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DISSIPATIVE DISTRIBUTED SYSTEMS

Jan C. Willems*

Abstract

A controllable distributed dynamical system described by a system of linear constant-coefficient partial differential equations is said to be conservative if for compact support trajectories the integral of the supply rate is zero. It is said to be dissipative if this integral is non-negative. The question that we consider is whether these global versions of conservation and dissipativeness are equivalent to local versions, involving a storage function and a dissipation rate. It is shown that this is indeed the case, provided we consider latent variable representations.

Keywords: Distributed systems, partial differential equations, behaviors, supply, storage, flux, dissipation, polynomial factorization.

1 Introduction

It is a pleasure to contribute an article to this Festschrift dedicated to Sanjoy Mitter on the occasion of his 65-th birthday, as an expression of thanks for his warm friendship and collegiality. His advice and opinion always meant a great deal to me, as a source of sound judgment and perspective in a field that is continuously torn by forces coming from mathematical virtuosity and technological euphoria. I first interacted with Sanjoy during his sabbatical leave at MIT in 1969-70. At that occasion he gave a course on distributed parameter systems. The breadth in scope and the mathematical sophistication of these lectures left a lasting impression. My article also deals with distributed systems, as a tribute to Sanjoy's scientific influence on me.

This paper reports on research done in collaboration with dr. Harish Pillai. It is preliminary in nature. It builds on his earlier results that appeared in [6]. An

*Mathematics Institute, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. e-mail: J.C.Willems@math.rug.nl.

extensive version containing proofs of the results reported here will be submitted elsewhere.

A word about notation. The dimension of a vector space is usually denoted by the same symbol as a generic element of that vector space, say $w \in \mathbb{R}^{\mathtt{w}}$, and analogously for function spaces. If the dimension is unimportant (but, of course, finite), we denote it as $\bullet$. We often assume without explicit mention that in vector and matrix multiplication, the vectors and matrices have suitable dimensions.

A *1-D dynamical system* Σ is a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} \subset \mathbb{R}$ the time-set, $\mathbb{W}$ the signal space, and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ the *behavior.* The intuition behind this definition is that $\mathbb{T}$ is the set of relevant time-instances; $\mathbb{W}$ is the set in which the signals, whose dynamic relation Σ models, take on their values; the behavior $\mathfrak{B}$ specifies which signals $w : \mathbb{T} \to \mathbb{W}$ obey the laws of the system. The time-set $\mathbb{T}$ equals for example $\mathbb{R}$ or $\mathbb{R}_+$ in continuous-time, and $\mathbb{Z}$ or $\mathbb{Z}_+$ in discrete-time systems. There is much interest in generalization from a time-set that is a subset of $\mathbb{R}$ to domains with more independent variables (e.g., time and space). These 'dynamical' systems have $\mathbb{T} \subset \mathbb{R}^n$, and are referred to as *n-D* systems. This paper deals with such systems, more specifically with systems described by linear constant-coefficient partial differential equations.

Define a *distributed differential system* as an *n-D* system $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, with behavior $\mathfrak{B}$ consisting of the solution set of a system of partial differential equations

$$R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = 0 \tag{1}$$

viewed as an equation in the functions

$$(x_1, \ldots, x_n) = x \in \mathbb{R}^n \mapsto (w_1(x), \ldots, w_{\mathtt{w}}(x)) = w(x) \in \mathbb{R}^{\mathtt{w}}.$$

Here, $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_n]$ is a matrix of polynomials in $\mathbb{R}[\xi_1, \ldots, \xi_n]$. The behavior of this system of partial differential equations is defined as

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid (1) \text{ is satisfied}\}.$$

Important properties of these systems are their *linearity* (meaning that $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{R}^n}$), and *shift-invariance* (meaning $\sigma^x \mathfrak{B} = \mathfrak{B}$ for all $x \in \mathbb{R}^n$, where σ^x denotes the $x-$shift, defined by $(\sigma^x f)(x') = f(x' + x)$). The $\mathfrak{C}^\infty$-assumption in the definition of $\mathfrak{B}$ is made for convenience only, and there is much to be said for using distributions instead. We denote the behavior of (1) as defined above by $\ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}))$, and the set of distributed differential systems thus obtained by $\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. Note that we may as well write $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, instead of $\Sigma \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$, since the set of independent variables ($\mathbb{R}^n$) and the signal space ($\mathbb{R}^{\mathtt{w}}$) are evident from this notation. Whence $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ means that there exists a matrix of polynomials $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_n]$ such that $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}))$. We call (1) a *kernel representation* of $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})))$. We will meet other representations later.

