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ADAPTIVE STABILIZATION OF MULTIVARIABLE LINEAR SYSTEMS

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Abstract

A ''Universal adaptive stabilization algorithm is presented which, for any m, stabilizes all minimum phase mxm multivariable systems having an invertible high frequency gain.

Introduction

Recently, much progress has been made in the design of universal adaptive controllers for stabilizing minimum phase plants in the presence of a more limited knowledge of certain discrete invariants than had previously been thought necessary (see e.g. [1]-[4]). Specifically, in [4] a parameter adjustment scheme for the gain parameter in a constant proportional output-feedback law was shown to be globally stabilizing for minimum phase systems of relative degree one. This scheme is based on classical, frequency domain design using constant gain feedback, where the sign of the gain is dictated by a generalization of a switching-law strategy recently used by Nussbaum [3] in the case of first-order systems. Perhaps most significant, however, is the fact that, because the underlying feedback law is an output-feedback law, no a priori knowledge or estimate of the McMillan degree of the system is required. This same approach has been extended to scalar systems of relative degree not exceeding two by Morse [2] and thus several of the standard assumptions (see [1]) required for the more traditional adaptive stabilization schemes now appear superfluous.

The purpose of this note is to extend these techniques to the multivariable setting. Explicitly, we show the existence - for mxm strictly proper linear systems - of a universal adaptive controller, which globally stabilizes any minimum phase plant having an invertible high frequency gain. This specializes, when m=1, to the main result in [4] and indicates that the recent output feedback based, adaptive stabilization techniques for scalar systems ought to extend, mutatis mutandis, to the multivariable setting. We emphasize that the controller proposed here is of a more existential contribution and expect to have more to say on, e.g. the high-gain features of universal controllers, in a future paper. It is a pleasure to thank our promovendi Bengt Martenson and Harry Trentleman for useful discussions, particularly on the material in sections 1 and 2.

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1. Adaptive Stabilization of Multivariable Systems with Stable Instantaneous Gain.

In this section we consider the mxm multivariable linear system, described by strictly proper transfer functions

$$y(s) = G(s)u(s) = N(s)D(s)^{-1}u(s) \qquad (1.1)$$

where G(s) is assumed to satisfy the conditions

(H1) $\det N(s) = 0 \Rightarrow \mathrm{Re}(s) < 0$;
(H2) $\mathrm{spec}\,(G_1) \quad \mathbb{C}^-$, where

$$G(s) = \sum_{i=1}^{\infty} G_i s^{-i}$$

(H1) and (H2) are, of course, the multivariable analogues of the minimum phase condition and of the condition, $cb < 0$, for scalar systems. Indeed, any square system satisfying (H1)-(H2) possesses a minimum realization of the form, with state $x = \begin{bmatrix} x_1 \\ y \end{bmatrix}$

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}y \\ \dot{y} &= A_{21}x_1 + A_{22}y + G_1u \end{aligned} \qquad (1.2)$$

where spec (A_{11}) coincides with the locus of det N(s). In particular A_{11} and G_1 are stable systems. We claim that for any such system the control law

$$k = ||y||^2 \quad , \quad u = ky \qquad (1.3)$$

stabilizes (1.2). More precisely,

Theorem 1.1. Suppose the multivariable linear system (1.1) satisfies (H1), (H2). Then, the closed loop system corresponding to the feedback strategy (1.3) satisfies, for all initial data (x_0, k_0)

(i) k_t converges as $t \to \infty$;

(ii) $\lim_{t\to\infty} x_t = 0$

Proof. Since k_t is monotone nondecreasing, k_t will converge to a limit, k_∞, provided k_t remains bounded. We note that, from the form of the closed loop equations:

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}y \\ \dot{y} &= A_{21}x_1 + A_{22}y + kGy \\ \dot{k} &= ||y||^2 \end{aligned} \qquad (1.4)$$

assertion (ii) also follows from the boundedness of k_t. For if k_t is bounded, $y_t \in L^2(R^+, R^m)$. Since A_{11} is stable, it then follows that x_1, $\dot{x}_1$ are square integrable. Since k is bounded, kGy and therefore $\dot{y}$ is square integrable. Thus,

$$x, \dot{x} \in L^2(R^+, R^n)$$

and we have $\lim_{t\to\infty} x(t) = 0$.

