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Ports and Terminals

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Abstract

A terminal of an electrical circuit is a wire that allows the circuit to interact with its environment through a potential and a current. Interconnection is defined as variable sharing: two terminals share the same potential and current. A port of an electrical circuit is a set of terminals that satisfy port-KCL (Kirchhoff's current law). Power and energy that enter a circuit is defined for ports. Terminals are for interconnection, ports are for energy transfer. A port of a mechanical system is a set of terminals that satisfy port-KFL (Kirchhoff's force law).

1 Introduction

It is a pleasure to contribute an article to this Festschrift in honor of Yutaka Yamamoto on the occasion of his 60-th birthday. I had the privilege to develop a fruitful research collaboration with him over the last decade, leading to a number of articles [11]–[17] combining ideas from behavioral theory with system representations in terms of rational and pseudorational symbols. I am also grateful to him for hosting me on several pleasant extended visits to Kyoto University over this period.

The aim of this article is to explain the distinction that should be made in physical systems between interconnection of systems on the one hand, and energy transfer between systems on the other hand. Interconnection happens via terminals, while energy transfer happens via ports. We consider systems that interact through terminals, as wires for electrical circuits, or pins for mechanical systems. We develop the ideas mainly in the context of electrical circuits, but, towards the end of the paper, we also study mechanical systems.

2 Behavioral circuit theory

We view a circuit as follows. An electrical circuit is a device, a black-box, with wires, called terminals, through which the circuit can interact with its environment. This interaction takes place through two

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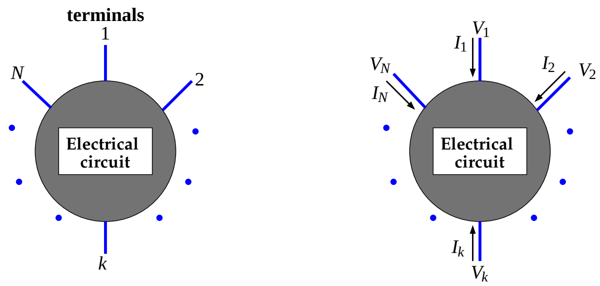
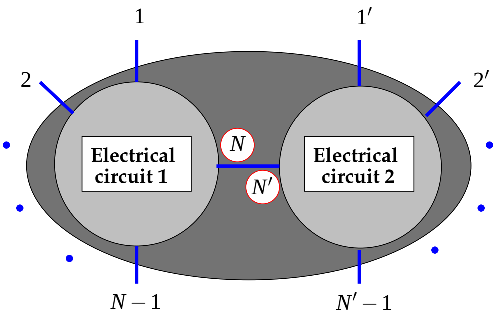




real variables, *a potential and a current*, at each terminal. The current is counted positive when it flows into the circuit. For the basic concepts of circuit theory, see [2], [6], or [1]. The setting developed in [**?**] and [6] has the same flavor as our approach.

The *behavior* of N-terminal circuit is a subset $\mathscr{B} \subseteq \left(\mathbb{R}^{2N}\right)^{\mathbb{R}}$; $(V,I) \in \mathscr{B}$ means that the time-function $(V,I) = (V_1,V_2,\ldots,V_N,I_1,I_2,\ldots,I_N) : \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}^N$ is compatible with the architecture and the element values of the circuit.

Circuit properties are conveniently defined in terms of the behavior.
A circuit obeys *Kirchhoff's voltage law* (KVL) if $(V_1,\ldots,V_N,I_1,\ldots,I_N) \in \mathscr{B}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ imply $(V_1+\alpha,\ldots,V_N+\alpha,I_1,\ldots,I_N) \in \mathscr{B}$.
A circuit obeys *Kirchhoff's current law* (KCL) if $(V_1,\ldots,V_N,I_1\ldots,I_N) \in \mathscr{B}$ implies $I_1+\cdots+I_N = 0$.

