepsilon_k} \in \mathfrak{B}_{\varepsilon_k}, k \in \mathbb{N}$, $\lim_{k\to\infty} \varepsilon_k = 0$, and $\lim_{k\to\infty} w_{\varepsilon_k} = w_0$ (pointwise convergence), then $w_0 \in \mathfrak{B}_0$

and (ii) whenever $w_0 \in \mathfrak{B}_0$ then $\exists\, w_\varepsilon \in \mathfrak{B}_\varepsilon$ such that $\lim_{\varepsilon\to 0} w_\varepsilon = w_0$.

On $\mathbb{R}^{\bullet\times q}[s,s^{-1}]$ we will use the following notion of convergence. Let $R_\varepsilon(s,s^{-1}) \in \mathbb{R}^{g_\varepsilon \times q}[s,s^{-1}]$, $\varepsilon \geq 0$. Then $\lim_{\varepsilon\to 0} R_\varepsilon = R_0$

if (i) $g_\varepsilon = g_0$ for ε sufficiently small

(ii) $R(s,s^{-1}) = R^\varepsilon_{L_\varepsilon} s^{L_\varepsilon} + R^\varepsilon_{L_\varepsilon - 1} s^{L_\varepsilon - 1} + \ldots + R^\varepsilon_{\ell_\varepsilon} s^{\ell_\varepsilon}$ satisfy $\ell \leq \ell_\varepsilon \leq L_\varepsilon \leq L$ for all $\varepsilon \geq 0$

and (iii) $\lim_{\varepsilon\to 0} R^\varepsilon_k = R^0_k$ for all $\ell \leq k \leq L$. This last convergence is componentwise in the entries of the matrices.

Thus system convergence means that convergent time-series from the behavior of the convergent systems approach a limit in the behavior of the limit system and, conversely, that each time-series in the behavior of the limit can be approximated by elements in the behavior of the convergent systems. Polynomial matrix convergence simply means convergence of the matrix coefficients.

It is not possible to prove in full generality that the parametrization of the previous theorem is a continuous one. For one thing, we need to restrict our attention to *full row rank* polynomial matrices with q columns. We will denote these as $\mathbb{R}_f^{\bullet\times q}[s,s^{-1}]$. A refinement of the previous theorem allows us to conclude [4,5]:

THEOREM: $(\mathbb{R}_f^{\bullet\times q}[s,s^{-1}],\pi)$ ***defines a parametrization of*** $\mathfrak{L}^q$.

Actually, we call behavioral equation representation $R(\sigma,\sigma^{-1})w = o$ of $\Sigma = (\mathbb{Z},\mathbb{R}^q,\mathfrak{B}) \in \mathfrak{L}^q$ a ***minimal*** *AR*-representation if $R(s,s^{-1}) \in \mathbb{R}_f^{\bullet\times q}[s,s^{-1}]$. Equivalently, (and this is what we have taken as our definition [5]) if the number of rows in R (that is the number of *scalar AR*-equations) is as small as possible subject, of course, to the constraint that it must represent Σ. It can be shown that if $R(\sigma,\sigma^{-1})w = o$ is one minimal *AR*-representation of Σ, then the transformation group (the *unimodular group*) $R \to UR$, where $U(s,s^{-1})$ ranges over the unimodular polynomial matrices, generates *precisely* all *minimal* *AR*-representations of Σ. This tremendous non-uniqueness of behavioral equation representations is, among other things, the source of difficulty in continuity considerations.

In order to state our result from [1] on continuous parametrization, we also need to introduce the notion of the *memory span* of an element $(\mathbb{Z},\mathbb{R}^q,\mathfrak{B}) \in \mathfrak{L}^q$. It can be shown that $\mathfrak{B}$ has the property that it has *finite memory span*, that is, that there exists a $\Delta \in \mathbb{Z}_+$ such that $w_1, w_2 \in \mathfrak{B}$ and $w_1(t) = w_2(t)$ for $0 \leq t \leq \Delta$ implies that $w_1 \wedge w_2 \in \mathfrak{B}$, where $w_1 \wedge w_2$ denotes the *concatenation* of w_1 and w_2, defined as $(w_1 \wedge w_2)(t) := w_1(t)$ for $t < 0$ and $(w_1 \wedge w_2)(t) := w_2(t)$ for $t \geq 0$. The smallest of such numbers $\Delta \in \mathbb{Z}$ will be called the ***memory span*** of Σ.

