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# Behaviors Described by Rational Symbols and the Parametrization of the Stabilizing Controllers

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**Summary.** We present a behavioral theory of linear systems described by differential equations involving matrices of rational functions. Representations of controllable and stabilizable systems that are left coprime over the ring of proper, stable, or proper stable rational functions are discussed. These representations lead to effective parametrizations of the set of stabilizing controllers for a plant.

**Keywords:** Behaviors, rational symbols, controllability, stabilizability, observability, regular controllers, superregular controllers, parametrization of stabilizing controllers.

**AMS Subject classifications**: 93A05, 93C15, 34A30, 93B05, 93B07, 93C35.

## 1.1 Introduction

It is a pleasure to contribute an article to this Festschrift in honor of Mathukumalli Vidyasagar on the occasion of his 60-th birthday. As the subject of our article, we have chosen the parametrization of stabilizing controllers for linear systems. This topic goes back to the pioneering contributions of Kučera [2] and Youla-Bongiorgio-Jabr [7], and is commonly known as the Kučera-Youla parametrization of the set of stabilizing controllers. This parametrization issue and the algebraic structure that underpins its solution are main topics discussed in Vidyasagar's book [4], one of the few books in the field of Systems & Control that can truly be termed 'Algebraic System Theory'. This book served as the inspiration for the present paper.

Our approach is somewhat different from the usual one in that we do not view a linear system as defined by a transfer function. Rather, we view a system in the behavioral sense, that is, as a family of trajectories. All relevant system properties, such as controllability, stabilizability, observability, and detectability, are defined in terms of the behavior. Control is viewed as restricting the plant behavior by intersecting it with the controller behavior.

The behavior of a linear time-invariant differential system is defined as the set of solutions of a system of linear constant-coefficient differential equations. However, these behaviors can be represented in many other ways, for example, as the set of solutions of a system of equations involving a differential operator in a matrix of rational functions, rather than in a matrix of polynomials. The problem of parametrizing the set of stabilizing controllers leads to the question of determining all controller behaviors which, when intersected with the given plant behavior, yield a stable system. The representation of behaviors in terms of rational symbols turns out to be an effective representation that leads to a parametrization of the set of stabilizing controllers.

In the classical approach [2, 7, 4], systems with the same transfer function are identified. By taking a trajectory-based definition of a system, the behavioral point of view is able to carefully keep track of all trajectories, also of the non-controllable ones. Loosely speaking, the stable coprime factorizations of the transfer-function based approach manage to avoid unstable pole-zero cancellations. Our approach avoids introducing, as well as cancelling, common poles and zeros. Since the whole issue of coprime factorizations over the ring of proper stable rational functions started from a need to deal carefully with pole-zero cancellations, we feel that our trajectory-based mode of thinking offers a useful point of view.

A few words about the notation and nomenclature used. We use standard symbols for the sets $\mathbb{R}, \mathbb{N}, \mathbb{Z}$, and $\mathbb{C}$. $\overline{\mathbb{C}}_+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) \geq 0\}$ denotes the closed right-half of the complex plane. We use $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc. for vectors and matrices. When the number of rows or columns is immaterial (but finite), we use the notation $^\bullet$, $^{\bullet\times\bullet}$, etc. Of course, when we then add, multiply, or equate vectors or matrices, we assume that the dimensions are compatible. $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ denotes the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{n}}$. The symbol $I$ denotes the identity matrix, and 0 the zero matrix. When we want to emphasize the dimension, we write $I_{\mathtt{n}}$ and $0_{\mathtt{n}_1\times\mathtt{n}_2}$. A matrix is said to be of *full row rank* if its rank is equal to the number of rows. Full column rank is defined analogously.

$\mathbb{R}[\xi]$ denotes the set of polynomials with real coefficients in the indeterminate $\xi$, and $\mathbb{R}(\xi)$ denotes the set of real rational functions in the indeterminate $\xi$. $\mathbb{R}[\xi]$ is a ring and $\mathbb{R}[\xi]^{\mathtt{n}}$ a finitely generated $\mathbb{R}[\xi]$-module. $\mathbb{R}(\xi)$ is a field and $\mathbb{R}(\xi)^{\mathtt{n}}$ is an $\mathtt{n}$-dimensional $\mathbb{R}(\xi)$-vector space. The polynomials $p_1, p_2 \in \mathbb{R}[\xi]$ are said to be *coprime* if they have no common zeros. $p \in \mathbb{R}[\xi]$ is said to be *Hurwitz* if it has no zeros in $\overline{\mathbb{C}}_+$. The *relative degree* of $f \in \mathbb{R}(\xi)$, $f = n/d$, with $n, d \in \mathbb{R}[\xi]$, is the degree of the denominator $d$ minus the degree of the numerator $n$; $f \in \mathbb{R}(\xi)$ is said to be *proper* if the relative degree is $\geq 0$, *strictly proper* if the relative degree is $> 0$, and *biproper* if the relative degree is equal to 0. The rational function $f \in \mathbb{R}(\xi)$, $f = n/d$, with $n, d \in \mathbb{R}[\xi]$ coprime, is said to be *stable* if $d$ is Hurwitz, and *miniphase* if $n$ and $d$ are both Hurwitz.

We only discuss the main ideas. Details and proofs may be found in [6]. The results can easily be adapted to other stability domains, but in this article, we only consider the Hurwitz domain for concreteness.

## 1.2 Rational symbols

We consider behaviors $\mathscr{B} \subseteq (\mathbb{R}^\bullet)^{\mathbb{R}}$ that are the set of solutions of a system of linear-constant coefficient differential equations. In other words, $\mathscr{B}$ is the solution set of

$$R\left(\tfrac{d}{dt}\right) w = 0, \tag{$\mathscr{R}$}$$

where $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$. We shall deal with infinitely differentiable solutions only. Hence ($\mathscr{R}$) defines the dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}^\bullet, \mathscr{B})$ with

$$\mathscr{B} = \left\{ w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \;\middle|\; R\left(\tfrac{d}{dt}\right) w = 0 \right\}.$$

We call this system (or its behavior) a *linear time-invariant differential system*. Note that we may as well denote this behavior as $\mathscr{B} = \mathtt{kernel}\big(R\left(\tfrac{d}{dt}\right)\big)$, since $\mathscr{B}$ is actually the kernel of the differential operator

$$R\left(\tfrac{d}{dt}\right) : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\texttt{column dimension}(R)}) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\texttt{row dimension}(R)}).$$

We denote the set of linear time-invariant differential systems or their behaviors by $\mathscr{L}^\bullet$ and by $\mathscr{L}^{\mathtt{w}}$ when the number of variables is $\mathtt{w}$.