A typical example of a distributed dynamical system is given by Maxwell's equations, which describe the possible realizations of the fields $\vec{E} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \vec{B} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3, \vec{j} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, and $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$. Maxwell's equations

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho, \\ \nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\ \nabla \cdot \vec{B} &= 0, \\ c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}, \end{aligned}$$

with ε_0 the dielectric constant of the medium and c^2 the speed of light in the medium, define a distributed differential system

$$\Sigma = (\mathbb{R}^4, \mathbb{R}^{10}, \ker(R(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}))) \in \mathfrak{L}^{10}_4,$$

with the matrix of polynomials $R \in \mathbb{R}^{8 \times 10}[\xi_1, \xi_2, \xi_3, \xi_4]$ easily deduced from the above equations. This defines the system $(\mathbb{R} \times \mathbb{R}^3, \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, \mathfrak{B})$, with $\mathfrak{B}$ the set of all $(\vec{E}, \vec{B}, \vec{j}, \rho)$ that satisfy Maxwell's equations.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. Hence, by definition, there exists a matrix of polynomials $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_n]$ such that $\mathfrak{B} = \ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}))$. However, while R thus defines $\mathfrak{B}$ uniquely, the converse is not true, because, for example,

$$\ker(R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})) = \ker((UR)(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}))$$

for any suitably sized matrix U of polynomials in $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_n]$ that is unimodular (*unimodular* means that $\det(U)$ is a non-zero element of $\mathbb{R}[\xi_1, \ldots, \xi_n]$ of degree zero, whence that U has an inverse that is also a matrix of polynomials). This situation calls for a more intrinsic definition of $\mathfrak{B}$. This can be done by considering the annihilators.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. A vector of polynomials $n \in \mathbb{R}^{\mathtt{w}}[\xi_1, \ldots, \xi_n]$ is said to be an *annihilator* for $\mathfrak{B}$ if $n^T(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\mathfrak{B} = 0$. Denote by $\mathcal{N}_{\mathfrak{B}} \subset \mathbb{R}^{\mathtt{w}}[\xi_1, \ldots, \xi_n]$ the set of annihilators of $\mathfrak{B}$. It is easy to see that $\mathcal{N}_{\mathfrak{B}}$ defines a submodule of $\mathbb{R}^{\mathtt{w}}[\xi_1, \ldots, \xi_n]$ viewed as a module over the ring $\mathbb{R}[\xi_1, \ldots, \xi_n]$. Consequently, since every submodule of $\mathbb{R}^{\mathtt{w}}[\xi_1, \ldots, \xi_n]$ is finitely generated, there is a one-to-one relationship between $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ and the set of submodules of $\mathbb{R}^{\mathtt{w}}[\xi_1, \ldots, \xi_n]$. This situation has led some authors to define dynamical systems as submodules, a practice that we find a bit unfortunate, the definition in which a system is viewed as a subset of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{R}^n}$ being a much more intrinsic one. The issue of how to define a behavior $\mathfrak{B}$ and its annihilators, such that, in situations other than in the linear shift-invariant case, there is a one-to-one relation between a system and its annihilators, has been pursued in [4, 3].

Consider a system $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, and let $\{1', \ldots, \mathtt{w}'\}$ be a subset of $\{1, \ldots, \mathtt{w}\}$. We call the variables $(w_{1'}, \ldots, w_{\mathtt{w}'})$ *free* in $\mathfrak{B}$ if for any $w' \in \mathfrak{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}'})$ there exists a $w \in \mathfrak{B}$ which has w' as its $(1', \ldots, \mathtt{w}')$-th components. Denote by $\mathtt{m}(\mathfrak{B})$ the maximal number of free variables that may be obtained this way by selecting the subset $\{1', \ldots, \mathtt{w}'\}$; $\mathtt{m}(\mathfrak{B})$ is called the *input cardinality* of $\mathfrak{B}$. In the *1-D* case, it can be shown that $\mathfrak{B}$ admits a kernel representation $\ker(R(\frac{d}{dt}))$ such that $\mathtt{m}(\mathfrak{B}) = \mathtt{w} - \text{rowdim}(R)$. However, in the *1-D* case, this may not be possible. For example for the behavior described by Maxwell's equations, $\mathtt{m}(\mathfrak{B}) = 3$ and $\mathtt{w} = 10$, but the minimal rowdim(R) equals 8, as is the case in Maxwell's equations.

More background on behavioral systems can be found in [10, 11, 7]. A very nice recent paper that treats systems defined by (1) from a behavioral perspective is [6]. Other authors who have discussed systems from related perspectives are Fliess [1, 2], Oberst [4, 3], and Pommaret [8, 9], and their co-workers.

2 Elimination

Mathematical models of complex systems are usually obtained by viewing the system (often in a hierarchical fashion) as an interconnection of subsystems. This principle of *tearing* and *zooming*, combined with *modularity*, lies at the basis of what is called *object-oriented* modeling, a very effective computer-assisted way of model building used in many engineering domains. An important aspect of these object-oriented modeling procedures is that they lead to models that relate the variables whose dynamic relation one wants to model (we call these *manifest* variables) to auxiliary variables (we call these *latent* variables) that have been introduced in the modeling process, for example as variables that specify the interconnection constraints. For distributed differential systems this leads to equations of the form

$$R(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell, \tag{2}$$

with R and M matrices of polynomials in $\mathbb{R}[\xi_1, \ldots, \xi_n]$. This equation relates the (vector of) manifest variables w to the (vector of) latent variables ℓ. Define the *full behavior* of this system as

$$\mathfrak{B}_{\text{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\dim(\ell)}) \mid (2) \text{ holds}\}$$

and the *manifest behavior* as

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}) \mid \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\dim(\ell)}) \text{ such that (2) holds}\}$$

We call (2) a *latent variable* representation of $\mathfrak{B}$. The question occurs whether $\mathfrak{B}$ is in $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$. This is the case indeed.

