ELIMINATION OF LATENT VARIABLES IN DIFFERENTIAL ALGEBRAIC SYSTEMS

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Abstract

First principles modelling invariably leads to models which contain latent variables in addition to the manifest variables that the model aims at describing. The problem of elimination of these latent variables and specifying the manifest behavior for real differential algebraic dynamical systems is posed.

Keywords: Elimination, latent variables, differential algebraic systems.

1 Introduction

The purpose of this note is to attract attention to what is called the *elimination problem*, which we believe to be an important but largely open problem in mathematical modelling. A typical way of modelling dynamical systems is by the method of hierarchical *tearing* and *zooming:* a system is decomposed into subsystems until a level is reached where the subcomponents have mathematical models that are assumed to be "known". This procedure is described for example in [17]. Electrical circuits and robotic kinematic chains form the prototype examples of this sort of first principles modelling. These ideas also lie at the basis of modelling concepts as bond-graphs and object-oriented computer-assisted procedures used frequently for instance in chemical process modelling.

As a consequence of the introduction of internal interconnections, the resulting model invariably contains more variables than those that the model aims

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at describing. We call these additional variables *latent variables*, in order to distinguish them from the *manifest variables*, as we call the variables of primary interest in the modelling process. The problem discussed in this note is the extent to which the latent variables can be eliminated.

We now formulate this question in mathematical terms for differential algebraic systems. Let f be a vector of polynomials. We will be interested in polynomials over both the real and complex field, since the results will be quite different. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The vector of polynomials f is used to define a system of differential equations in the variables w and ℓ. The w's are the manifest variables, and the ℓ's are the latent variables, and the fact that they both appear is of the essence. The order of this differential equation is denoted by n. The assumption that each of the differential equations has the same order n in both w and ℓ can be achieved by putting the appropriate coefficients to zero. This is reflected in the following notation for the variables in f, the indeterminates:

$$w, w^{(1)}, w^{(2)}, \cdots, w^{(n)}, \ell, \ell^{(1)}, \cdots, \ell^{(n)} \tag{1}$$

where each of the $w^{(k)}$'s consist of q variables, and each of the $\ell^{(k)}$'s consists of d variables. This leads to

$$f(w, w^{(1)}, \cdots, w^{(n)}, \ell, \ell^{(1)}, \cdots, \ell^{(n)}), \tag{2}$$

a vector of polynomials with coefficients in $\mathbb{K}$ in $(n+1)(q+d)$ indeterminates. Expression (2) leads to the system of differential equations

$$f \circ (w, \frac{d}{dt}w, \cdots, \frac{d^n}{dt^n}w, \ell, \frac{d}{dt}\ell, \cdots, \frac{d^n}{dt}\ell) = 0. \tag{3}$$

The end result of the tearing and zooming modelling process referred to above is typically such a system of differential equations, assuming that each of the subsystems are described by polynomial differential expressions. Of course, often more complicated functions than polynomials appear, but for the purposes of this note, we concentrate on polynomial functions. Note that static (i.e., algebraic) equations can be accommodated in (3) by having differential equations of order zero. This is important, since first principles modelling invariably leads, because of the interconnection equations, to a predominance of algebraic as compared to dynamic equations.

2 Elimination of latent variables

Equation (3) defines a *dynamical system with latent variables* as this term is defined in [18]:

$$\Sigma_f = (\mathbb{R}, \mathbb{K}^q, \mathbb{K}^d, \mathfrak{B}_f) \tag{4}$$

where $\mathbb{R}$ is the time-axis, $\mathbb{K}^q$ the space of manifest variables, $\mathbb{K}^d$ the space of latent variables, and the *full behavior* $\mathfrak{B}_f$ is defined as

$$\mathfrak{B}_f = \{(w,\ell) : \mathbb{R} \to \mathbb{K}^q \times \mathbb{K}^d)|\ (3) \text{ is satisfied}\} \qquad (5)$$

We are deliberate vague about the solution concept that is involved in the definition of $\mathfrak{B}_f$, since flexibility in this respect will be required in order to obtain satisfactory results. When we suppress the latent variables in (4), we obtain the *manifest system* induced by (4), defined as

$$\Sigma = (\mathbb{R}, \mathbb{K}^q, \mathfrak{B}) \qquad (6)$$

with *manifest behavior*

$$\mathfrak{B} = \{w : \mathbb{R} \to \mathbb{K}^q\ |\exists\ \ell : \mathbb{R} \to \mathbb{K}^d \text{ such that } (w,\ell) \in \mathfrak{B}_f\} \qquad (7)$$

We view (6) as a system in which the latent variables have been eliminated. The question which we address is:

What sort of equations, formulae, describe the manifest behavior $\mathfrak{B}$?