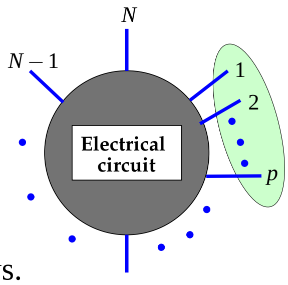
KVL means that the potentials are defined up to an arbitrary additive constant (that may change in time), while KCL means that the circuit stores no net charge.

3 Interconnection

We view interconnection as the connection of two terminals, as shown in the figure below. We start with two circuits, one with N and one with N' terminals. We assume that one terminal (terminal N) of the first circuit is connected to another terminal (terminal N') of the second circuit. The interconnection equations are

$$V_N = V_{N'} \quad \text{and} \quad I_N + I_{N'} = 0.$$

This yields a new circuit with $N+N'-2$ terminals, with behavior $\mathscr{B}_1 \sqcap \mathscr{B}_2$ defined in terms of the behavior $\mathscr{B}_1$ of the first circuit and $\mathscr{B}_2$ of the second (we consider the connected terminals as internal



to the interconnected circuit) as follows.

$$\mathscr{B}_1 \sqcap \mathscr{B}_2 := \{(V_1,V_2,\ldots,V_{N-1},V_{1'},V_{2'},\ldots,V_{N'-1},I_1,I_2,\ldots,I_{N-1},I_{1'},I_{2'},\ldots,I_{N'-1}) \\ \mid \exists\, V,I \text{ such that } (V_1, V_2, \ldots, V_{N-1},V,I_1,I_2, \ldots,I_{N-1}, I) \in \mathscr{B}_1 \\ (V_{1'},V_{2'},\ldots,V_{N'-1},V,I_{1'},I_{2'},\ldots,I_{N'-1},-I) \in \mathscr{B}_2\}.$$

The idea is that the connected terminals share the voltage and the current (up to a sign) of after interconnection. Note that the product of the shared variables has the dimension of power. The same idea of interconnection applies to the interconnection of two terminals of the same circuit, and to the connection of more terminals of two or more circuits way by connecting one pair of terminals at the time.

Interconnection preserves many circuit properties. In particular, if $\mathscr{B}_1$ and $\mathscr{B}_2$ obey KVL, or KCL, then so does $\mathscr{B}_1 \sqcap \mathscr{B}_2$.

4 Ports

In this section, we introduce a notion that is essential to the energy exchange of a circuit with its environment and between circuits. Consider a circuit with N terminals, and single out p terminals, which, for simplicity, we take to be the first p terminals.

Terminals $\{1,2,\ldots,p\}$ form a (electrical) *port* $:\Leftrightarrow$

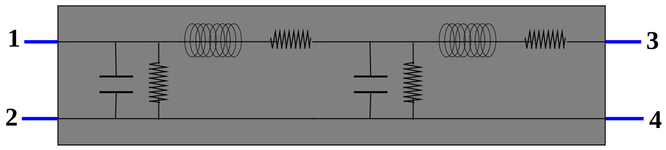
$$\big(V_1,V_2,\ldots,V_p,V_{p+1},\ldots,V_N,I_1,I_2,\ldots,I_p,I_{p+1},\ldots,I_N\big) \in \mathscr{B} \\ \Rightarrow I_1 + I_2 + \cdots + I_p = 0.$$

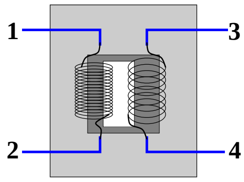
We call this relation *port KCL*. KCL implies that all the terminals combined form a port. It can be shown that for linear passive circuits satisfying KVL and KCL, port KCL is *equivalent* to *port KVL*, defined by

$$\big(V_1,\ldots,V_p,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N\big) \in \mathscr{B}, \text{ and } \alpha : \mathbb{R} \to \mathbb{R} \\ \Rightarrow \quad \big(V_1+\alpha,\ldots,V_p+\alpha,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N\big) \in \mathscr{B}.$$