Theorem 1 (Elimination theorem) : *For any real matrices of polynomials* (R, M) *in* $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_n]$ *with* $\text{rowdim}(R) = \text{rowdim}(M)$, *there exists a matrix of polynomials* R' *in* $\mathbb{R}[\xi_1, \xi_2, \ldots, \xi_n]$ *such that the manifest behavior of (2) has kernel representation* $R'(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w = 0$.

The theoretical basis that underlies this elimination theorem is the *fundamental principle*. This gives necessary and sufficient conditions for solvability for the unknown $x \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^\bullet)$ in the equation $F(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})x = y$, with $F \in \mathbb{R}^{\bullet\times\bullet}[\xi_1, \ldots, \xi_n]$ and $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^\bullet)$ given. Define the *annihilators* of F as $\mathcal{K}_F := \{n \in \mathbb{R}^{\text{rowdim}(F)}[\xi_1, \ldots, \xi_n] \mid n^T F = 0\}$. The fundamental principle states that $F(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})x = y$ is solvable if and only if $n^T(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})y = 0$ for all $n \in \mathcal{K}_F$. Obviously, $\mathcal{K}_F$ is a submodule of $\mathbb{R}^{\text{rowdim}(F)}[\xi_1, \ldots, \xi_n]$, and hence there exists a matrix of polynomials $N \in \mathbb{R}^{\text{rowdim}(F)\times\bullet}[\xi_1, \ldots, \xi_n]$ such that $n \in \mathcal{K}_F$ if and only if $n = Nd$ for some $d \in \mathbb{R}^{\text{coldim}(N)}[\xi_1, \ldots, \xi_n]$. Whence, $n^T(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})y = 0$ for all $n \in \mathcal{K}_F$ if and only if $N^T(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})y = 0$. This immediately yields the elimination theorem. Indeed, let N be such that its columns are a set of generators of $\mathcal{K}_M$, and define $R' = N^T R$.

The above theorem implies that a distributed differential system $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ admits not only many kernel, but also many latent variable representations. Latent variable representations are very useful. Not only because first principles models usually come in this form, but also because latent variables routinely enter in representation questions. As we shall see in this paper, they allow to express conservation and dissipation laws in terms of local storage functions and dissipation rates.

As an illustration of the elimination theorem, consider the elimination of $\vec{B}$ and ρ from Maxwell's equations. The following equations describe the possible realizations of the fields $\vec{E}$ and $\vec{j}$:

$$\varepsilon_0 \frac{\partial}{\partial t} \nabla\cdot\vec{E} + \nabla\cdot\vec{j} = 0,$$

$$\varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{E} + \varepsilon_0 c^2 \nabla\times\nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{j} = 0.$$

3 Controllability and observability

An important property in the analysis and synthesis of dynamical systems is controllability. Controllability refers to be ability of transferring a system from one mode of operation to another. It has been discussed for many classes of systems. We now explain the generalization to linear constant-coefficient partial differential equations. A system $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ and for all bounded open subsets O_1, O_2 of $\mathbb{R}^n$ with disjoint closure, there exists $w \in \mathfrak{B}$ such that $w|_{O_1} = w_1|_{O_1}$ and $w|_{O_2} = w_2|_{O_2}$. We denote the set of controllable elements of $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$ by $\mathfrak{L}_{\mathtt{n,cont}}^{\mathtt{w}}$.

Note that it follows from the elimination theorem that the manifest behavior of a system in image representation, i.e., a latent variable system of the special form

$$w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell \tag{3}$$

belongs to $\mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$. Whence, for any matrix of polynomials $M \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi_1, \ldots, \xi_n]$, $(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}}, \mathrm{im}(M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}))) \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$, with $M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ viewed as an operator from $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\dim(\ell)})$ to $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. In other words, every image of a constant coefficient linear partial differential operator is the kernel of a constant coefficient linear partial differential operator. However, not every kernel of a constant coefficient linear partial differential operator is the image of a constant coefficient linear partial differential operator. The following theorem, obtained in [6], shows that it are precisely the controllable systems that admit an image representation.

Theorem 2 (Controllability) : *The following statements are equivalent for* $\mathfrak{B} \in \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}}$*:*

1. $\mathfrak{B}$ *defines a controllable system,*
2. $\mathfrak{B}$ *admits an image representation,*
3. *The trajectories of compact support are dense in* $\mathfrak{B}$*.*

It is a simple consequence of this theorem that a scalar partial differential equation in one function only (i.e., with $\mathrm{rowdim}(R) = \mathrm{coldim}(R) = 1$) with $R \neq 0$ cannot be controllable. It can be shown, on the other hand, that Maxwell's equations define a controllable distributed differential system.

