STATE AND FIRST ORDER REPRESENTATIONS

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0.1 Abstract

We conjecture that the solution set of a system of linear constant coefficient PDE's is Markovian if and only if it is the solution set of a system of first order PDE's. An analogous conjecture regarding state systems is also made.
Keywords: Linear differential systems, Markovian systems, state systems, kernel representations.

0.2 Description of the problem

0.2.1 Notation

First, we introduce our notation for the solution sets of linear PDE's in the $\mathtt{n}$ real independent variables $x = (x_1, \ldots, x_{\mathtt{n}})$. Let $\mathfrak{D}'_{\mathtt{n}}$ denote, as usual, the set of real distributions on $\mathbb{R}^{\mathtt{n}}$, and $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ the linear subspaces of $(\mathfrak{D}'_{\mathtt{n}})^{\mathtt{w}}$ consisting of the solutions of a system of linear constant coefficient PDE's in the $\mathtt{w}$ real-valued dependent variables $w = \mathrm{col}(w_1, \ldots, w_{\mathtt{w}})$. More precisely, each element $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ is defined by a polynomial matrix $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$, with $\mathtt{w}$ columns, but any number of rows, such that

$$\mathfrak{B} = \{w \in (\mathfrak{D}'_{\mathtt{n}})^{\mathtt{w}} \mid R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})w = 0\}.$$

We refer to elements of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ as *linear differential* $\mathtt{n}$-*D systems.* The above PDE is called a *kernel representation* of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. Note that each $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ has many kernel representations. For an in depth study of $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, see [1] and [2].

Next, we introduce a class of special three-way partitions of $\mathbb{R}^{\mathtt{n}}$. Denote by $\mathfrak{P}$ the following set of partitions of $\mathbb{R}^{\mathtt{n}}$:

$$[(S_-, S_0, S_+) \in \mathfrak{P}] :\Leftrightarrow [(S_-, S_0, S_+ \text{ are disjoint subsets of } \mathbb{R}^{\mathtt{n}}) \\ \wedge (S_- \cup S_0 \cup S_+ = \mathbb{R}^{\mathtt{n}}) \wedge (S_- \text{ and } S_+ \text{ are open, and } S_0 \text{ is closed})].$$

Finally, we define concatenation of maps on $\mathbb{R}^{\mathtt{n}}$. Let $f_-, f_+ : \mathbb{R}^{\mathtt{n}} \to \mathfrak{F}$, and let $\pi = (S_-, S_0, S_+) \in \mathfrak{P}$. Define the map $f_- \wedge_\pi f_+ : \mathbb{R}^{\mathtt{n}} \to \mathfrak{F}$, called the *concatenation* of (f_-, f_+) along π, by

$$(f_- \wedge_\pi f_+)(x) := \begin{cases} f_-(x) & \text{for} \quad x \in S_- \\ f_+(x) & \text{for} \quad x \in S_0 \cup S_+ \end{cases}$$

0.2.2 Markovian systems

Define $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ to be *Markovian* $:\Leftrightarrow$

$$[(w_-, w_+ \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})) \wedge (\pi = (S_-, S_0, S_+) \in \mathfrak{P}) \\ \wedge (w_-|_{S_0} = w_+|_{S_0})] \Rightarrow [(w_- \wedge_\pi w_+ \in \mathfrak{B}].$$

Think of S_- as the *'past'*, S_0 as the *'present'*, and S_+ as the *'future'*. Markovian means that if two solutions of the PDE agree on the present, then their pasts and futures are compatible, in the sense that the past (and present) of one, concatenated with the (present and) future of the other, is also a solution. In the language of probability: the past and the future are independent given the present.

We come to our first conjecture:

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ is Markovian if and only if it has a kernel representation that is first order.

I.e., it is conjectured that a Markovian system admits a kernel representation of the form

$$R_0 w + R_1 \frac{\partial}{\partial x_1} w + R_2 \frac{\partial}{\partial x_2} w + \cdots R_{\mathtt{n}} \frac{\partial}{\partial x_{\mathtt{n}}} w = 0.$$

Oberst [2] has proven that there is a one-to-one relation between $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ and the submodules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$, each $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ being identifiable with its set of *annihilators*

$$\mathfrak{N}_{\mathfrak{B}} := \{n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}] \mid n^\top(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}.$$

Markovianity is hence conjectured to correspond exactly to those $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ for which the submodule $\mathfrak{N}_{\mathfrak{B}}$ has a set of first order generators.

0.2.3 State systems

In this section we consider systems with two kind of variables: $\mathtt{w}$ real-valued *manifest* variables, $w = \mathrm{col}(w_1, \ldots, w_{\mathtt{w}})$, and $\mathtt{z}$ real-valued *state* variables, $z = \mathrm{col}(z_1, \ldots, z_{\mathtt{z}})$. Their joint behavior is again assumed to be the solution set of a system of linear constant coefficient PDE's. Thus we consider behaviors in $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}+\mathtt{z}}$, whence each element $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}+\mathtt{z}}$ is described in terms of two polynomial matrices $(R, M) \in \mathbb{R}^{\bullet \times (\mathtt{w}+\mathtt{z})}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]$ by

$$\mathfrak{B} = \{(w, z) \in (\mathfrak{D}'_{\mathtt{n}})^{\mathtt{w}} \times (\mathfrak{D}'_{\mathtt{n}})^{\mathtt{z}} \mid R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})w + M(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_{\mathtt{n}}})z = 0\}.$$

Define $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}+\mathtt{z}}$ to be a *state system* with state z $:\Leftrightarrow$

$$[((w_-, z_-), (w_+, z_+) \in \mathfrak{B} \cap \mathfrak{C}^{\infty}(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}+\mathtt{z}})) \wedge (\pi = (S_-, S_0, S_+) \in \mathfrak{P}) \wedge \ (z_-|_{S_0} = z_+|_{S_0})] \Rightarrow [(w_-, z_-) \wedge_{\pi} (w_+, z_+) \in \mathfrak{B}].$$

Think again of S_- as the *'past'*, S_0 as the *'present'*, S_-+ as the *'future'*. State means that if the state components of two solutions agree on the present, then their pasts and futures are compatible, in the sense that the past of one solution (involving both the manifest and the state variables), concatenated with the present and future of the other solution, is also a solution. In the language of probability: the present state *'splits'* the past and the present plus future of the manifest and the state trajectory combined.

We come to our second conjecture:

$\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}+\mathtt{z}}$ is a state system
if and only if
it has a kernel representation
that is first order in the state variables **z**
and zero-th order in the manifest variables **w**.

I.e., it is conjectured that a state system admits a kernel representation of the form

$$R_0 w + M_0 z + M_1 \frac{\partial}{\partial x_1} z + M_2 \frac{\partial}{\partial x_2} z + \cdots M_{\mathtt{n}} \frac{\partial}{\partial x_{\mathtt{n}}} z = 0.$$

0.3 Motivation and history of the problem

These open problems aim at understanding state and state construction for $\mathtt{n}$-D systems.

Maxwell's equations constitute an example of a Markovian system. The diffusion equation and the wave equation are non-examples.

0.4 Available results

It is straightforward to prove the 'if'-part of both conjectures. The conjectures are true for $\mathtt{n} = 1$, i.e. in the ODE case, see [3].

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