If terminals $\{1,2,\ldots,p\}$ form a port, then we define the *power* that flows into the circuit at time t along these p terminals to be equal to

$$\mathit{power} \;=\; V_1(t)I_1(t) + V_2(t)I_2(t) + \cdots + V_p(t)I_p(t),$$





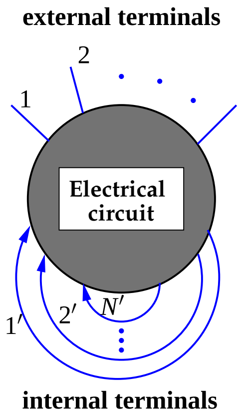
and the *energy* that flows into the circuit along these p terminals during the time-interval $[t_1, t_2]$ to be equal to

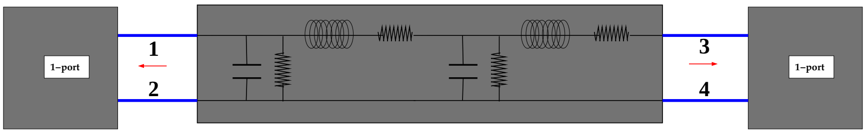
$$energy \;=\; \int_{t_1}^{t_2} (V_1(t)I_1(t) + V_2(t)I_2(t) + \cdots + V_p(t)I_p(t))\, dt.$$

Note that port KCL implies that the additive constant from KVL does not appear in the expressions of power and energy.

The above formulas for power and energy are not valid *unless these terminals form a port!* In particular, it is not possible to speak about the energy that flows into the circuit along a single wire — a conclusion that is physically quite obvious. Power and energy flow are not 'local' physical entities, but they involve action at a distance. Note that the terminals of a 2-terminal circuit that internally consists of the interconnection of circuits that all satisfy KVL and KCL form a 1-port, since KVL and KCL are preserved under interconnection. In particular, a 2-terminal circuit that is composed of resistors, capacitors, inductors, transformers, gyrators, memristors, etc. forms a 1-port. However, a pair of terminals of a circuit with more than two terminals rarely forms a 1-port. In particular, for the circuit shown below, the terminals $\{1,2,3,4\}$ form a port, but there is no reason why the terminal pairs $\{1,2\}$ and $\{3,4\}$ should form ports.

An example of an element that consists of more than one port is a transformer.





The behavioral equations of an ideal transformer are

$$V_1 - V_2 = n(V_3 - V_4),\ I_3 = -nI_1,\ I_1 + I_2 = 0,\ I_3 + I_4 = 0, \quad \text{with } n \text{ the } \textit{turns ratio}.$$

Clearly $\{1,2\}$ and $\{3,4\}$ form ports, and the energy that flows into the port $\{1,2\}$ is equal to the energy that flows out of the port $\{3,4\}$.

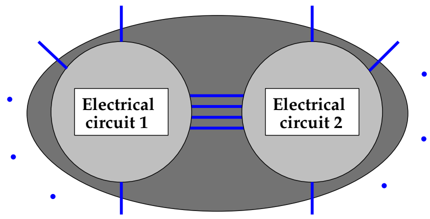
5 Internal ports

In order to study the energy flow inside a circuit, we introduce in this section circuits with both external and internal terminals. Consider a circuit with N external terminals and also N' internal terminals. Assume that the internal terminals are directed.

We can define the behavior of this circuit analogously as we did for circuits with only external terminals. A set of terminals, say $\{1', 2', \ldots, p'\}$, forms an *internal port* $:\Leftrightarrow$ for all elements of the behavior, $I_{1'} + I_{2'} + \cdots + I_{p'} = 0$. A circuit has in general *external ports*, consisting of only external terminals, *internal ports*, consisting of only internal terminals, and *mixed ports*, consisting of both external and internal terminals. The internal ports allow to consider the power and energy flow between parts of a circuit. For example, it is possible this way to consider the energy transferred into the ports formed by terminals $\{1,2\}$ and $\{3,4\}$ of the circuit below, since these pairs form internal ports.