Note that an image representation corresponds to what in mathematical physics is called a *potential function* with ℓ the potential and $M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ the partial differential operator that generates elements of the behavior from the potential.

An interesting aspect of the above theorem therefore is the fact that it identifies the existence of a potential function with the system theoretic property of controllability and concatenability of trajectories in the behavior. In the case of Maxwell's equations, an image representation is given by

$$
\begin{aligned}
\vec{E} &= -\frac{\partial}{\partial t}\vec{A} - \nabla\phi, \\
\vec{B} &= \nabla \times \vec{A}, \\
\vec{j} &= \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} - \varepsilon_0 c^2 \nabla^2 \vec{A}, \\
\rho &= \frac{\varepsilon_0}{c^2}\frac{\partial^2}{\partial t^2}\phi - \varepsilon_0 \nabla^2 \phi,
\end{aligned}
$$

where $\phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ is a scalar, and $\vec{A} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ a vector potential. Note that Maxwell's equations consist of 8 equations in 10 variables. As already mentioned, the number of free variables is 3. In the above image representation there are 4 free latent variables. This can actually be reduced to 3, say by putting one component of $\vec{A}$ to zero. A more elegant way of reducing the freedom in the latent variables is by imposing a *gauge*, for example, restricting $\vec{A}$ and ϕ to satisfy $c^2\nabla \cdot \vec{A} + \frac{\partial}{\partial t}\phi = 0$. Imposing this gauge retains the symmetry, but the resulting set of equations yields a latent variable representation of the behavior, not an image representation. The representation of systems in image representation is being pursued very vigorously, using the terminology *flatness*, by Fliess [2] and co-workers.

The notion of observability was introduced hand in hand with controllability. In the context of the input/state/output systems, it refers to the possibility of deducing, using the laws of the system, the state from observation of the input and the output. The definition that is used in the behavioral context is more universal in that the variables that are observed and the variables that need to be deduced are kept general. In the context of distributed differential systems the notion of observability is as follows.

Let $\Sigma = (\mathbb{R}^n, \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2}, \mathfrak{B}) \in \mathfrak{L}_n^{\mathtt{w}_1+\mathtt{w}_2}$. We call w_2 *observable* from w_1 in $\mathfrak{B}$ if $(w_1, w_2'),\ (w_1, w_2'') \in \mathfrak{B}$ implies $w_2' = w_2''$.

The theory of observability runs parallel to that of controllability. We mention only the result that for distributed differential systems, w_1 is observable from w_2 if and only if there exists a set of annihilators of the behavior of the following form that puts observability into evidence: $w_1 = R_2'(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})w_2$, with $R_2' \in \mathbb{R}^{\dim(\mathtt{w}_1)\times\dim(\mathtt{w}_2)}[\xi_1, \ldots, \xi_n]$. In this paper, we will only pursue observability for latent variable systems (2). We call this latent variable representation of the manifest behavior *observable* if ℓ is observable from w in its full behavior. We call it *weakly observable*, if to every $w \in \mathfrak{B}$ of compact support, there corresponds a unique ℓ

that is also of compact support.

For *1-D* systems it is easy to show that every controllable $\mathfrak{B} \in \mathfrak{L}_1^w$ admits an observable image representation. This, however, does not hold for *n-D* systems, and hence the representation of controllable systems in image representation (i.e., with potential functions) may require the introduction of latent variables that are 'hidden', in the sense that $M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell = 0$ has solutions $\ell \neq 0$. This means that however one represents $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\text{cont}}$ as $w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell$, there may not exist an $N \in \mathbb{R}^{\mathtt{w}\times\bullet}[\xi_1, \dots, \xi_n]$ such that $w = M(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})\ell$ implies $\ell = N(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})w$. The latent variables do not be recoverable from the manifest ones by a 'local' differential operator. However, it is always possible to represent $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\text{cont}}$ by an image representation that has $\dim(\ell) = \mathtt{m}(\mathfrak{B})$. These image representations are weakly observable.

For example, the image representation of the behavior defined by Maxwell's equations in terms of the vector potential $\vec{A}$ and the scalar potential ϕ, is not observable (neither is the latent variable representation obtained after imposing the gauge, but then the resulting latent variable representation is weakly observable). In fact, Maxwell's equations are an example of a controllable system that does not allow an observable image representation.

4 An example: heat diffusion

Although in this paper we are mainly interested in the construction of storage functions for conservative and dissipative distributed differential systems and quadratic supply rates, we start with an example that does not fall into this category, in order to illustrate the nature of the problems that we have in mind. Consider the diffusion equation that describes, in suitable units, the evolution of the temperature profile along a uniform heat conducting bar:

$$\frac{\partial}{\partial t}T = \frac{\partial^2}{\partial x^2}T + q, \qquad T > 0,$$

where $T(x,t) \in \mathbb{R}$ denotes the absolute temperature at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}$, and $q(x,t) \in \mathbb{R}$ denotes the rate of heat supplied to the bar at time $t \in \mathbb{R}$ and position $x \in \mathbb{R}$.