We will extend the above definition of a behavior defined by a differential equation involving a polynomial matrix to a 'differential equation' involving a matrix of rational functions. In order to do so, we first recall the terminology of factoring a matrix of rational functions in terms of polynomial matrices. The pair $(P, Q)$ is said to be a *left factorization over* $\mathbb{R}[\xi]$ of $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ if (i) $P \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_1}$ and $Q \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$, (ii) $\mathtt{determinant}(P) \neq 0$, and (iii) $M = P^{-1}Q$. $(P, Q)$ is said to be a *left-coprime factorization over* $\mathbb{R}[\xi]$ of $M$ if, in addition, (iv) $P$ and $Q$ are left coprime over $\mathbb{R}[\xi]$. Recall that $P$ and $Q$ are said to be *left coprime over* $\mathbb{R}[\xi]$ if for every factorization $\left[P\ Q\right] = F\left[P'\ Q'\right]$ with $F \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_1}, F$ is $\mathbb{R}[\xi]$-unimodular. It is easy to see that a left-coprime factorization over $\mathbb{R}[\xi]$ of $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ is unique up to premultiplication of $P$ and $Q$ by an $\mathbb{R}[\xi]$-unimodular polynomial matrix $U \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_1}$.

Consider the system of 'differential equations'

$$G\left(\tfrac{d}{dt}\right) w = 0, \tag{$\mathscr{G}$}$$

with $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, called the *symbol* of ($\mathscr{G}$). Since $G$ is a matrix of rational functions, it is not clear when $w : \mathbb{R} \to \mathbb{R}^\bullet$ is a solution of ($\mathscr{G}$). This is not a matter of smoothness, but a matter of giving a meaning to the equality, since $G\left(\tfrac{d}{dt}\right)$ is not a differential operator, and not even a map.

We define solutions as follows. Let $(P, Q)$ be a left-coprime matrix factorization over $\mathbb{R}[\xi]$ of $G = P^{-1}Q$. Define

$$[\![ w : \mathbb{R} \to \mathbb{R}^\bullet \text{ is a solution of } (\mathscr{G}) ]\!] :\Leftrightarrow [\![ Q\left(\tfrac{d}{dt}\right) w = 0 ]\!].$$

Hence ($\mathscr{G}$) defines the system

$$\Sigma = \left(\mathbb{R}, \mathbb{R}^\bullet, \mathtt{kernel}\left(Q\left(\tfrac{d}{dt}\right)\right)\right) \in \mathscr{L}^\bullet.$$

It follows from this definition that $G\left(\frac{d}{dt}\right)$ is *not* a map on $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$. Rather, $w \mapsto G\left(\frac{d}{dt}\right)w$ is the point-to-set map that associates with $w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ the set $v' + v$, with $v' \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ a particular solution of $P\left(\frac{d}{dt}\right)v' = Q\left(\frac{d}{dt}\right)w$ and $v \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ any function that satisfies $P\left(\frac{d}{dt}\right)v = 0$. This set of $v$'s is a finite-dimensional linear subspace of $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ of dimension equal to the degree of $\mathtt{determinant}(P)$. Hence, if $P$ is not an $\mathbb{R}[\xi]$-unimodular polynomial matrix, equivalently, if $G$ is not a polynomial matrix, $G\left(\frac{d}{dt}\right)$ is not a point-to-point map. Viewing $G\left(\frac{d}{dt}\right)$ as a point-to set map leads to the definition of its kernel as

$$\mathtt{kernel}\left(G\left(\tfrac{d}{dt}\right)\right) := \{w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \mid 0 \in G\left(\tfrac{d}{dt}\right)w\},$$

i.e. $\mathtt{kernel}\left(G\left(\frac{d}{dt}\right)\right)$ consists of the set of solutions of ($\mathscr{G}$), and of its image as

$$\mathtt{image}\left(G\left(\tfrac{d}{dt}\right)\right) := \{v \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet) \mid v \in G\left(\tfrac{d}{dt}\right)w \text{ for some } w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)\}.$$

Hence ($\mathscr{G}$) defines the system

$$\Sigma = \left(\mathbb{R}, \mathbb{R}^\bullet, \mathtt{kernel}\left(G\left(\tfrac{d}{dt}\right)\right)\right) := \left(\mathbb{R}, \mathbb{R}^\bullet, \mathtt{kernel}\left(Q\left(\tfrac{d}{dt}\right)\right)\right) \in \mathscr{L}^\bullet.$$

Three main theorems in the theory of linear time-invariant differential systems are (i) the elimination theorem, (ii) the one-to-one relation between annihilators and submodules or subspaces, and (iii) the equivalence of controllability and existence of an image representation. Results involving (ii) and (iii) are discussed in later sections.

The *elimination theorem* states that if $\mathscr{B} \in \mathscr{L}^{\mathtt{w}_1+\mathtt{w}_2}$, then

$$\mathscr{B}_1 := \{w_1 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \mid \exists w_2 \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \text{ such that } (w_1, w_2) \in \mathscr{B}\}$$

belongs to $\mathscr{L}^{\mathtt{w}_1}$. In other words, $\mathscr{L}^\bullet$ is closed under projection. The elimination theorem implies that $\mathscr{L}^\bullet$ is closed under addition, intersection, projection, and under action and inverse action with $F\left(\frac{d}{dt}\right)$, where $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$.

## 1.3 Input, output, and state cardinality

The integer invariants $\mathsf{w}, \mathsf{m}, \mathsf{p}, \mathsf{n}$ are maps from $\mathscr{L}^\bullet$ to $\mathbb{Z}_+$ that play an important role in the theory of linear time-invariant differential systems. Intuitively,

$\mathsf{w}(\mathscr{B})$ equals the number of variables in $\mathscr{B}$,
$\mathsf{m}(\mathscr{B})$ equals the number of input variables in $\mathscr{B}$,
$\mathsf{p}(\mathscr{B})$ equals the number of output variables in $\mathscr{B}$, and
$\mathsf{n}(\mathscr{B})$ equals the number of state variables in $\mathscr{B}$.

The integer invariant $\mathsf{w}$ is defined by $[\![\mathsf{w}(\mathscr{B}) := \mathtt{w}]\!] \iff [\![\mathscr{B} \in \mathscr{L}^{\mathtt{w}}]\!]$.

The other integer invariants are most easily captured by means of representations. A behavior $\mathscr{B} \in \mathscr{L}^\bullet$ admits an *input/output representation*

$$P\left(\tfrac{d}{dt}\right)y = Q\left(\tfrac{d}{dt}\right)u, \quad w = \Pi \begin{bmatrix} u \\ y \end{bmatrix} \tag{i/o}$$

with $P \in \mathbb{R}(\xi)^{\mathsf{p}(\mathscr{B})\times\mathsf{p}(\mathscr{B})}$, $\mathtt{determinant}(P) \neq 0$, $Q \in \mathbb{R}(\xi)^{\mathsf{p}(\mathscr{B})\times\mathsf{m}(\mathscr{B})}$, and $\Pi \in \mathbb{R}^{\mathsf{w}(\mathscr{B})\times\mathsf{w}(\mathscr{B})}$ a permutation matrix. This input/output representation of $\mathscr{B}$ defines $\mathsf{m}(\mathscr{B})$ and $\mathsf{p}(\mathscr{B})$ uniquely. It follows from the conditions on $P$ and $Q$ that $u$ is free, that is, that for any $u \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathsf{m}(\mathscr{B})})$, there exists a $y \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathsf{p}(\mathscr{B})})$ such that $P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$. The permutation matrix $\Pi$ shows how the input and output components are chosen from the components of $w$, and results in an input/output partition of $w$.