Let $R(s,s^{-1}) = R_L s^L + R_{L-1} s^{L-1} + \ldots + R_\ell s^\ell \in \mathbb{R}^{\bullet\times q}[s,s^{-1}]$, have $R_L \neq 0$ and $R_\ell \neq 0$. Then we call $L - \ell$ the ***degree*** of R.

Let us denote by $\mathbb{R}^{\bullet\times q}_{f,\Delta}[s,s^{-1}]$ the collection of elements of $\mathbb{R}_f^{\bullet\times q}[s,s^{-1}]$ with degree $\leq \Delta$. Also, let us denote by $\mathfrak{L}^q_\Delta$ those elements of $\mathfrak{L}^q$ with memory span $\leq \Delta$. In [1] we have proven the following interesting continuity result:

THEOREM: $(\mathbb{R}_f^{\bullet\times q}[s,s^{-1}],\pi)$ ***defines a continuous parametrization of*** $\mathfrak{L}^q$, ***in the following sense:***

1. Assume that $R_\varepsilon(s,s^{-1}) \in \mathbb{R}_f^{g\times q}[s,s^{-1}]$, $\varepsilon \geq 0$, satisfy $R_\varepsilon \xrightarrow[\varepsilon\to 0]{} R_0$. Let $\mathfrak{B}_\varepsilon := \ker R_\varepsilon(\sigma,\sigma^{-1})$. Then $\mathfrak{B}_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{B}_0$.
2. Assume that $\mathfrak{B}_\varepsilon$, belonging to $\mathfrak{L}^q_\Delta$, $\forall \varepsilon \geq 0$, satisfy $\mathfrak{B}_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{B}_0$. Then there is a $\Delta' \in \mathbb{Z}_+$ and there exist $R_\varepsilon(s,s^{-1}) \in \mathbb{R}^{\bullet\times q}_{f,\Delta'}[s,s^{-1}]$, $\forall \varepsilon \geq 0$, such that $R_\varepsilon \xrightarrow[\varepsilon\to 0]{} R_0$ and $\mathfrak{B}_\varepsilon = \ker R_\varepsilon(\sigma,\sigma^{-1})$.

Thus with the restrictions imposed by the above theorem, linear time-invariant complete dynamical systems converge if and only if their *AR*-representations converge.

In [2] we have obtained the continuous-time analog of this result. However, we need to be a bit more restrictive about the behavior of (DE) than was advocated in Section 3. We will define the behavior of

$$R\left(\frac{d}{dt}\right)w = o \qquad \textbf{(DE)}$$

as $\tilde{\mathfrak{B}} := \{w \in C^\infty(\mathbb{R};\mathbb{R}^q) \mid R\left(\frac{d}{dt}\right)w = o$ and $\forall n \in \mathbb{Z}_+$ there exists $\alpha \in \mathbb{R}$ such that $\int_{-\infty}^{+\infty} \| \frac{d^n w}{dt^n}(t) \| e^{\alpha|t|} dt < \infty\}$. In order we will be restricting attention to C^∞ solutions of (DE) which together with its derivatives are of exponential growth. Let us denote the collection of dynamical systems $\Sigma=(\mathbb{R},\mathbb{R}^q,\tilde{\mathfrak{B}})$ thus obtained by $\tilde{\mathfrak{L}}^q$.