To see that k_t is bounded, choose $Q = Q^T$ such that

$$G^TQ + QG = -I$$

and note

$$\frac{d}{dt}(y^TQy) = 2 \langle A_{21}x_1, Qy \rangle + y_t^T(QA_{22} + A_{22}^TQ)y_t - k\,||y||^2 \quad (1.5)$$

Viewing $A_{21}x_1$ as the output of the stable system

$$\dot{x}_1 = A_{11}x_1 + A_{12}Q^{-1}(Qy)$$

driven by the input Qy we have

Lemma 1.1. ([4]) $\int_0^T \langle A_{21}x_1, Qy \rangle\, dt \le c_0 + c_1\int_0^T ||y||^2 dt$

where c_0 depends only on $x_1(0)$.

Integrating both sides of (1.5) we have

$$y^TQy|_0^T \le c_0 + c_1\int_0^T ||y||^2dt + c_2\int_0^T ||y||^2dt - \int_0^T k||y||^2dt$$

and recalling (1.3), we have

$$\gamma_1 \le \gamma_2 k(T) - k^2(T)$$

for γ_1, γ_2 constant. In particular, k(T) is bounded from above.

2. Stabilization by Static Precompensation.

In this section, we consider the following matrix-theoretic problem: Is there a finite collection $\mathcal{K}$ of mxm matrices such that for any $B \in GL(m,R)$ at least one of the matrices

$$BK \quad , \quad K \in \mathcal{K}$$

is stable?

For example, if m=1 then we can take $K = \{\pm 1\}$. If m=2, then a more elaborate argument (which we owe to H. Trentleman) shows that one can take

$$K = \{\mathrm{diag}\,(\varepsilon_i):\varepsilon_i = \pm 1\}.$$

Unfortunately, for $m \ge 3$ the class of signature matrices no longer suffices and, as far as we are aware, even for m=3 the question we pose is open.

Conjecture: There exists a finite subset, $\mathcal{K} = M_m(R)$, such that for all $B \in GL(M,R)$, at least one of

$$BK \quad , \quad K \in \mathcal{K}$$

is (asymptotically) stable.

If our conjecture were true, the universal stabilizer we construct in section 3 would of course have a simpler form. For our purposes, however, it is sufficient to prove the following result.

Proposition 2.1. For any m, there is a countable collection $\mathcal{K}$ of real mxm matrices, such that for any $B \in GL(m,R)$ at least one of

$$GK \quad , \quad K \in \mathcal{K}$$

is asymptotically stable.

Proof. Taking $q \in Q$, it suffices to prove:

Lemma 2.2. For any $q > 0$ there is a finite set $\mathcal{K}$ of stabilizing matrices for the subset

$$B_q = \{B \in GL(m,R): ||B^{-1}|| \le q\}.$$

Proof of Lemma 2.2. Denoting by S_m the unit sphere in the space of mxm matrices, the subset $S_m \cap B_q$ is closed in S_m and hence compact. For any $K \in GL(m,R)$ the subset

$$U'_K = \{B \in GL(m,R):\ BK \text{ is asymptotically stable}\}$$

is open and therefore

$$U_K = U'_K \cap S_m \cap B_q$$

is open in $S_m \cap B_q$. Noting that B is stable if, and only if, λB is stable for positive scalars λ, there exists a finite set $\mathcal{K}$ of stabilizing matrices by the Heine-Borel Theorem.

3. Global Stabilization by Means of a Switching Law.

In the section we will construct a feedback control law which stabilizes the linear system

$$\dot{y} = Ay + Bu \quad (3.1)$$

$u \in R^m$, $y \in R^m$, $A \in R^{mxm}$, $B \in R^{mxm}$, with A and B unknown, under the a priori knowledge that $B \in \mathcal{B} \subset R^{mxm}$. We will assume that $\mathcal{B}$ has the property that there exists a finite or countably infinite set of matrices $\mathcal{K} \subset R^{mxm}$ such that for each $B \in \mathcal{B}$ at least one matrix of the set $\{BK | K \in \mathcal{K}\}$ has its eigenvalues in the open left half plane. As we have seen in the previous section, this will be the case with $\mathcal{K}$ finite if for example

$$\mathcal{B} = Gl(m) \text{ with m=1 or 2}$$

$$\text{or } \mathcal{B} = \{B \in Gl(m) | FM\ \ ||B^{-1}|| \le M\}$$

and, with $\mathcal{K}$ countably infinite, if

$$\mathcal{B} = Gl(m)$$

We will treat the case that $\mathcal{K}$ is finite explicitely and afterwards indicate the extension of this result to the countably infinite case.