The matrix $G = P^{-1}Q \in \mathbb{R}(\xi)^{\mathsf{p}(\mathscr{B})\times\mathsf{m}(\mathscr{B})}$ is called the *transfer function* corresponding to this input/output partition. In fact, it is possible to choose this partition such that $G$ is proper. It is worth mentioning that in general $P\left(\frac{d}{dt}\right)y = Q\left(\frac{d}{dt}\right)u$ has a different behavior than $y = P^{-1}Q\left(\frac{d}{dt}\right)u$. The difference is due to the fact that $\mathscr{B}$ may not be controllable, as discussed in the next section.

A behavior $\mathscr{B} \in \mathscr{L}^\bullet$ also admits an observable *input/state/output representation*

$$\tfrac{d}{dt}x = Ax + Bu,\ y = Cx + Du, \quad w = \Pi \begin{bmatrix} u \\ y \end{bmatrix}, \tag{i/s/o}$$

with $A \in \mathbb{R}^{\mathsf{n}(\mathscr{B})\times\mathsf{n}(\mathscr{B})}$, $B \in \mathbb{R}^{\mathsf{n}(\mathscr{B})\times\mathsf{m}(\mathscr{B})}$, $C \in \mathbb{R}^{\mathsf{p}(\mathscr{B})\times\mathsf{n}(\mathscr{B})}$, $D \in \mathbb{R}^{\mathsf{p}(\mathscr{B})\times\mathsf{m}(\mathscr{B})}$, $\Pi \in \mathbb{R}^{\mathsf{w}(\mathscr{B})\times\mathsf{w}(\mathscr{B})}$ a permutation matrix, and $(A,C)$ an observable pair. By eliminating $x$, the $(u,y)$-behavior defines an linear time-invariant differential system, with behavior denoted by $\mathscr{B}'$. This behavior is related to $\mathscr{B}$ by $\mathscr{B} = \Pi\mathscr{B}'$. It can be shown that this input/state/output representation of $\mathscr{B}$, including the observability of $(A,C)$, defines $\mathsf{m}(\mathscr{B})$, $\mathsf{p}(\mathscr{B})$, and $\mathsf{n}(\mathscr{B})$ uniquely.

## 1.4 Controllability, stabilizability, observability, and detectability

The behavior $\mathscr{B} \in \mathscr{L}^\bullet$ is said to be *controllable* if for all $w_1, w_2 \in \mathscr{B}$, there exists $T \geq 0$ and $w \in \mathscr{B}$, such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geq T$. $\mathscr{B}$ is said to be *stabilizable* if for all $w \in \mathscr{B}$, there exists $w' \in \mathscr{B}$, such that $w'(t) = w(t)$ for $t < 0$ and $w'(t) \to 0$ as $t \to \infty$.

In words, controllability means that it is possible to switch between any two trajectories in the behavior, and stabilizability means that every trajectory can be steered to zero asymptotically.

Until now, we have dealt with representations involving the variables $w$ only. However, many models, such as first principles models obtained by interconnection and state models, include auxiliary variables in addition to the variables the model aims at. We call the latter *manifest variables*, and the auxiliary variables *latent variables*. In the context of rational models, this leads to the model class

$$R\left(\tfrac{d}{dt}\right)w = M\left(\tfrac{d}{dt}\right)\ell \tag{LV}$$

with $R, M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. By the elimination theorem, the *manifest behavior* of (LV), defined as

$$\{w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet) \mid \exists \ell \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet) \text{ such that (LV) holds}\},$$

belongs to $\mathscr{L}^\bullet$.

The latent variable system (LV) is said to be *observable* if, whenever $(w,\ell_1)$ and $(w,\ell_2)$ satisfy (LV), then $\ell_1 = \ell_2$. (LV) is said to be *detectable* if, whenever $(w,\ell_1)$ and $(w,\ell_2)$ satisfy (LV), then $\ell_1(t) - \ell_2(t) \to 0$ as $t \to \infty$.

In words, observability means that the latent variable trajectory can be deduced from the manifest variable trajectory, and detectability means that the latent variable trajectory can be deduced from the manifest variable trajectory asymptotically. The notions of observability and detectability apply to more general situations, but here we use them only in the context of latent variable systems.

It is easy to derive tests to verify these properties in terms of kernel representations and the zeros of the associated symbol. We first recall the notion of poles and zeros of a matrix of rational functions.

$M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ can be brought into a simple canonical form, called the *Smith-McMillan form* by pre- and postmultiplication by $\mathbb{R}[\xi]$-unimodular polynomial matrices. Let $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$. There exist $U \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_1}, V \in \mathbb{R}[\xi]^{\mathtt{n}_2\times\mathtt{n}_2}$, both $\mathbb{R}[\xi]$-unimodular, $\Pi \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_1}$, and $Z \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$ such that $M = U\Pi^{-1}ZV$, with

$$\Pi = \mathtt{diagonal}(\pi_1,\pi_2,\cdots,\pi_{\mathtt{n}_1}), Z = \begin{bmatrix} \mathtt{diagonal}(\zeta_1,\zeta_2,\cdots,\zeta_\mathtt{r}) & 0_{\mathtt{r}\times(\mathtt{n}_2-\mathtt{r})} \\ 0_{(\mathtt{n}_1-\mathtt{r})\times\mathtt{r}} & 0_{(\mathtt{n}_1-\mathtt{r})\times(\mathtt{n}_2-\mathtt{r})} \end{bmatrix}$$

with $\zeta_1,\zeta_2,\cdots,\zeta_\mathtt{r},\pi_1,\pi_2,\cdots,\pi_{\mathtt{n}_1}$ non-zero monic elements of $\mathbb{R}[\xi]$, the pairs $\zeta_\mathtt{k},\pi_\mathtt{k}$ coprime for $\mathtt{k} = 1,2,\ldots,\mathtt{r}$, $\pi_\mathtt{k} = 1$ for $\mathtt{k} = \mathtt{r}+1,\mathtt{r}+2\ldots,\mathtt{n}_1$, and with $\zeta_{\mathtt{k}-1}$ a factor of $\zeta_\mathtt{k}$ and $\pi_\mathtt{k}$ a factor of $\pi_{\mathtt{k}-1}$, for $\mathtt{k} = 2,\cdots,\mathtt{r}$. Of course, $\mathtt{r} = \mathtt{rank}(M)$. The roots of the $\pi_\mathtt{k}$'s (hence of $\pi_1$, disregarding multiplicity issues) are called the *poles* of $M$, and those of the $\zeta_\mathtt{k}$'s (hence of $\zeta_\mathtt{r}$, disregarding multiplicity issues) are called the *zeros* of $M$. When $M \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, the $\pi_\mathtt{k}$'s are absent (they are equal to 1). We then speak of the *Smith form*.