We call the problem of obtaining such formulae the *elimination problem*. In order to obtain a clean theory, it may be necessary to amend the definitions of $\mathfrak{B}_f$ and $\mathfrak{B}$ suitably, e.g., by allowing solutions defined on finite intervals only, or by allowing solutions that are distributions or hyper-functions.

2.1 The linear case

When f is a linear map (3) leads to the system of differential equations

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \qquad (8)$$

with R and M polynomial matrices over $\mathbb{K}$. In this case it is known [10] that after elimination of the latent variables we obtain again a system of linear differential equations. More precisely, for any pair of polynomial matrices R and M over $\mathbb{K}$, there exist a polynomial matrix R' over $\mathbb{K}$ such that the set

$$\{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{K}^q)|\ \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{K}^d) \text{ such that (8) holds}\} \qquad (9)$$

consists *exactly* of all $\mathfrak{C}^\infty$ solutions of

$$R'(\frac{d}{dt})w = 0 \qquad (10)$$

It is important to note that, in order to cope rigorously with lack of controllability and common factors, this elimination result requires more than just transfer function thinking. This elimination result has recently been generalized to constant coefficient PDE's [9] and to time-varying systems [8].

2.2 The complex case

A more complicated situation has been studied in [3] for complex differential algebraic systems, i.e., with f a vector of polynomials and $\mathbb{K} = \mathbb{C}$. In this case, elimination (using an algebraically motivated solution concept for algebraic differential equations) is shown to lead to a system of resultant equations, i.e., it is shown that $\mathfrak{B}$ leads to the finite set of $\mathfrak{B}_\alpha$'s (with $\alpha \in A$, and A a finite index set), each described by combined algebraic differential equations and inequations. The relation between $\mathfrak{B}$ and the $\mathfrak{B}_\alpha$'s appears to be that $\mathfrak{B}$ is contained in $\bigwedge_{\alpha \in A} \mathfrak{B}_\alpha$, where $\bigwedge_{\alpha \in A} \mathfrak{B}_\alpha$ denotes the *concatenation product* of the $\mathfrak{B}_\alpha$'s.

Each of the $\mathfrak{B}_\alpha$'s is specified by a finite set of polynomials

$$f_\alpha(w, w^{(1)}, \cdots, w^{(n_\alpha)}) \tag{11}$$

over $\mathbb{C}$ and one polynomial

$$g_\alpha(w, w^{(1)}, \cdots, w^{(n_\alpha)}) \tag{12}$$

over $\mathbb{C}$ such that

$$\begin{aligned} \mathfrak{B}_\alpha = \{w : \mathbb{R} \to \mathbb{C}^q \quad | \quad & f_\alpha \circ (w, \frac{d}{dt}w, \cdots, \frac{d^{n_\alpha}}{dt^{n_\alpha}}w) = 0 \\ & (g_\alpha \circ (w, \frac{d}{dt}w, \cdots, \frac{d^{n_\alpha}}{dt^{n_\alpha}}w))(t) \neq 0 \text{ for all } t \in \mathbb{R}\} \end{aligned} \tag{13}$$

In fact, [3] also describes an algorithm for passing from f to (f_α, g_α)'s. In addition, this result remains true in the more general case in which the starting point (3) contains also inequations. As such, it is much more logical to start in the complex case from the very beginning with a system of differential algebraic equations *and* differential algebraic inequations (or a finite union of these). And indeed, that is what is done in [3]. Note, however, that in general $\mathfrak{B}$ will be properly contained in $\bigwedge_{\alpha \in A} \mathfrak{B}_\alpha$. Moreover, in the dynamic case, when trajectories come from differential equations, and pass from one of the $\mathfrak{B}_\alpha$'s to another, certain *gluing conditions* have to be satisfied. We will return to these in the next section.