6 Terminals are for interconnection, ports for energy transfer

As explained before, interconnection means that certain terminals share the same potential and current (up to a sign). This is distinctly different from stating that the power or the energy flows from one side of an interconnection to the other side. Power and energy involve ports, and this requires consideration

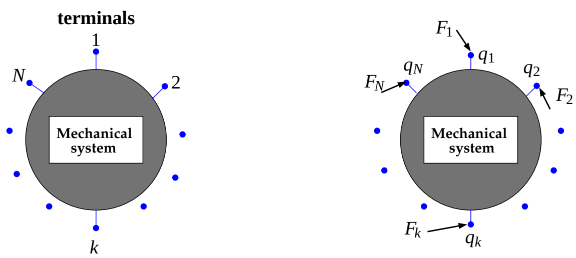


of more than one terminal at the time. For example, the two circuits in the figure below share four terminals, but it is not possible to speak of the energy that flows from circuit 1 to circuit 2, unless the connected terminals form internal ports. Similarly, it is not possible to speak about the energy that flows from the environment into circuit 1, or from the environment into circuit 2, unless the external terminals of system 1 and of system 2 form ports. Of course, assuming KVL and KCL, the external terminals of the interconnected system always form a port.

Setting up the behavioral equations of a circuit involves interconnection and variable sharing. Exchange of power and energy involves ports. Interconnections need not involve ports or power and energy transfer. These observations put into perspective power-based modeling methodologies of interconnected systems, as bond graphs [7, 3] and port-Hamiltonian systems [9, 4]. In [10] we propose a modeling methodology for interconnected systems based on *tearing, zooming, and linking*, which involves interconnection by sharing variables, but in which power considerations do not take a central place.

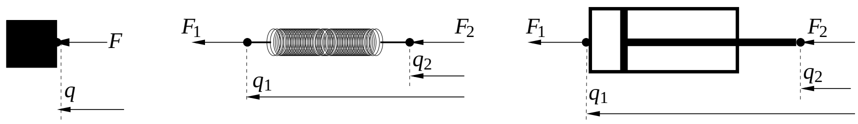
7 Mechanical systems

We view a mechanical system as a device, a black box, with pins, called terminals, through which the system can interact with its environment. This interaction takes place through two vectors, *a position and a force*, for each terminal. Even though angles and torques play an important role in mechanical systems, we do not consider these here. The position and the force are elements of $\mathbb{R}$ for rectilinear motion, or of $\mathbb{R}^2$ for motions in the plane, or of $\mathbb{R}^3$ for spatial motion. We indicate the fact that we want to leave open which of these cases we consider by the notation $q_k : \mathbb{R} \to \mathbb{R}^\bullet$ and $F_k : \mathbb{R} \to \mathbb{R}^\bullet$.



The *behavior* of the mechanical system is a subset $\mathscr{B} \subseteq ((\mathbb{R}^\bullet)^{2N})^{\mathbb{R}}$; $(q,F) \in \mathscr{B}$ means that the position/force time-function $(q,F) = (q_1,q_2,\ldots,q_N,F_1,F_2,\ldots,F_N) : \mathbb{R} \to (\mathbb{R}^\bullet)^N \times (\mathbb{R}^\bullet)^N$ is compatible with the architecture and the element values of the mechanical system.

Basic building blocks for mechanical systems under rectilinear motion are masses, springs, and



dampers. Their behavioral equations are

$$
\begin{array}{lrl}
\text{mass:} & M\frac{d^2}{dt^2}q = & F, \\
\text{spring:} & q_1 - q_2 = & \rho(F_1), \quad F_1 + F_2 = 0, \\
\text{damper:} & \frac{d}{dt}q_1 - \frac{d}{dt}q_2 = & d(F_1), \quad F_1 + F_2 = 0,
\end{array}
$$

with $\rho : \mathbb{R} \to \mathbb{R}$ the spring characteristic, and $d : \mathbb{R} \to \mathbb{R}$ the damper characteristic.