**Proposition 1**

1. ($\mathscr{G}$) is controllable if and only if $G$ has no zeros.
2. ($\mathscr{G}$) is stabilizable if and only if $G$ has no zeros in $\overline{\mathbb{C}}_+$.
3. (LV) is observable if and only if $M$ has full column rank and has no zeros.
4. (LV) is detectable if and only if $M$ has full column rank and has no zeros in $\overline{\mathbb{C}}_+$.

■

Consider the following special case of (LV)

$$w = M\left(\tfrac{d}{dt}\right)\ell \qquad (\mathscr{M})$$

with $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$. Note that, with $M\left(\frac{d}{dt}\right)$ viewed as a point-to-set map, the manifest behavior of ($\mathscr{M}$) is equal to $\mathtt{image}\left(M\left(\frac{d}{dt}\right)\right)$. ($\mathscr{M}$) is hence called an *image representation* of its manifest behavior. In the observable case, that is, if $M$ is of full column rank and has no zeros, $M$ has a polynomial left inverse, and hence ($\mathscr{M}$) defines a differential operator mapping $w$ to $\ell$. In other words, in the observable case, there exists an $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that ($\mathscr{M}$) has the representation

$$w = M\left(\tfrac{d}{dt}\right)\ell, \quad \ell = F\left(\tfrac{d}{dt}\right)w.$$

The well-known relation between controllability and image representations for polynomial symbols remains valid in the rational case.

**Theorem 2** The following are equivalent for $\mathscr{B} \in \mathscr{L}^{\bullet}$.

1. $\mathscr{B}$ is controllable.
2. $\mathscr{B}$ admits an image representation ($\mathscr{M}$) with $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$.
3. $\mathscr{B}$ admits an observable image representation ($\mathscr{M}$) with $M \in \mathbb{R}(\xi)^{\bullet\times\bullet}$.

■

Let $\mathscr{B} \in \mathscr{L}^{\bullet}$. The *controllable part* of $\mathscr{B}$ is defined as

$$\mathscr{B}_{\text{controllable}} := \{w \in \mathscr{B} \mid \forall\, t_0, t_1 \in \mathbb{R}, t_0 \leq t_1,$$
$$\exists\, w' \in \mathscr{B} \text{ with compact support such that } w(t) = w'(t) \text{ for } t_0 \leq t \leq t_1\}.$$

In words, $\mathscr{B}_{\text{controllable}}$ consists of the trajectories in $\mathscr{B}$ that can be steered to zero in finite time. It is easy to see that $\mathscr{B}_{\text{controllable}} \in \mathscr{L}^{\bullet}$ and that it is controllable. In fact, $\mathscr{B}_{\text{controllable}}$ is the largest controllable behavior contained in $\mathscr{B}$.

The controllable part induces an equivalence relation on $\mathscr{L}^{\bullet}$, called *controllability equivalence*, by setting

$$[\![\mathscr{B}' \sim_{\text{controllability}} \mathscr{B}'']\!] :\Leftrightarrow [\![\mathscr{B}'_{\text{controllable}} = \mathscr{B}''_{\text{controllable}}]\!].$$

It is easy to prove that $\mathscr{B}' \sim_{\text{controllable}} \mathscr{B}''$ if and only if $\mathscr{B}'$ and $\mathscr{B}''$ have the same compact support trajectories, or, for that matter, the same square integrable trajectories. Each equivalence class modulo controllability contains exactly one controllable behavior. This controllable behavior is contained in all the other behaviors that belong to the equivalence class modulo controllability.

The system $G\left(\frac{d}{dt}\right)w = 0$, where $G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and $F\left(\frac{d}{dt}\right)G\left(\frac{d}{dt}\right)w = 0$ are controllability equivalent if $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ is square and nonsingular. In particular, two input/output systems (i/o) have the same transfer function if and only if they are controllability equivalent.

If $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}$ have full row rank, then the behavior defined by $G_1\left(\frac{d}{dt}\right)w = 0$ is equal to the behavior defined by $G_2\left(\frac{d}{dt}\right)w = 0$ if there exists a $\mathbb{R}[\xi]$-unimodular matrix $U \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $G_2 = UG_1$. On the other hand, the behavior defined by $G_1\left(\frac{d}{dt}\right)w = 0$ has the same controllable part as the behavior defined by $G_2\left(\frac{d}{dt}\right)w = 0$ if and only if there exists an $F \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, square and nonsingular, such that $G_2 = FG_1$. If $G_1$ and $G_2$ are full row rank polynomial matrices, then equality of the behaviors holds if and only if $G_2 = UG_1$. This illustrates the subtle distinction between equations that have the same behavior, versus behaviors that are controllability equivalent.

## 1.5 Rational annihilators

Obviously, for $n \in \mathbb{R}(\xi)^\bullet$ and $w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^\bullet)$, the statements $n\left(\frac{d}{dt}\right)^\top w = 0$, and, hence, for $\mathscr{B} \in \mathscr{L}^\bullet$, $n\left(\frac{d}{dt}\right)^\top \mathscr{B} = 0$, meaning $n\left(\frac{d}{dt}\right)^\top w = 0$ for all $w \in \mathscr{B}$, are well-defined, since we have given a meaning to ($\mathscr{G}$).

Call $n \in \mathbb{R}[\xi]^\bullet$ a *polynomial annihilator* of $\mathscr{B} \in \mathscr{L}^\bullet$ if $n\left(\frac{d}{dt}\right)^\top \mathscr{B} = 0$, and call $n \in \mathbb{R}(\xi)^\bullet$ a *rational annihilator* of $\mathscr{B} \in \mathscr{L}^\bullet$ if $n\left(\frac{d}{dt}\right)^\top \mathscr{B} = 0$.

Denote the set of polynomial and of rational annihilators of $\mathscr{B} \in \mathscr{L}^\bullet$ by $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ and $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$, respectively. It is well known that for $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$, $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ is an $\mathbb{R}[\xi]$-module, indeed, a finitely generated one, since all $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$ are finitely generated. However, $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is also an $\mathbb{R}[\xi]$-module, but a submodule of $\mathbb{R}(\xi)^{\mathtt{w}}$ viewed as an $\mathbb{R}[\xi]$-module (rather than as an $\mathbb{R}(\xi)$-vector space). The $\mathbb{R}[\xi]$-submodules of $\mathbb{R}(\xi)^{\mathtt{w}}$ are not necessarily finitely generated.

The question occurs when $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is a vector space. This question has a nice answer, given in the following theorem.