Two thermodynamic laws govern this system: the first law, conservation of energy, and the second law, which implies irreversibility: heat cannot be freely transported from cold to hot areas. In order to express these laws, consider for every $T_0 > 0$, the behavior $\mathfrak{B}_{T_0}$ consisting of all $(T,q) \in \mathfrak{C}^\infty(\mathbb{R}^2, \mathbb{R})$ that satisfy the diffusion equation and such that $T(t,x) = T_0$ for all (t,x) outside a compact subset of $\mathbb{R}^2$. The first and second laws require that for all T_0 and for all $(T,q) \in \mathfrak{B}_{T_0}$, we must have:

$$\int_{\mathbb{R}^2} q(t,x)\; dx\; dt = 0,$$
$$\int_{\mathbb{R}^2} \frac{q(t,x)}{T(x,t)}\; dx\; dt \leq 0.$$

We interpret the first law as stating that the diffusion equation defines a system that is conservative with respect to the supply rate q, and the second law as stating that it is dissipative with respect to the supply rate $-\frac{q}{T}$. The question is if these 'global' versions of the first and second law can be expressed as 'local' laws. This is the case indeed. Define the following latent variables:

$$\begin{aligned}
E &: \mathbb{R}^2 \to \mathbb{R}, \text{ the stored energy density,}\\
F_E &: \mathbb{R}^2 \to \mathbb{R}, \text{ the energy flux,}\\
S &: \mathbb{R}^2 \to \mathbb{R}, \text{ the entropy density,}\\
F_S &: \mathbb{R}^2 \to \mathbb{R}, \text{ the entropy flux, and,}\\
D_S &: \mathbb{R}^2 \to \mathbb{R}, \text{ the rate of entropy production.}
\end{aligned}$$

Relate these to T as follows:

$$\begin{aligned}
E &= T,\\
F_E &= -\frac{\partial}{\partial x}T,\\
S &= \ln(T),\\
F_S &= -\frac{1}{T}\frac{\partial}{\partial x}T,\\
D_S &= (\frac{1}{T}\frac{\partial}{\partial x}T)^2.
\end{aligned}$$

The first and second laws can be deduced from the following local versions:

$$\begin{aligned}
\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}F_E &= q,\\
\frac{\partial}{\partial t}S + \frac{\partial}{\partial x}F_S &= \frac{q}{T} + D_S.
\end{aligned}$$

This example shows that for the one-dimensional diffusion equation, it is possible to express the first and second law in terms of equations that are local in time and space. Note that in this case, the stored energy density, E, and the entropy density, S, are given as differential operators on the manifest variables (T,q). This is, as we shall see, not possible in general.

5 Conservative and dissipative systems

We are interested in this paper in distributed dynamical systems that are conservative or dissipative with respect to a supply rate that is a quadratic function of the manifest variables and their partial derivatives. These are defined by matrices $\Phi_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}, k_1,\ldots,k_n,\ell_1,\ldots,\ell_n \in \mathbb{Z}_+$, with all but a finite number of these matrices equal to zero. We call the map from $\mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R})$ defined by

$$w \in \mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}}) \mapsto \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} (\frac{\partial^{k_1}}{\partial x_1^{k_1}}\cdots\frac{\partial^{k_n}}{\partial x_n^{k_n}}w)^T \Phi_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} (\frac{\partial^{\ell_1}}{\partial x_1^{\ell_1}}\cdots\frac{\partial^{\ell_n}}{\partial x_n^{\ell_n}}w)$$

a *quadratic differential form* on $\mathfrak{C}^\infty(\mathbb{R}^n,\mathbb{R}^{\mathtt{w}})$. Note that a quadratic differential form is completely specified by the $(\mathtt{w}\times\mathtt{w})$-matrix Φ of $2n$-variable polynomials in $\mathbb{R}[\zeta_1,\ldots,\zeta_n,\eta_1,\ldots,\eta_n]$, defined by

$$\Phi(\zeta_1,\ldots,\zeta_n,\eta_1,\ldots,\eta_n) = \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n} \Phi_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n}\zeta_1^{k_1}\cdots\zeta_n^{k_n}\eta_1^{\ell_1}\cdots\eta_n^{\ell_n}.$$

We denote the quadratic differential form that corresponds to the matrix of polynomials Φ by Q_Φ. Define $\Phi^\star$, the matrix of $2n-$variable polynomials, by

$$\Phi^\star(\zeta_1,\ldots,\zeta_n,\eta_1,\ldots,\eta_n) = \Phi^T(\eta_1,\ldots,\eta_n,\zeta_1,\ldots,\zeta_n).$$

If $\Phi = \Phi^\star$, we call Φ *symmetric*. We may (and will) assume, since obviously $Q_\Phi = Q_{\Phi^\star} = Q_{\frac{1}{2}(\Phi+\Phi^\star)}$, that in a quadratic differential form the matrix of polynomials Φ is symmetric.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\text{cont}}$ and $\in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\ldots,\zeta_n,\eta_1,\ldots,\eta_n]$. Define $\mathfrak{B}$ to be *conservative* with respect to the supply rate Q_Φ if

$$\int_{\mathbb{R}^n} Q_\Phi(w) = 0$$

for all $w \in \mathfrak{B}$ of compact support, and *dissipative* if

$$\int_{\mathbb{R}^n} Q_\Phi(w) \geq 0$$

for all $w \in \mathfrak{B}$ of compact support.