Our control law consists of a high gain feedback law, modulated by a switching gain policy. More precisely, with $\mathcal{K} = \{K_1, K_2, \ldots, K_N\}$ a finite set, we use the control law

$$\dot{k} = ||y||^2 \quad (3.2a)$$

$$y = kK_{s(k)} \quad (3.2b)$$

with $s: R \to \{1, 2, \ldots, N\}$ a suitable switching law. In particular, we will take for s(k) a switching policy which rotates the gain among the different K_i's. Specifically, take

$$\tau(k) = i$$

for $\tau_{1N+i} \le k < \tau_{1N+i+1}$ $\quad 1 = 0, 1, 2, \ldots$ $\quad i = 1, 2, \ldots, N$

with the thresholds $0 < \tau_1, < \tau_2 < \ldots$ chosen such that for every $i \in \{1, 2, \ldots, N\}$ the Cesaro mean

$$\liminf_{k\to\infty} \frac{1}{k}\int_0^k f_i(\gamma)d = \infty \tag{3.3}$$

where $f_i(\sigma) = \begin{cases} +1 & \text{whenever } s(\sigma) = i \\ -1 & \text{whenever } s(\sigma) \neq i \end{cases}$

An example of such a switching policy is

$$\tau_{n+1} = \tau_n + e^{n^2} \qquad n = 0, 1, 2, \ldots$$

We will prove:

<u>Proposition 3.1</u>: For any $k_0 \in R$, $y_0 \in R^m$, $A \in R^{mxm}$, and $B \in \mathcal{B} \subset R^{mxm}$, the system (3.1) - (3.2), with (3.3) holding, satisfies:

(i) $\lim_{t\to\infty} k_t = k_\infty < \infty$

(ii) $\lim_{t\to\infty} y_t = 0$

<u>Proof</u>: Since one of the matrices $\{BK_1, BK_2, \ldots, BK_N\}$ is asymptotically stable, there exists an $i \in \{1, 2, \ldots, K\}$ and a $Q = Q^T > 0$ such that

$$K_i^T B^T Q + QBK_i = -I$$

Let $\alpha > 0$ be such that

$$K_j^T B_j^T Q + QBK_j \leq \alpha I \quad \text{for } j \neq i$$

and β be such that

$$A^T Q + QA \leq \beta I$$

Clearly k_t is monotone increasing along solutions of (3.2). We will prove that $\lim_{t\to\infty} k_t < \infty$. Assume the contrary and consider the behavior of $y^T Qy$ along solutions of (3.2). Then for t sufficiently large

$$\frac{d}{dt} y_t^T Q y_t \leq \begin{cases} (\beta - k_t)||y_t||^2 & \text{when } s(k_t) = i \\ (\beta + \alpha k_t)||y_t||^2 & \text{when } s(k_t) \neq i \end{cases}$$

Equivalently,

$$\frac{d}{dt} y_t^T Q y_t \leq \begin{cases} (\beta - k_t)\dot{k}_t & \text{when } s(k_t) = i \\ (\beta + \alpha k_t)\dot{k}_t & \text{when } s(k_t) \neq i \end{cases}$$

Integrating this equation yields

$$y_t^T Q y_t \leq y_0^T Q y_0 + \int_0^{k_t} (\beta - \min(1,\alpha)\ f_i(\sigma))d\sigma \tag{3.4}$$

Now (3.3) implies that for any $\alpha', \beta' \in R$ with $\alpha' > 0$

$$\liminf_{k\to\infty} \int_0^k (\beta' - \alpha'\ f_i(\sigma))d\sigma = -\infty \tag{3.5}$$

This implies that, as $k_t \to \infty$,

$$y_0^T Q y_0 + \int_0^{k_t} (\beta - \min(1,\alpha)\ f_i(\sigma))d\sigma$$

will become negative. Since however it equals $y_t^T Q y_t$ this is impossible and consequently $k_t \xrightarrow[t\to\infty]{} k_\infty < \infty$.

Now, by (3.2a) this implies that the solution y of (3.1) - (3.2) satisfies $y \in L_2([0,\infty); R^m)$, while (3.2b) yields $u \in L_2([0,\infty); R^m)$, and (3.1) yields $\dot{y} \in L_2([0,\infty); R^m)$. Now $\{y, \dot{y} \in L_2([0,\infty); R^m)\} \supset \{\lim_{t\to\infty} y_t = 0\}$, and the proposition is proven.