**Theorem 3** *Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$.*

1. *$\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is an $\mathbb{R}[\xi]$-submodule of $\mathbb{R}(\xi)^{\mathtt{w}}$.*
2. *$\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is an $\mathbb{R}(\xi)$-vector subspace of $\mathbb{R}(\xi)^{\mathtt{w}}$ if and only if $\mathscr{B}$ is controllable.*
3. *Denote the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$ by $\mathfrak{M}^{\mathtt{w}}$. There is a bijective correspondence between $\mathscr{L}^{\mathtt{w}}$ and $\mathfrak{M}_{\mathtt{w}}$, given by*

$$\mathscr{B} \in \mathscr{L}^{\mathtt{w}} \mapsto \mathscr{B}^{\perp_{\mathbb{R}[\xi]}} \in \mathfrak{M}^{\mathtt{w}},$$

$$\mathbb{M} \in \mathfrak{M}^{\mathtt{w}} \mapsto \{w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid n\left(\tfrac{d}{dt}\right)^\top w = 0 \ \forall n \in \mathbb{M}\}.$$

4. *Denote the linear $\mathbb{R}(\xi)$-subspaces of $\mathbb{R}(\xi)^{\mathtt{w}}$ by $\mathfrak{L}^{\mathtt{w}}$. There is a bijective correspondence between $\mathscr{L}^{\mathtt{w}}_{\text{controllable}}$, the controllable elements of $\mathscr{L}^{\mathtt{w}}$, and $\mathfrak{L}_{\mathtt{w}}$ given by*

$$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\text{controllable}} \mapsto \mathscr{B}^{\perp_{\mathbb{R}(\xi)}} \in \mathfrak{L}^{\mathtt{w}},$$

$$\mathbb{L} \in \mathfrak{L}^{\mathtt{w}} \mapsto \{w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid n\left(\tfrac{d}{dt}\right)^\top w = 0 \ \forall n \in \mathbb{L}\}.$$

■

This theorem shows a precise sense in which a linear time-invariant system can be identified by a module, and a controllable linear time-invariant differential system (an infinite dimensional subspace of $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ whenever $\mathscr{B} \neq \{0\}$) can be identified with a *finite-dimensional* vector space (of dimension $\mathtt{p}(\mathscr{B})$). Indeed, through the polynomial annihilators, $\mathscr{L}^{\mathtt{w}}$ is in one-to-one correspondence with the $\mathbb{R}[\xi]$-submodules of $\mathbb{R}[\xi]^{\mathtt{w}}$, and, through the rational annihilators, $\mathscr{L}^{\mathtt{w}}_{\text{controllable}}$ is in one-to-one correspondence with the $\mathbb{R}(\xi)$-subspaces of $\mathbb{R}(\xi)^{\mathtt{w}}$.

Consider the system $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ and its rational annihilators $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$. In general, this is an $\mathbb{R}[\xi]$-submodule, but not $\mathbb{R}(\xi)$-vector subspace of $\mathbb{R}(\xi)^{\mathtt{w}}$. Its polynomial elements, $\mathscr{B}^{\perp_{\mathbb{R}[\xi]}}$ always form an $\mathbb{R}[\xi]$-submodule over $\mathbb{R}[\xi]^{\mathtt{w}}$, and this module determines $\mathscr{B}$ uniquely. Therefore, $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ also determines $\mathscr{B}$ uniquely. Moreover, $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ forms an $\mathbb{R}(\xi)$-vector space if and only if $\mathscr{B}$ is controllable. More generally, the $\mathbb{R}(\xi)$-span of $\mathscr{B}^{\perp_{\mathbb{R}(\xi)}}$ is exactly $\mathscr{B}_{\text{controllable}}^{\perp_{\mathbb{R}(\xi)}}$. Therefore the $\mathbb{R}(\xi)$-span of the rational annihilators of two systems are the same if and only if they have the same controllable part. We state this formally.

**Theorem 4** *Let $\mathscr{B}_1$ be given by $G_1\left(\frac{d}{dt}\right)w = 0$ and $\mathscr{B}_2$ by $G_2\left(\frac{d}{dt}\right)w = 0$, with $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\mathtt{w}}$. The rows of $G_1$ and $G_2$ span the same $\mathbb{R}[\xi]$-submodule of $\mathbb{R}(\xi)^{\mathtt{w}}$ if and only if $\mathscr{B}_1 = \mathscr{B}_2$. The rows of $G_1$ and $G_2$ span the same $\mathbb{R}(\xi)$-vector subspace of $\mathbb{R}(\xi)^{\mathtt{w}}$ if and only if $\mathscr{B}_1$ and $\mathscr{B}_2$ have the same controllable part, that is, if and only if $\mathscr{B}_1 \sim_{\text{controllable}} \mathscr{B}_2$.* ∎

## 1.6 Left-prime representations

In order to express system properties and to parametrize the set of stabilizing controllers effectively, we need to consider representations with matrices of rational functions over certain special rings. We now introduce the relevant subrings of $\mathbb{R}(\xi)$.

1. $\mathbb{R}(\xi)$ itself, the rational functions,
2. $\mathbb{R}[\xi]$, the polynomials,
3. $\mathbb{R}(\xi)_{\mathscr{P}}$, the set elements of $\mathbb{R}(\xi)$ that are proper,
4. $\mathbb{R}(\xi)_{\mathscr{S}}$, the set elements of $\mathbb{R}(\xi)$ that are stable,
5. $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}} = \mathbb{R}(\xi)_{\mathscr{P}} \cap \mathbb{R}(\xi)_{\mathscr{S}}$, the proper stable rational functions.

We can think of these subrings in terms of poles. Indeed, these subrings are characterized by, respectively, arbitrary poles, no finite poles, no poles at $\{\infty\}$, no poles in $\overline{\mathbb{C}}_+$, and no poles in $\overline{\mathbb{C}}_+ \cup \{\infty\}$. It is easy to identify the unimodular elements (that is, the elements that have an inverse in the ring) of these rings. They consist of, respectively, the non-zero elements, the non-zero constants, the biproper elements, the miniphase elements, and the biproper miniphase elements of $\mathbb{R}(\xi)$.

We also consider matrices over these rings. Call an element of $\mathbb{R}(\xi)^{\bullet\times\bullet}$ *proper*, *stable*, or *proper stable* if each of its entries is. The square matrices over these rings are unimodular if and only if the determinant is unimodular. For $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\bullet\times\bullet}$, define $M^\infty := \lim_{x\in\mathbb{R}, x\to\infty} M(x)$. Call the matrix $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$ *biproper* if it has an inverse in $\mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}\times\mathtt{n}}$, that is, if $\mathtt{determinant}(M^\infty) \neq 0$, and call $M \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}\times\mathtt{n}}$ *miniphase* if it has an inverse in $\mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}\times\mathtt{n}}$, that is, if $\mathtt{determinant}(M^\infty) \neq 0$ is miniphase.

