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PARADIGMS AND PUZZLES IN MODELLING DYNAMICAL SYSTEMS

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The purpose of this lecture is to present in an informal way a new framework and a mathematical language for modelling dynamical systems. We will take the point of view that mathematical concepts shouldstart as close as possible to concrete examples, to practical situations. As such, the starting point taken in control theory where one assumes an input/output structure may not be a natural one in many applications. For example, when terminating an electrical network with another impedance, it is not clear that and how one should define inputs and outputs. Similarly, when terminating a mechanical device with a damper it is all but clear how one can view this as a control law in the conventional sense.

Our approach allows very general dynamical structures to be considered. As such, it offers a viable framework in which to study general dynamical decision and design processes as, for example, discrete event systems.

We view a dynamical system as an entity Σ which interacts (through observations, controls, disturbances, etc.) with its environment. The attributes of the signals through which this interaction takes place take on their values in a space called the *signal space*. The time instances relevant to our problem yield a subset T of $\mathbb{R}$. Under a *signal* we will understand a map $w: T \rightarrow W$, i.e., a time function taking on its values in the signal space. In mathematical notation: $w \in W^T$ (W^T denotes the collection of all maps from T to W). Now, the laws governing the dynamical system will tell us that some signals are and some are not feasible, are or are not compatible with these laws. The collection of all compatible signals will be called the *behavior* of the dynamical system. The behavior is consequently a subset of W^T and will be denoted by $\mathfrak{B}$. Summarizing:

Definition: A *dynamical system* Σ is a triple $\Sigma = (T, W, \mathfrak{B})$ with $T \subseteq \mathbb{R}$ the *time axis*, W the *signal space*, and $\mathfrak{B} \subseteq W^T$ the *behavior*.

Models which we write down from first principles or laws which are postulated from data will invariably involve other time series in addition to the signals which our model aims at describing. We will call the attributes of those auxiliary time series *latent variables*. Formalizing the laws linking the signals to the latent variables will yield, analogously as in the above basic definition of a dynamical system, a system $\Sigma_a = (T, W, A, \mathfrak{B}_a)$ with A the set of latent variables and $\mathfrak{B}_a \subseteq (W \times A)^T$ the *full behavior*. Of course, we can and may want to eliminate these latent variables. This will induce the *external behavior* $\mathfrak{B} := \{w: T \rightarrow W \mid$ there exists $a: T \rightarrow A$ such that $(w, a) \in \mathfrak{B}_a\}$. Formally:

Definition: A *dynamical system with latent variables* is defined as $\Sigma_a = (T, W, A, \mathfrak{B}_a)$ with $T \subset \mathbb{R}$ the *time axis*, W the *signal space*, A the *space of latent variables*, and $\mathfrak{B}_a \subset (W \times A)^T$ the *full behavior*. The (*intrinsic*) *dynamical system* induced by Σ_a is given by $\Sigma = (T, W, \mathfrak{B})$ with $\mathfrak{B} := \{w \mid$ there exists a such that $(w, a) \in \mathfrak{B}_a\}$; $\mathfrak{B}$ is called the *external behavior*.

Latent variables may sound abstract. Let us illustrate it by means of a number of examples:

- When writing down a model of an electrical circuit, an electrical engineer will need to introduce the currents through and the voltages accross the internal branches of the circuit in order to express the constraints imposed by the constitutive laws of the elements and Kirchhoff's current and voltage laws. These internal voltages and currents can be viewed as latent variables.

- When setting up a model for the dynamics of the positions of the moving parts of a machine, a mechanical engineer may find it convenient to introduce the momenta as latent variables.

- When postulating a relation for the time evolution of the demand and supply of a scarce resource, an economist may want to introduce the price of this resource as an auxiliary variable. The price can be considered as a latent variable.

- When explaining the scores on tests, a psychologist will find is useful to consider intelligence as a latent variable.

- When axiomatizing what is the memory of a dynamical system or when studying its stability, a mathematician will be led to write the equations in state form. The state becomes a latent variable.

Within this framework we will introduce *state space systems*. These are dynamical models in which the state is a latent variable having some special properties related to the memory structure of the system. Similarly we will study *input/output structures*. The crucial property of the input is that it is a *free* variable.

Next, we will discuss interconnection, interaction of systems. We will distinguish two types of *interconnections*: interconnections which can be viewed as *control laws* and interconnections which should be viewed as *compliance conditions*.

This theory will be illustrated particularly in the context of linear time invariant dynamical systems.

We will also address the question of modelling from observations. We will emphasize the central role which should be played in this context by the concepts of *falsification* and of *the most powerful unfalsified model*. We will also briefly touch the question of *approximate modelling*.

More details can be found in:

[1] J.C. WILLEMS, From Time Series to Linear System:
[3a] Part I: Finite Dimensional Linear Time Invariant Systems;
[3b] Part II: Exact Modelling;
[3c] Part III: Approximate Modelling.
Automatica, Vol. 22, No. 5, pp. 561–580, 1986;
Vol. 22, No. 6, pp. 675–694, 1986;
Vol. 23, No. 1, pp. 87–115, 1987.

[2] J.C. WILLEMS, Modelling Linear Systems, ***Proceedings of the 10th World Congress on Automatic Control***, Münich, July 27–31, Pergamon Press, Vol. 9. pp. 1–10, 1987.

An early reference for this approch is:

[3] J.C. WILLEMS, System Theoretic Models for the Analysis of Physical Systems, ***Richerche di Automatica***, Vol. 10, No. 2, pp. 71–106, 1979.

A detailed mathematical exposition is given in

[4] J.C. WILLEMS, Models for Dynamics, ***Dynamics Reported***, to appear.