6 Local version of a conservation law

The following result shows in what sense a conservation law can be expressed as a local law.

Theorem 3 (Local version of a conservation law) : *Consider the controllable n-D distributed dynamical system* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n,cont}}$ *and the* supply rate *defined by the quadratic differential form* Q_Φ*. Let* $w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell$ *be an image representation of* $\mathfrak{B}$*. Then* $\mathfrak{B}$ *is conservative with respect to* Q_Φ *if and only if there exist an* $n-$*vector of quadratic differential forms* $Q_\Psi = (Q_{\Psi_1}, \ldots, Q_{\Psi_n})$ *on* $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$*, called the* flux density*, such that*

$$\nabla \cdot Q_\Psi(w, \ell) = Q_\Phi(w)$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$*, the full behavior of* $w = M(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})\ell$*.*

When the first independent variable is time, and the others are space variables, then the local version of the conservation law can be expressed a bit more intuitively in terms of a quadratic differential form Q_S, the *storage density*, and a 3-vector of quadratic differential forms Q_F, the *spatial flux density*, as

$$\frac{\partial}{\partial t} Q_S(w, \ell) + \nabla \cdot Q_F(w, \ell) = Q_\Phi(w)$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$, the full behavior of $w = M(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})\ell$, an image representation of $\mathfrak{L}^{\mathtt{w}}_{4,\mathrm{cont}}$.

In the *1-D* case, the introduction of latent variables is unnecessary, and we can simply claim the existence of a quadratic differential form Q_Ψ on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, such that $\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w)$ for all $w \in \mathfrak{B}$. However, in the *n-D* case, the introduction of latent variables cannot be avoided, because not every controllable distributed parameter system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n}}$ admits an observable image representation.

The idea behind the proof of the above theorem is as follows. Using a weakly observable image representation for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n,cont}}$ shows that is suffices to prove the result for the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. Next, observe that $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ is conservative with respect to Q_Φ if and only if $\partial(\Phi) = 0$. This in turn is easily seen to be equivalent to the solvability of the equation

$$\begin{aligned}\Phi(\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n) = {} & (\zeta_1 + \eta_1)\Psi_1(\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n) \\ & + \ldots + (\zeta_n + \eta_n)\Psi_n(\zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n) \quad (4)\end{aligned}$$

for $(\Psi_1, \ldots, \Psi_n)$. Note that in the *n-D* case, contrary to the *1-D* case, the solution $(\Psi_1, \ldots, \Psi_n)$ to this equation is not unique, and hence the flux density is in general not uniquely specified by the dynamics and the supply rate.

As an illustration of the above theorem, consider Maxwell's equations. This defines a conservative distributed dynamical system with respect to the supply rate $-\vec{E} \cdot \vec{j}$, the rate of electric energy supplied to the electro-magnetic field. In other words, for all $(\vec{E}, \vec{j}) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ of compact support that satisfy the partial differential equations equations obtained from Maxwell's equations after elimination of the fields $\vec{B}$ and ρ, there holds $\int_{\mathbb{R}} \int_{\mathbb{R}^3} \vec{E} \cdot \vec{j}\; dxdydz\; dt = 0$. A local version of the law of conservation of energy is provided by introducing the *stored energy density,* S, and the *energy flow* (the flux density), $\vec{F}$, the *Poynting vector.* These are related to $\vec{E}$ and $\vec{B}$ by

$$\begin{aligned} S(\vec{E}, \vec{B}) &= \frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{\varepsilon_0 c^2}{2} \vec{B} \cdot \vec{B}, \\ \vec{F}(\vec{E}, \vec{B}) &= \varepsilon_0 c^2 \vec{E} \times \vec{B}. \end{aligned}$$

As is well-known, there holds,

$$\frac{\partial}{\partial t} S(\vec{E}, \vec{B}) + \nabla \cdot \vec{F}(\vec{E}, \vec{B}) + \vec{E} \cdot \vec{j} = 0 \tag{5}$$

along the behavior defined by Maxwell's equations. Note that the local version of conservation of energy involves $\vec{B}$ in addition to $\vec{E}$ and $\vec{j}$, the variables that define the rate of energy supplied. Whence $\vec{B}$ plays the role of a latent variable, and it is not possible to express conservation of energy in terms of $\vec{E}, \vec{j}$, and their partial derivatives. A more subtle issue is the uniqueness of the energy density and the energy flow. As we have seen above, in a conservative distributed dynamical system, the storage density and the flux density need not be uniquely specified by the behavior and the supply rate. This non-uniqueness is not purely a mathematical issue. It is also the case for conservation of energy in Maxwell's equations. In fact, it is not known how much energy is stored in an electro-magnetic field, and $\frac{\varepsilon_0}{2} \vec{E} \cdot \vec{E} + \frac{\varepsilon_0 c^2}{2} \vec{B} \cdot \vec{B}$, is only one of many possible choices. As long as the stored energy density only needs to satisfy the local version of the law of conservation of energy for some energy flow, there are many possible choices.