Let $\mathscr{R}$ denote any of the rings $\mathbb{R}(\xi)$, $\mathbb{R}[\xi]$, $\mathbb{R}(\xi)_{\mathscr{P}}$, $\mathbb{R}(\xi)_{\mathscr{S}}$, $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$. $M \in \mathscr{R}^{\mathtt{n}_1\times\mathtt{n}_2}$ is said to be *left prime* over $\mathscr{R}$ if for every factorization of $M$ the form $M = FM'$ with $F \in \mathscr{R}^{\mathtt{n}_1\times\mathtt{n}_1}$ and $M' \in \mathscr{R}^{\mathtt{n}_1\times\mathtt{n}_2}$, $F$ is unimodular over $\mathscr{R}$. It is easy to characterize the left-prime elements. $M \in \mathbb{R}(\xi)^{\mathtt{n}_1\times\mathtt{n}_2}$ is the prime over $\mathscr{R}$ if and only if

1. $M$ is of full row rank when $\mathscr{R} = \mathbb{R}(\xi)$,
2. $M \in \mathbb{R}[\xi]^{\mathtt{n}_1\times\mathtt{n}_2}$ and $M(\lambda)$ is of full row rank for all $\lambda \in \mathbb{C}$ when $\mathscr{R} = \mathbb{R}[\xi]$,

3. $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ and $M^\infty$ is of full row rank when $\mathscr{R} = \mathbb{R}(\xi)_{\mathscr{P}}$,
4. $M$ is of full row rank and has no poles and no zeros in $\overline{\mathbb{C}}_+$ when $\mathscr{R} = \mathbb{R}(\xi)_{\mathscr{S}}$,
5. $M \in \mathbb{R}(\xi)_{\mathscr{P}}^{\mathtt{n}_1 \times \mathtt{n}_2}$, $M^\infty$ is of full row rank, and $M$ has no poles and no zeros in $\overline{\mathbb{C}}_+$, when $\mathscr{R} = \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$.

Controllability and stabilizability can be linked to the existence of left-prime representations over these subrings of $\mathbb{R}(\xi)$.

1. $\mathscr{B} \in \mathscr{L}^\bullet$ admits a representation ($\mathscr{R}$) with $R$ of full row rank, and a representation ($\mathscr{G}$) with $G$ of full row rank and $G \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\bullet \times \bullet}$, that is, with all its elements proper and stable, meaning that they have no poles in $\overline{\mathbb{C}}_+$.
2. $\mathscr{B}$ admits a representation ($\mathscr{G}$) with $G$ left prime over $\mathbb{R}(\xi)$, that is, with $G$ of full row rank.
3. $\mathscr{B}$ is controllable if and only if it admits a representation ($\mathscr{G}$) with $G$ left prime over $\mathbb{R}(\xi)$, that is, $G$ has full row rank and has no zeros.
4. $\mathscr{B}$ is controllable if and only if it admits a representation ($\mathscr{R}$) with $R \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ left prime over $\mathbb{R}[\xi]$, that is, with $R(\lambda)$ of full row rank for all $\lambda \in \mathbb{C}$.
5. $\mathscr{B}$ is controllable if and only if it admits a representation ($\mathscr{G}$) that is left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$, that is, all elements of $G$ are proper and $G^\infty$ of full row rank, and $G$ has no zeros.
6. $\mathscr{B}$ admits a representation ($\mathscr{G}$) with $G$ left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$, that is, all elements of $G$ are proper and $G^\infty$ has full row rank.
7. $\mathscr{B}$ is stabilizable if and only if it admits a representation ($\mathscr{G}$) with $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\bullet \times \bullet}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$, that is, $G$ has full row rank and no poles and no zeros in $\overline{\mathbb{C}}_+$.
8. $\mathscr{B}$ is stabilizable if and only if it admits a representation ($\mathscr{G}$) with $G \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\bullet \times \bullet}$ left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$, that is, $G^\infty$ has full row rank and $G$ has no poles and no zeros in $\overline{\mathbb{C}}_+$.

These results illustrate how system properties can be translated into properties of rational symbols. Roughly speaking, every $\mathscr{B} \in \mathscr{L}^\bullet$ has a full row rank polynomial and a full row rank proper and/or stable representation. As long as we allow a non-empty region where to put the poles, we can obtain a representation with a rational symbol with poles confined to that region. The zeros of the representation are more significant. No zeros correspond to controllability. No unstable zeros correspond to stabilizability. In [6] an elementary proof is given that does not involve complicated algebraic arguments of the characterization of stabilizability in terms of a representation that is left-prime over the ring of proper stable rational functions. Analogous results can also be obtained for image representations.

Note that a left-prime representation over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$ exists if and only if the behavior is stabilizable. This result can be compared with the classical result obtained by Vidyasagar in his book [4], where the aim is to obtain a proper stable left-prime representation of a system that is given as a transfer function, $y = F\left(\frac{d}{dt}\right)u$, where $F \in \mathbb{R}(\xi)^{\mathtt{p} \times \mathtt{m}}$. This system is a special case of ($\mathscr{G}$) with $G = \begin{bmatrix} I_{\mathtt{p}} & -F \end{bmatrix}$, and, since it has no zeros, $y = F\left(\frac{d}{dt}\right)u$ is controllable, and hence stabilizable. Therefore, a system defined by a transfer function admits a representation $G_1\left(\frac{d}{dt}\right)y = G_2\left(\frac{d}{dt}\right)u$

with $G_1, G_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}_{\mathscr{P}\mathscr{S}}$, and $\begin{bmatrix} G_1 & G_2 \end{bmatrix}$ left coprime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$. This is an important, classical, result. However, in the controllable case, we can obtain a representation that is left prime over $\mathbb{R}(\xi)_{\mathscr{P}}$, and such that $\begin{bmatrix} G_1 & G_2 \end{bmatrix}$ has no zeros at all. The main difference of our result from the classical left-coprime factorization results over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$ is that we faithfully preserve the exact behavior and not only the controllable part of a behavior, whereas in the classical approach all stabilizable systems with the same transfer function are identified. We thus observe that the behavioral viewpoint provides a more intrinsic approach for discussing pole-zero cancellation. Indeed, since the transfer function is a rational function, poles and zeros can — by definition — be added and cancelled *ad libitum*. Transfer functions do not provide the correct framework in which to discuss pole-zero cancellations. Behaviors defined by rational functions do.

## 1.7 Control

We refer to [5, 1] for an extensive treatment of control in a behavioral setting. In terms of the notions introduced in these references, we shall be concerned with full interconnection only, meaning that the controller has access to all the system variables. We refer to [1] for a nice discussion of the concepts involved.

In the behavioral approach, control is viewed as the interconnection of a plant and a controller. Let $\mathscr{P}$ (henceforth $\in \mathscr{L}^{\mathtt{w}}$) be called the *plant*, $\mathscr{C}$ (henceforth $\in \mathscr{L}^{\mathtt{w}}$) the *controller*, and their interconnection $\mathscr{P} \cap \mathscr{C}$ (hence also $\in \mathscr{L}^{\mathtt{w}}$), the *controlled system*. This signifies that in the controlled system, the trajectory $w$ has to obey both the laws of $\mathscr{P}$ and $\mathscr{C}$, which leads to the point of view that control means restricting the plant behavior to a subset, the intersection of the plant and the controller.