A family of dynamical systems $\Sigma_\varepsilon=(\mathbb{R},\mathbb{R}^q,\tilde{\mathfrak{B}}_\varepsilon)\in\tilde{\mathfrak{L}}^q$, $\varepsilon>0$ is said to *converge* to $\Sigma_0=(\mathbb{R},\mathbb{R}^q,\tilde{\mathfrak{B}}_0)\in\tilde{\mathfrak{L}}^q$ if $w_\varepsilon\in\tilde{\mathfrak{B}}_\varepsilon$, $\varepsilon>0$, and $w_\varepsilon \xrightarrow[\varepsilon\to 0]{} w_0$ implies $w_0\in\tilde{\mathfrak{B}}_0$ and if $w_0\in\tilde{\mathfrak{B}}_0$ implies the existence of $w_\varepsilon\in\mathfrak{B}_\varepsilon$, $\varepsilon>0$, such that $w_\varepsilon \xrightarrow[\varepsilon\to 0]{} w_0$. The convergence $w_\varepsilon \xrightarrow[\varepsilon\to 0]{} w_0$ is taken to be pointwise convergence, uniform on compact subsets of $\mathbb{R}$. Convergence of polynomial matrices $R_\varepsilon(s)\in\mathbb{R}^{q_\varepsilon\times q}[s]$ is defined completely analogously to the case over $\mathbb{R}[s,s^{-1}]$.

The notion of *minimal* DE-representation requires, as in the AR-case, that $R(s)\in\mathbb{R}_f^{\bullet\times q}[s]$, i.e., that it is of full row rank. The *memory span* can now be defined as the smallest L for which there exists a representation DE of $\Sigma\in\tilde{\mathfrak{L}}^q$ with $R(s)=R_Ls^L+R_{L-1}s^{L-1}+\ldots+R_0$.

Denote by $\tilde{\mathfrak{L}}^q_\Delta$ the elements of $\tilde{\mathfrak{L}}^q$ with memory span $\leq\Delta$ and by $\mathbb{R}^{\bullet\times q}_{f,\Delta}[s]$ with degree $\leq\Delta$. In [2] we have proven the following generalization of the previous theorem to the continuous-time case.

THEOREM: $(\mathbb{R}_f^{\bullet\times q}[s],\pi)$ ***defines a continuous parametrization of*** $\tilde{\mathfrak{L}}^q$.

Thus with the restrictions imposed by the above theorem, linear time-invariant differential dynamical systems converge if and only if their DE-representations converge.

5. CONTINUITY OF ARMA-MODELS. Let $(\mathbb{T},\mathbb{W},\mathbb{L},\mathfrak{B}_f)$ be a latent variable dynamical system. We will say that it is ***observable*** if $\{(w',a'),\ (w'',a'')\in\mathfrak{B}_f$, and $w'=w''\}$ imply $\{a'=a''\}$. Observability in other words implies that the latent variable trajectory a can be deduced from the manifest variable trajectory w. Equivalently, if the full behavior is the graph of a mapping from the manifest behavior into $\mathbb{L}^{\mathbb{T}}$.

Now consider the *ARMA*-model

$$\boxed{R(\sigma,\sigma^{-1})w=M(\sigma,\sigma^{-1})a} \qquad \textit{(ARMA)}$$

with $R(s,s^{-1})\in\mathbb{R}^{g\times q}[s,s^{-1}]$ and $M(s,s^{-1}\in\mathbb{R}^{g\times d}[s,s^{-1}]$. In this situation it is possible to derive a concrete test for observability in terms of R and M. Indeed, the following result is proven in [4]:

THEOREM: ***The following conditions are equivalent:***

1. ***The above ARMA-model is observable;***
2. ***The complex matrix*** $M(\lambda,\lambda^{-1})$ ***is of full column rank for all*** $0\neq\lambda\in\mathbb{C}$;
3. ***There exist polynomial matrices*** $R'(s,s^{-1})\in\mathbb{R}^{\bullet\times q}[s,s^{-1}]$ and $R''(s,s^{-1})\in\mathbb{R}^{d\times q}[s,s^{-1}]$ ***such that the full behavior of***

$$R'(\sigma,\sigma^{-1})w=o$$
$$a=R''(\sigma,\sigma^{-1})w$$

is equal to that of (ARMA).