7 Local version of a dissipation law

We now discuss dissipative dynamical systems. A quadratic differential form Q_Δ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$ is said to be *non-negative*, denoted $Q_\Delta \geq 0$, if $Q_\Delta(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}})$. The following theorem gives the local version of dissipativeness for distributed differential systems.

Theorem 4 (Local version of a dissipation law) : *Consider the controllable n-D distributed dynamical system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{n,cont}}$ and the* supply rate *defined by the quadratic differential form Q_Φ. Then $\mathfrak{B}$ is dissipative with respect to Q_Φ if and only if there exist:*

1. *a latent variable representation*

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell \tag{6}$$

   of $\mathfrak{B}$,

2. *an $n-$vector of quadratic differential forms $Q_\Psi = (Q_{\Psi_1}, \cdots, Q_{\Psi_n})$ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the* flux density,

3. *a non-negative quadratic differential Q_Δ on $\mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{w}+\dim(\ell)})$, called the* dissipation rate,

such that

$$\nabla \cdot Q_\Psi(w, \ell) = Q_\Phi(w) + Q_\Delta(w, \ell)$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$, the full behavior of (6).

When the first independent variable is time, and the others are space variables, then the local version of the dissipation law can be expressed a bit more intuitively in terms of a quadratic differential form Q_S, the *storage density*, a 3-vector of quadratic differential forms Q_F, the *spatial flux density*, and the quadratic differential form $Q_\Delta \geq 0$, the *dissipation rate*, such that

$$\frac{\partial}{\partial t} Q_S(w, \ell) + \nabla \cdot Q_F(w, \ell) = Q_\Phi(w) + Q_\Delta(w, l)$$

for all $(w, \ell) \in \mathfrak{B}_{\mathrm{full}}$, the full behavior of a suitable latent variable representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{4,\mathrm{cont}}$.

In the *1-D* case, the introduction of latent variable is once again unnecessary, and we can simply claim the existence of quadratic differential forms (Q_Ψ, Q_Δ) on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, with $Q_\Delta \geq 0$, such that $\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) + Q_\Delta(w)$ for all $w \in \mathfrak{B}$.

In order to see where the introduction of latent variables enters in the *n-D* case, we will briefly sketch the proof of the above theorem in the *1-D* case (see [12] for details) . It is easy to see, using an observable image representation, that it suffices to consider the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Next, use Fourier transforms to prove that the integral $\int_{-\infty}^{+\infty} Q_\Phi(w)\ dt$ is non-negative for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ of compact support if and only if the hermitian matrix $\Phi(i\omega, -i\omega) \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}$ is non-negative definite for all $\omega \in \mathbb{R}$. This in turn implies that the matrix of polynomials $\Phi(\xi, -\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be factored as $\Phi(\xi, -\xi) = D^T(-\xi)D(\xi)$ with $D(\xi) \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ a matrix of polynomials. The result then follows by taking

$$\begin{aligned}\Psi(\zeta, \eta) &= \frac{\Phi(\zeta, \eta) - D^T(\zeta)D(\eta)}{\zeta + \eta}, \text{ and} \\ \Delta(\zeta, \eta) &= D^T(\zeta)D(\eta).\end{aligned}$$

For dissipative systems the storage function Q_Ψ, and hence the dissipation rate, Q_Δ, are, in general, not unique, not even in the *1-D* case, since the factorization $\Phi(\xi, -\xi) = D^T(-\xi)D(\xi)$ is not unique.

The generalization of this proof to the *n-D* case fails on two accounts. Firstly, because there may not exist an observable image representation for $\mathfrak{B}$. Secondly, because the polynomial matrix factorization

$$\Phi(\xi_1, \dots, \xi_n, -\xi_1, \dots, -\xi_n) = D^T(-\xi_1, \dots, -\xi_n)D(\xi_1, \dots, \xi_n) \tag{7}$$

with $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1, \dots, \xi_n]$ may not be possible, whereas we still have that dissipativeness of $\mathfrak{B}$ with respect to the supply rate Q_Φ is equivalent to non-negative definiteness of the hermitian matrix $\Phi(i\omega_1, \dots, i\omega_n, -i\omega_1, \dots, -i\omega_n) \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}$ for all $\omega_1, \dots, \omega_n \in \mathbb{R}$. However, it turns out that a factorization as (7), with $D \in \mathbb{R}^{\bullet\times\mathtt{w}}(\xi_1, \dots, \xi_n)$ a matrix of rational functions in the variables $\xi_1, \dots, \xi_n$ does exist. This accounts for the need to introduce a latent variable (and not just an image) representation of $\mathfrak{B}$ in the above theorem.