The controller $\mathscr{C}$ is said to be a *regular controller* for $\mathscr{P}$ if

$$\mathsf{p}(\mathscr{P} \cap \mathscr{C}) = \mathsf{p}(\mathscr{P}) + \mathsf{p}(\mathscr{C}).$$

and *superregular* if, in addition,

$$\mathsf{n}(\mathscr{P} \cap \mathscr{C}) = \mathsf{n}(\mathscr{P}) + \mathsf{n}(\mathscr{C}).$$

The origin and the significance of these concepts is discussed in, for example, [1, section VII]. The classical input/state/output based sensor-output-to-actuator-input controllers that dominate the field of control are superregular. Controllers that are regular, but not superregular, are relevant in control, much more so than is appreciated, for example as PID controllers, or as control devices that do not act as sensor-output-to-actuator-input feedback controllers.

Superregularity means that the interconnection of the plant with the controller can take place at any moment in time. The controller $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is superregular for $\mathscr{P} \in \mathscr{L}^{\mathtt{w}}$ if and only if for all $w_1 \in \mathscr{P}$ and $w_2 \in \mathscr{C}$, there exists a $w \in (\mathscr{P} \cap \mathscr{C})^{\text{closure}}$ such that $w_1'$ and $w_2'$ defined by

$$w_1'(t) = \begin{cases} w_1(t) & \text{for } t \leq 0 \\ w(t) & \text{for } t > 0 \end{cases},$$

and

$$w_2'(t) = \begin{cases} w_2(t) & \text{for } t \leq 0, \\ w(t) & \text{for } t > 0 \end{cases}$$

belongs to $\mathscr{P}$ and $\mathscr{C}$, respectively. Hence, for a superregular interconnection, any distinct past histories in $\mathscr{P}$ and $\mathscr{C}$ can be continued as one and the same future trajectory in $\mathscr{P} \cap \mathscr{C}$. In [5] it has been shown that superregularity can also be viewed as feedback.

The controller $\mathscr{C}$ is said to be *stabilizing* if $\mathscr{P} \cap \mathscr{C}$ is stable, that is, if $w \in \mathscr{P} \cap \mathscr{C}$ implies $w(t) \to 0$ as $t \to \infty$. Note that we consider stability as a property of an autonomous behavior (a behavior $\mathscr{B}$ with $\mathtt{m}(\mathscr{B}) = 0$). In the input/output setting, as in [4], the interconnection of $\mathscr{P}$ and $\mathscr{C}$ is defined to be stable if the system obtained by injecting artificial arbitrary inputs at the interconnection terminals is bounded-input/bounded-output stable. Our stability definition requires that $w(t) \to 0$ for $t \to \infty$ in $\mathscr{P} \cap \mathscr{C}$. It turns out that bounded-input/bounded-output stability requires (i) our stability, combined with (ii) superregularity. Interconnections that are not superregular cannot be bounded-input/bounded-output stable. However, for physical systems these concepts (stability and superregularity) are quite unrelated. For example, the harmonic oscillator $M\frac{d^2}{dt^2}w_1 + Kw_1 = w_2$, with $M, K > 0$, is stabilized by the damper $w_2 = -D\frac{d}{dt}w_1$ if $D > 0$. In our opinion, it makes little sense to call this interconnection unstable, just because the interconnection is not superregular.

Regularity and superregularity can be expressed in terms of left-prime kernel representations with rational symbols.

**Proposition 5** *Consider the plant $\mathscr{P} \in \mathscr{L}^{\mathtt{w}}$. Assume that $\mathscr{P}$ is stabilizable. Let $\mathscr{P}$ be described by $P\left(\frac{d}{dt}\right)w = 0$ with $P \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. By stabilizability of $\mathscr{P}$ such a representation exists.*

1. *$\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is a regular stabilizing controller if and only if $\mathscr{C}$ admits a representation $C\left(\frac{d}{dt}\right)w = 0$ with $C \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$, and such that*

$$G = \begin{bmatrix} P \\ C \end{bmatrix}$$

*is square and $\mathbb{R}(\xi)_{\mathscr{S}}$-unimodular, that is, with $\mathtt{determinant}(G)$ miniphase.*

2. *$\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ is a superregular stabilizing controller if and only if $\mathscr{C}$ admits a representation $C\left(\frac{d}{dt}\right)w = 0$ with $C \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$ left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$, and such that*

$$G = \begin{bmatrix} P \\ C \end{bmatrix}$$

*is square and $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$-unimodular, that is, with $\mathtt{determinant}(G)$ biproper and miniphase.*

■

The equivalence of the following statements can be shown:

$$[\![\mathscr{P} \text{ is stabilizable}]\!] \Leftrightarrow [\![\exists \text{ a regular controller } \mathscr{C} \text{ that stabilizes } \mathscr{P}]\!]$$
$$\Leftrightarrow [\![\exists \text{ a superregular controller } \mathscr{C} \text{ that stabilizes } \mathscr{P}]\!].$$

Combining this with the previous theorem leads to the following result on matrices of rational functions.

**Corollary 6** *1. Assume that* $G \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ *is left prime over* $\mathbb{R}(\xi)_{\mathscr{S}}$*. Then there exists* $F \in \mathbb{R}(\xi)_{\mathscr{S}}^{(\mathtt{n}_2-\mathtt{n}_1)\times \mathtt{n}_2}$ *such that*

$$\begin{bmatrix} G \\ F \end{bmatrix}$$

*is* $\mathbb{R}(\xi)_{\mathscr{S}}$*-unimodular.*

2. *Assume that* $G \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{n}_1 \times \mathtt{n}_2}$ *is left prime over* $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$*. Then there exists* $F \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{(\mathtt{n}_2-\mathtt{n}_1)\times \mathtt{n}_2}$ *such that*

$$\begin{bmatrix} G \\ F \end{bmatrix} \text{ is}$$

$\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$*-unimodular.*

## 1.8 Parametrization of the set of regular stabilizing, superregular stabilizing, and dead-beat controllers

In this section, we parametrize the set of regular and superregular controllers that stabilize a given stabilizable plant $\mathscr{P} \in \mathscr{L}^\bullet$.

### 1.8.1 Regular stabilizing controllers

*Step 1.* The parametrization starts from a kernel representation $P\left(\frac{d}{dt}\right) w = 0$ of $\mathscr{P}$, with $P \in \mathbb{R}(\xi)^{\mathsf{p}(\mathscr{P})\times \mathsf{w}(\mathscr{P})}$ left prime over $\mathbb{R}(\xi)_{\mathscr{S}}$. By stabilizability of $\mathscr{P}$, such a representation exists.

*Step 2.* Construct a $P' \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathsf{m}(\mathscr{P})\times \mathsf{w}(\mathscr{P})}$ such that

$$\begin{bmatrix} P \\ P' \end{bmatrix}$$

is $\mathbb{R}(\xi)_{\mathscr{S}}$-unimodular. By corollary 6, such a $P'$ exists.