We now list some properties of mechanical systems that are conveniently defined in terms of the behavior.
A mechanical system is *invariant under uniform motions* if $(q_1, \ldots, q_N, F_1, \ldots, F_N) \in \mathscr{B}$ and $v : t \in \mathbb{R} \mapsto (a + bt) \in \mathbb{R}^\bullet$, $a, b \in \mathbb{R}^\bullet$, imply $(q_1 + v, \ldots, q_N + v, F_1, \ldots, F_N) \in \mathscr{B}$.
A mechanical system obeys *Kirchhoff's force law* (KFL) if $(q_1, q_2, \ldots, q_N, F_1, F_2, \ldots, F_N) \in \mathscr{B}$ implies $F_1 + F_2 + \cdots + F_N = 0$.
The spring and the damper obey KFL, but the mass does not. Invariance under uniform motions, a most basic premise of mechanics, is important in the sequel.

The interconnection of two mechanical systems is defined by interconnecting two terminals at the time, identifying the positions of the interconnected terminals, and putting the sum of the forces acting on the interconnected terminals equal to zero. The interconnecting equations are

$$q_N = q_{N'} \quad \text{and} \quad F_N + F_{N'} = 0.$$

Note that the product of the shared variables does not have the dimension of power.

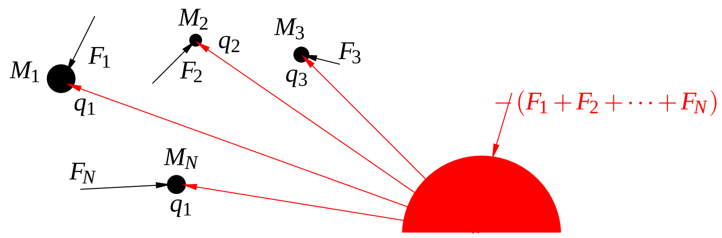
This yields, with notation analogous to the one used for circuits,

$$
\begin{aligned}
\mathscr{B}_1 \sqcap \mathscr{B}_2 := \{ & (q_1, q_2, \ldots, q_{N-1}, q_{1'}, q_{2'}, \ldots, q_{N'-1}, F_1, F_2, \ldots, F_{N-1}, F_{1'}, F_{2'}, \ldots, F_{N'-1}) \\
& \mid \exists\, q, F \text{ such that } (q_1, q_2, \ldots, q_{N-1}, q, F_1, F_2, \ldots, F_{N-1}, F) \in \mathscr{B}_1 \\
& \qquad (q_{1'}, q_{2'}, \ldots, q_{N'-1}, q, F_{1'}, F_{2'}, \ldots, F_{N'-1}, -F) \in \mathscr{B}_2 \}.
\end{aligned}
$$

This leads to interconnection of different terminals of the same mechanical system, and to interconnection of many pairs of terminals of two or more mechanical systems. Interconnection preserves invariance under uniform motion and KFL.

8 Mechanical ports

We now introduce conditions that allows to study power and energy flow in mechanical systems. Consider a mechanical system, and single out p terminals, which, for simplicity, we take to be the



first p terminals.

Terminals $\{1,2,\ldots,p\}$ form a (mechanical) *port* $:\Leftrightarrow$

$$(q_1,\ldots,q_p,q_{p+1},\ldots,q_N,F_1,\ldots,F_p,F_{p+1},\ldots,F_N)\in\mathscr{B},$$
$$\Rightarrow \; F_1+F_2+\cdots+F_p=0.$$

We call this relation *port KFL*. Note that KFL imply that all terminals combined form a port. Also, the external terminals of the interconnection of port devices form again a port. Note that including masses with external forces acting on them form a difficulty for KFL.