Let us examine this factorization question in a bit more detail. Let Γ be a $(\mathtt{w} \times \mathtt{w})$-matrix of $n-$variable polynomials in $\mathbb{R}[\xi_1, \dots, \xi_n]$. The problem is to factor Γ as

$$\Gamma(\xi_1, \dots, \xi_n) = F^T(-\xi_1, \dots, -\xi_n)F(\xi_1, \dots, \xi_n). \tag{8}$$

Both factorizations with F a matrix of polynomials in $\mathbb{R}[\xi_1, \dots, \xi_n]$, and with F a matrix of rational functions in $\mathbb{R}(\xi_1, \dots, \xi_n)$ are of interest. Two obvious necessary conditions for factorizability are:

(*i*) $\Gamma(\xi_1, \dots, \xi_n) = \Gamma^T(-\xi_1, \dots, -\xi_n)$,

(*ii*) the hermitian matrix $\Gamma(i\omega_1, \dots, i\omega_n) \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}$ is non-negative definite for all $\omega_1, \dots, \omega_n \in \mathbb{R}$.

The question is whether conditions (*i*) and (*ii*) are also sufficient for factorizability. In order to grasp the difficulties involved, consider the following four cases:
$n = 1, \mathtt{w} = 1$; $n = 1, \mathtt{w} > 1$; $n > 1, \mathtt{w} = 1$; and $n > 1, \mathtt{w} > 1$.

1. In the case $n = 1, \mathtt{w} = 1$, it is a trivial matter to see that (i) and (ii) are necessary and sufficient for factorizability of $\Gamma(\xi) = F(-\xi)F(\xi)$ with $F \in \mathbb{R}[\xi]$ a real polynomial.

2. In the case $n = 1, \mathtt{w} > 1$, it is well-known that (i) and (ii) are necessary and sufficient for factorizability of $\Gamma(\xi) = F^T(-\xi)F(\xi)$ with F a polynomial matrix, $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. This result, polynomial matrix factorization, is not at all simple to prove. In the case that F is allowed to be a matrix of rational functions, $F \in \mathbb{R}(\xi)$, (even when Γ is a matrix of polynomials), it is much easier to prove factorizability, and in this situation factorizability becomes in fact a simple consequence of factorizability in the case $n = 1, \mathtt{w} = 1$.

3. The case $n > 1, \mathtt{w} = 1$ is especially interesting from the mathematical point of view. In this case, the factorization of Γ, assuming that (ii) holds, can be brought back to the following question about real polynomials in many variables. Assume that $p \in \mathbb{R}[\xi_1, \ldots, \xi_n]$ satisfies $p \geq 0$, i.e., $p(x_1, \ldots, x_n) \geq 0$ for all $x_1, \ldots, x_n \in \mathbb{R}$. Does this imply that there exist polynomials $p_1, \ldots, p_N \in \mathbb{R}[\xi_1, \ldots, \xi_n]$ such that $p = p_1^2 + \ldots + p_N^2$? Hilbert proved in 1888 that this is not the case, although it took until 1966 until Motzkin displayed a concrete example, $1 + x_1^2x_2^4 + x_1^4x_2^2 - 3x_1^2x_2^2$, where this factorization is impossible. The problem of factoring $p \geq 0$ as $p = p_1^2 + \ldots + p_N^2$, with $p_1, \ldots, p_N \in \mathbb{R}(\xi_1, \ldots, \xi_n)$ rational functions is the subject of Hilbert's 17-th problem announced at the International Congress of Mathematicians in 1900. Hilbert himself had proven this factorizability for the case $n = 2$ in 1893. In 1926, E. Artin proved the result for general n. Factorizability with $N = 2^n$ was proven by Pfister in 1967. See [5] for an account of Hilbert's 17-th problem.

4. In the case $n > 1, \mathtt{w} > 1$, the factorization (8) with F a matrix of rational functions, $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1, \ldots, \xi_n]$, becomes again a consequence of factorizabilty in the case $n > 1, \mathtt{w} = 1$.

Summarizing, the construction of a dissipation rate for a system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n},\mathrm{cont}}$ that is dissipative with respect to the supply rate Q_Φ reduces to the factorization problem (7). In the case $n > 1$, this factorization reduces to Hilbert's 17-th problem and yields factorizability with a matrix of rational functions. By considering a suitable representation of $\mathfrak{B}$ and constructing the flux density by an equation as (4) yields a local version of the dissipation law.

8 Conclusions

In this paper, we studied conservative and dissipative systems in the context of distributed dynamical systems described by constant-coefficient partial differential equations. For such systems, it is possible to express a global conservation or dissipation law as a local one, involving the flux density and the dissipation rate. There are two interesting aspects of the construction of the flux density and the dissipation rate. The first one is the relation with Hilbert's 17-th problem on the factorization of real non-negative rational functions in many variables as a sum of squares of real rational functions. The second interesting aspect is that local conservation or dissipation laws necessarily involve 'hidden' latent variables.

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