3 Open problems

The question raised is to study the case that $\mathbb{K} = \mathbb{R}$. It is easy to see that in addition to inequations also inequalities are needed in this case. A concrete conjecture therefore is that the resulting equations (7) will then consist of a finite union of systems of differential equations, differential inequations, and differential inequalities. Hence each of the resulting $\mathfrak{B}_\alpha$'s will be specified by f_α's, g_α's, and h_α's, with the f_α's and g_α's as (11, 12), but with real coefficients, and

$$h_\alpha(w, w^{(1)}, \cdots, w^{(n_\alpha)}) \tag{14}$$

a system of polynomials over $\mathbb{R}$, such that

$$\begin{aligned} \mathfrak{B}_\alpha = \{w : \mathbb{R} \to \mathbb{R}^q \quad | \quad & f_\alpha \circ (w, \frac{d}{dt}w, \cdots, \frac{d^{n_\alpha}}{dt^{n_\alpha}}w) = 0 \\ & (g_\alpha \circ (w, \frac{d}{dt}w, \cdots \frac{d^{n_\alpha}}{dt^{n_\alpha}}w))(t) \neq 0 \text{ for all } t \in \mathbb{R} \\ & (h_\alpha \circ (w, \frac{d}{dt}w, \cdots \frac{d^{n_\alpha}}{dt^{n_\alpha}}w))(t) \geq 0 \text{ for all } t \in \mathbb{R}\} \end{aligned} \tag{15}$$

and $\mathfrak{B} \subset \bigwedge_{\alpha \in A} \mathfrak{B}_\alpha$.

Of course, once this result has been established, it follows that in the real case, it is more natural to start, instead of with (3), with a vector of differential algebraic equations, inequations, and inequalities (or a finite union of these), and take the development from there. An excellent reference that gives a starting point for the problem put forward here is [5].

The $\mathfrak{B}_\alpha$'s contain the manifest behavior $\mathfrak{B}$, in the sense that $\mathfrak{B} \subset \bigwedge_{\alpha \in A} \mathfrak{B}_\alpha$. The question occurs what has to be added in order to specify $\mathfrak{B}$ exactly. The concatenations cannot occur freely, in the sense that for example the concatenation of $w_1 \in \mathfrak{B}_{\alpha_1}$ and $w_2 \in \mathfrak{B}_{\alpha_2}$ at $t = 0$ will be an element of $\mathfrak{B}$ only if certain conditions matching the derivatives of w_1 at $t = 0^-$ with those of w_2 at $t = 0^+$ are satisfied. These relations are called the *gluing conditions*.

In conclusion, the open problem consists of the following parts.

(i) Establish the resulting f_α's, g_α's, h_α's, in the real case (in particular, prove that there are only a finite number of these).

(ii) Prove (both in the complex and in the real case) that each $w \in \mathfrak{B}$ is the concatenation of trajectories from the $\mathfrak{B}_\alpha$'s.

(iii) Establish (both in the complex and in the real case) the gluing conditions.

The important system theoretic implication of all this is that nonlinear differential equations form not a particularly natural starting point for the manifest behavior of a nonlinear dynamical system. Since each system is in some sense the result of interconnecting subsystems, it is unclear how the problem of elimination of latent (interconnection) variables can be avoided. Inequalities and inequations will be introduced, not necessarily because of "hard" constraints that may be present in a system, but because they are introduced during the process of elimination.

The problems presented here are natural in the development of the study of algebraic difference or differential equations. Such systems where introduced in the system theory literature by Sontag in his thesis (see [12]), and further developed in [13, 14] (see [15] for a tutorial exposition). Other system theory work in this areas centers around the names of Fliess [6, 7] and Glad [4, 5]. It is interesting to observe that the gluing conditions can been seen as additional motivation for hybrid systems work as reported for example in [1, 16].

We remark in closing that the motivation of this open problem comes from an awareness of the urgency to generalize the algorithmic aspects of the behavioral theory (elimination, controllability, observability, control, etc., etc.) from linear systems to the differential algebraic systems. Differential algebraic systems have been studied from a control perspective for instance by Fliess and Glad [6, 7]. It is likely that Ritt's algorithm [5, 4, 11] and Gröbner bases techniques [2] can be used effectively in the problems proposed.

4 References

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