In order to extend Proposition 3.1 to the countable case $K = \{K_1, K_2, \ldots, K_N, \ldots\}$, we can set up, instead of the periodic switching pattern of the finite case, a switching pattern as shown below

$$\begin{array}{cccccccc} K_1 & K_2 & K_3 & K_1 & \ldots & K_1 & \ldots \\ K_2 & K_2 & K_2 & K_2 & \ldots & K_2 & \ldots \\ K_3 & K_3 & K_3 & K_3 & \ldots & K_3 & \ldots \\ K_4 & K_4 & K_4 & K_4 & \ldots & K_4 & \ldots \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \end{array}$$

Now by using switching thresholds $\tau_1, \tau_2, \ldots$ which are progressively sufficiently high we can obtain the extension of the result of the countable case. The details are omitted.

Finally, we would like to emphasize strongly that Proposition 3.1 should only be considered as an existence result for a globally stabilizing control law. We make no claim whatsoever that (3.2) gives a reasonable adaptive control law.

4. <u>A Universal Stabilization Algorithm for Minimum Phase Systems with Invertible High Frequency Gain</u>.

Combining the algorithms developed in the previous sections, we obtain a universal controller which is globally stabilizing for minimum phase systems with invertible high frequency gain. Explicitly, consider an mxm system (1.1) satisfying the minimum phase condition (H1) and (H2)' det $G_1 \neq 0$, where $G(s) = \sum_{i=1}^{\infty} G_i s^{-i}$. As in sections 2, 3 we choose a collection $\mathcal{K}$ of stabilizing matrices for $G_1 \in GL(m,R)$ and form the strategy (3.2)

$$k = ||y||^2, \quad u = kK_{s(k)}$$

with s(k) an appropriate switching law; for example, having the form described in (3.3).

<u>Theorem 4.1</u>. Suppose the multivariable linear system (1.1) satisfies (H1) and (H2)'. Then, the closed loop system corresponding to the feedback strategy (1.3) satisfies, for all initial data (x_0,k_0):

(i) $\lim_{t\to\infty} k_t$ exists;

(ii) $\lim_{t\to\infty} x_t = 0$.

<u>Proof</u>. As before, it suffices to verify (i). Choose i such that G_1K_i is stable and solve Lyapunov's equation

$$QG_1K_i + K_i^T G_1^T Q = -I$$

Modifying the arguments in sections 1 and 3, from the closed-loop equations (1.2)

$$\dot{x}_1 = A_{11}x_1 + A_{12}y$$

$$\dot{y} = A_{21}x_1 + A_{22}y + kG_1K_{s(k)}y$$

$$\dot{k} = ||y||^2$$

we deduce

$$y_t^T Q y_t \Big|_0^\tau \leq 2 \int_0^\tau \langle A_{21}x_1, Qy \rangle \, dt + \int_0^k (\beta - \alpha t f_i(t)) dt$$

for α, $\beta \in R$ with $\alpha > 0$.

From Lemma 1.1 we conclude the existence of constants such that

$$\gamma \leq \delta k(T) + \int_0^{k_t} (\beta - \alpha t f_i(t)) dt$$

or, equivalently,

$$\gamma \leq \int_0^{k_t} (\beta' - \alpha t f_i(t)) dt \qquad (4.1)$$

Since the left-hand side of (4.1) has, by choice of the switching law s(k), limit infinum $-\infty$ as $k \to \infty$, k_t is bonded.

Bibliography

[1] A. S. Morse, ''Recent Problems in Parameter Adaptive Control,'' Outils et Modeles Mathematiques pour l'Automatique, l'Analyse de Systemes et le Traitement du Signal (I.D. Landau, ed.) Editions CNRS Vol. 3 (1983) 733-740.

[2] A. S. Morse, ''A 3-dimensional 'universal' controller for the adaptive stabilization of any strictly proper, minimum phase system with relative degree not exceeding two,'' preprint, August 1984.

[3] R. D. Nussbaum, ''Some Remarks on a Conjecture in Parameter Adaptive Control,'' Systems and Control Letters, 3(1983) 243-246.

[4] J. C. Willems and C. I. Byrnes, ''Global Adaptive Stabilization in the Absence of Information on the Sign of the High Frequency Gain,'' Proc. INRIA Conf. on Analysis and Optimization of Systems, Springer-Verlag, 1984, 49-57.