If terminals $\{1,2,\ldots,p\}$ form a port, then we define the *power* that flows into the mechanical system at time t along these p terminals and the *energy* that flows into the circuit along these p terminals on the time-interval $[t_1,t_2]$ to be equal to

$$\begin{aligned} \textit{power} &= F_1(t)^\top \frac{d}{dt}q_1(t)+F_2(t)^\top \frac{d}{dt}q_2(t)+\cdots+F_p(t)^\top \frac{d}{dt}q_p(t), \\ \textit{energy} &= \int_{t_1}^{t_2}\left(F_1(t)^\top \frac{d}{dt}q_1(t)+F_2(t)^\top \frac{d}{dt}q_2(t)+\cdots+F_p(t)^\top \frac{d}{dt}q_p(t)\right)dt. \end{aligned}$$

The above formulas for power and energy are not valid *unless these terminals form a mechanical port!* Note that port KFL implies that power and energy are invariant under the additive constant that can be added to the velocities due to the port invariance under uniform motion. A mass, a spring and a damper obey invariance under uniform motion. A spring and a damper form a mechanical port, but a mass does not. The inerter [8] is a mass-like device that is a port. In order to be able to consider the energy that flows into a mechanical system, we should make sure that the total external force acting on the masses is zero. This can be obtained, albeit in a physically artificial way, by introducing a *'ground'*, an infinite mass that cannot be accelerated, on which the negative of the total force acts, and with respect to which positions are measured, as illustrated below.

We now compute the kinetic energy stored in N moving masses with masses $M_1,M_2,\ldots,M_N$, positions $q_1,q_2,\ldots,q_N\in\mathbb{R}^3$, and with forces $F_1,F_2,\ldots,F_N\in\mathbb{R}^3$ acting on them. By Newton's second law, $M_k\frac{d^2}{dt^2}q_k=F_k$. If we assume that KFL is satisfied, $F_1+F_2+\cdots+F_N=0$, then it is readily verified that

$$\frac{d}{dt}\left(\frac{1}{4}\sum_{i,j\in\{1,2,\ldots,N\}}\frac{M_i\,M_j}{M_1+M_2+\cdots+M_N}\left|\left|\frac{d}{dt}q_i-\frac{d}{dt}q_j\right|\right|^2\right) = \sum_{i\in\{1,2,\ldots,N\}}F_i^\top\frac{d}{dt}q_i.$$

Hence the kinetic energy equals

$$\mathscr{E}_{\text{kinetic}}=\frac{1}{4}\sum_{i,j\in\{1,2,\ldots,N\}}\frac{M_i\,M_j}{M_1+M_2+\cdots+M_N}\left|\left|\frac{d}{dt}q_i-\frac{d}{dt}q_j\right|\right|^2.$$

$\mathscr{E}_{\text{kinetic}}$ is invariant under uniform motions, as a physically meaningful quantity should be. The expression for $\mathscr{E}_{\text{kinetic}}$ can also be justified by computing the energy that can be stored in a spring or dissipated in a damper, mounted between the masses, while bringing all the masses to the same velocity. This expression is distinct from the classical expression of the kinetic energy,

$$\mathscr{E}_{\text{classical}} = \frac{1}{2} \sum_{i\in\{1,2,\ldots,N\}} M_i \, ||\frac{d}{dt}q_i||^2.$$

In fact, without requiring KFL, there holds

$$\frac{d}{dt}\left(\frac{1}{2} \sum_{i\in\{1,2,\ldots,N\}} M_i \, ||\frac{d}{dt}q_i||^2\right) = \sum_{i\in\{1,2,\ldots,N\}} F_i^\top \frac{d}{dt}q_i.$$

The classical expression $\mathscr{E}_{\text{classical}}$ for the kinetic energy can be made compatible with the expression for $\mathscr{E}_{\text{kinetic}}$ by assuming the presence of an infinite mass at rest on which the force $-(F_1 + F_2 + \cdots + F_N)$ acts without accelerating it, and applying the formula for $\mathscr{E}_{\text{kinetic}}$.

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