*Step 3.* The set of regular stabilizing controllers $\mathscr{C} \in \mathscr{L}^{\mathsf{w}(\mathscr{P})}$ is given as the systems with kernel representation $C(\frac{d}{dt})w = 0$, where

$$C = F_1 P + F_2 P',$$

with $F_1 \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathsf{m}(\mathscr{P})\times \mathsf{p}(\mathscr{P})}$ is free and $F_2 \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathsf{m}(\mathscr{P})\times \mathsf{m}(\mathscr{P})}$ is $\mathbb{R}(\xi)_{\mathscr{S}}$-unimodular, that is, with $\mathtt{determinant}(F_2)$ miniphase.

*Step 3'*. This parametrization may be further simplified using controllability equivalence, by identifying controllers that have the same controllable part, that is, by considering controllers up to controllability equivalence. The set of controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}(\mathscr{P})}$ with kernel representation $C(\frac{d}{dt})w = 0$ and $C$ of the form

$$C = FG + G',$$

with $F \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{m}(\mathscr{P}) \times \mathtt{p}(\mathscr{P})}$ free, consists of regular stabilizing controllers, and contains an element of the equivalence class modulo controllability of each regular stabilizing controller for $\mathscr{P}$.

### 1.8.2 Superregular stabilizing controllers

*Step 1*. The parametrization starts from a kernel representation $P\left(\frac{d}{dt}\right)w = 0$ of $\mathscr{P}$, with $P \in \mathbb{R}(\xi)^{\mathtt{p}(\mathscr{P}) \times \mathtt{w}(\mathscr{P})}$ left prime over $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$. By stabilizability of $\mathscr{P}$, such a representation exists.

*Step 2*. Construct a $P' \in \mathbb{R}(\xi)_{\mathscr{S}}^{\mathtt{m}(\mathscr{P}) \times \mathtt{w}(\mathscr{P})}$ such that

$$\begin{bmatrix} P \\ P' \end{bmatrix}$$

is $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$-unimodular. By corollary 6, such a $P'$ exists.

*Step 3*. The set of superregular stabilizing controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}(\mathscr{P})}$ is given as the systems with kernel representation $C(\frac{d}{dt})w = 0$, where

$$C = F_1 P + F_2 P',$$

with $F_1 \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{P}) \times \mathtt{p}(\mathscr{P})}$ free and $F_2 \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{P}) \times \mathtt{m}(\mathscr{P})}$ $\mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}$-unimodular, that is, with $\mathtt{determinant}(F_2)$ biproper and miniphase.

*Step 3'*. This parametrization may be further simplified using controllability equivalence, by identifying controllers that have the same controllable part, that is, by considering controllers up to controllability equivalence. The set of controllers $\mathscr{C} \in \mathscr{L}^{\mathtt{w}(\mathscr{P})}$ with kernel representation $C(\frac{d}{dt})w = 0$ and $C$ of the form

$$C = FG + G,'$$

with $F \in \mathbb{R}(\xi)_{\mathscr{P}\mathscr{S}}^{\mathtt{m}(\mathscr{P}) \times \mathtt{p}(\mathscr{P})}$ free, consists of superregular stabilizing controllers, and contains an element of the equivalence class modulo controllability of each superregular stabilizing controller for $\mathscr{P}$.

It is of interest to compare these parametrizations with the one obtained in [3]. We now show a very simple example to illustrate the difference between the parametrizations obtained in step 3 and step 3'.

*Example:* Consider the plant $y = 0u$, hence $P = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and the superregular stabilizing controller $u + \alpha \frac{d}{dt} u = 0$, with $\alpha \geq 0$. Take $P' = \begin{bmatrix} 0 & 1 \end{bmatrix}$ in the parametrizations. The set of (super)regular stabilizing controllers is given by $C\left(\frac{d}{dt}\right) u = 0$, with

$C \in \mathbb{R}(\xi)$ miniphase in the regular case, and miniphase and biproper in the super-regular case. Taking $F_2(\xi) = (1+\alpha\xi)/(1+2\alpha\xi)$, for example, yields the controller $u + \alpha\frac{d}{dt}u = 0$, with $\alpha \geq 0$. The parametrization in step 3' yields only the controller $u = 0$, which is indeed the controllable part of $u + \alpha\frac{d}{dt}u = 0$.

This example illustrates that the parametrization in step 3' does not yield all the (super)regular stabilizing controllers, although it yields all the stabilizing controller transfer functions. Note that the parametrization of step 3 does exclude the destabilizing controller $u + \alpha\frac{d}{dt}u = 0$, with $\alpha < 0$.

The trajectory-based parametrization is not only more general, but it also give sharper results. It yields all stabilizing controllers, without having to resort to equivalence modulo controllability.

## 1.9 Acknowledgments

The SISTA Research program is supported by the Research Council KUL: GOA AMBioRICS, CoE EF/05/006 Optimization in Engineering, several PhD/postdoc & fellow grants; by the Flemish Government: FWO: PhD/postdoc grants, projects, G.0407.02 (support vector machines), G.0197.02 (power islands), G.0141.03 (Identification and cryptography), G.0491.03 (control for intensive care glycemia), G. 0120.03 (QIT), G.0452.04 (new quantum algorithms), G.0499.04 (Statistics), G.0211.05 (Nonlinear), G.0226.06 (cooperative systems and optimization), G.0321.06 (Tensors), G.0302.07 (SVM/Kernel, research communities (ICCoS, ANMMM, MLDM); by IWT: PhD Grants, McKnow-E, Eureka-Flite2; and by the Belgian Federal Science Policy Office: IUAP P6/04 (Dynamical systems, Control and Optimization, 2007-2011).

This research is also supported by the Japanese Government under the 21st Century COE (Center of Excellence) program for research and education on complex functional mechanical systems, and by the JSPS Grant-in-Aid for Scientific Research (B) No. 18360203, and also by Grand-in-Aid for Exploratory Research No. 17656138.

## References

1. M. Belur and H.L. Trentelman, Stabilization, pole placement, and regular implementability, *IEEE Transactions on Automatic Control*, volume 47, pages 735–744, 2002.
2. V. Kučera, Stability of discrete linear feedback systems, paper 44.1, *Proceedings of the 6-th IFAC Congress*, Boston, Massachusetts, USA, 1975.
3. M. Kuijper, Why do stabilizing controllers stabilize?, *Automatica*, volume 34, pages 621–625, 1995.
4. M. Vidyasagar, *Control System Synthesis*, The MIT Press, 1985.
5. J.C. Willems, On interconnections, control and feedback, *IEEE Transactions on Automatic Control*, volume 42, pages 326–339, 1997.
6. J.C. Willems and Y.Yamamoto, Behaviors defined by rational functions, *Linear Algebra and Its Applications*, volume 425, pages 226-241, 2007.
7. D.C. Youla, J.J. Bongiorno, and H.A. Jabr, Modern Wiener-Hopf design of optimal controllers, Part I: The single-input case, Part II: The multivariable case, *IEEE Transactions on Automatic Control*, volume 21, pages 3–14 and 319–338,1976.