For a full discussion of the notion of observability and the companion notion of controllability in our behavioral context, and their relation to the classical versions of these notions, we refer the reader to [3,4,5].

Now consider a family of *ARMA*-models

$$R_\varepsilon(\sigma,\sigma^{-1})w=M_\varepsilon(\sigma,\sigma^{-1})a$$

depending on a real parameter $\varepsilon\geq 0$. Let $\mathfrak{B}_f^\varepsilon$ denote its full behavior and $\mathfrak{B}^\varepsilon$ its manifest behavior. From the results from [1] mentioned in Section 4 we know that if $[R_0(s,s^{-1})\,\vdots\,-M_0(s,s^{-1})]$ is of full row rank and if $\lim_{\varepsilon\to 0} R_\varepsilon=R_0$ and $\lim_{\varepsilon\to 0} M_\varepsilon=M_0$, then $\mathfrak{B}_f^\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{B}_f^0$. The question which we wish to address is *if and when* this (parameter or behavior) convergence of the *full* behavior implies the convergence $\mathfrak{B}^\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{B}^0$ of the *manifest* behavior. The example in Section 2 shows that this implication will not be automatic and it is refreshing to take note of the fact that observability provides the key to a positive result in this direction!

> **THEOREM (MAIN RESULT):** ***Assume that the ARMA-system***
> $$R_0(\sigma,\sigma^{-1})w=M_0(\sigma,\sigma^{-1})a$$
> ***is minimal (i.e., that*** $[R_0(s,s^{-1})\,\vdots\,-M_0(s,s^{-1})]$ ***is of full row rank) and observable (i.e., that*** $M(\lambda,\lambda^{-1})$ ***is of full column rank for all*** $0\neq\lambda\in\mathbb{C}$***). Then if*** $\lim_{\varepsilon\to 0} R_\varepsilon(s,s^{-1})=R_0(s,s^{-1})$ ***and*** $\lim_{\varepsilon\to 0} M_\varepsilon(s,s^{-1})=M_0(s,s^{-1})$, ***the full behavior*** $\mathfrak{B}_f^\varepsilon$ ***as well as the manifest behavior*** $\mathfrak{B}^\varepsilon$ ***will converge:***
> $$\lim_{\varepsilon\to 0}\mathfrak{B}_f^\varepsilon=\mathfrak{B}_f^0 \quad \textit{and} \quad \lim_{\varepsilon\to 0}\mathfrak{B}^\varepsilon=\mathfrak{B}^0$$

Proof: See [6].

Note that in the above theorem observability is a sufficient condition for a convergence. It is, however, certainly not a necessary condition.

For further ramifications of our main result the reader is also referred to [6].

REFERENCES

[1] J.W. NIEUWENHUIS and J.C. WILLEMS
Continuity of Dynamical System: A System Theoretic Approach
MATHEMATICS OF CONTROL, SIGNALS, AND SYSTEMS
Vol. 1, pp. 147–165, 1988.

[2] J.W. NIEUWENHUIS and J.C. WILLEMS
Continuity of Dynamical Systems: The Continuous-Time Case
Submitted for publication.

[3] J.C. WILLEMS
From Time Series to Linear System –
Part I. Finite Dimensional Linear Time Invariant Systems
Part II. Exact Modelling
Part III. Approximate Modelling
AUTOMATICA
Vol. 22, pp. 561–580, 1986; Vol. 22, pp. 675–694, 1986; Vol. 23, pp. 87–115, 1987.

[4] J.C. WILLEMS
Models for Dynamics
DYNAMICS REPORTED
Vol. 2, pp. 171–269, 1989.

[5] J.C. WILLEMS
Paradigms and Puzzles in the Theory of Dynamical Systems
IEEE TRANSACTIONS ON AUTOMATIC CONTROL
To appear.

[6] J.C. WILLEMS and J.W. NIEUWENHUIS
Continuity of Latent Variable Models,
IEEE TRANSACTIONS ON AUTOMATIC CONTROL
To appear