Balanced State Representations with Polynomial Algebra*

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1 Abstract

Algorithms are derived that pass directly from the differential equation describing the behavior of a finite-dimensional linear system to a balanced state representation.

Keywords: Linear systems, behaviors, image representation, state representation, controllability gramian, observability gramian, balancing, model reduction.

2 Introduction

The algorithms for model reduction are among the most useful achievements of linear system theory. A low order model that incorporates the important features of a high order one offers many advantages: it reduces the computational complexity, it filters out irrelevant details, it smooths the data, etc. Two main classes of algorithms for model reduction have been developed: (i) model reduction by *balancing*, and (ii) model reduction in the *Hankel norm* (usually called *AAK model reduction*). The implementation of these algorithms typically starts from the finite-dimensional state space system

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du,$$

commonly denoted as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{1}$$

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In the context of model reduction, it is usually assumed that this system is minimal (i.e., controllable and observable) and stable (i.e., the matrix A is assumed to be *Hurwitz*, meaning that all its eigenvalues have negative real part).

However, a state space system is seldom the end product of a first principles modeling procedure. Typically one obtains models involving a combination of (many) algebraic equations, (high order) differential equations, transfer functions, auxiliary variables, etc. Since model reduction procedures aim at systems of high dynamic complexity, it may not be an easy matter to transform the first principles model to state form. It is therefore important to develop algorithms that pass directly from model classes different from state space models to reduced models, without passing through a state representation.

There are, in fact, some interesting subtle algorithms that do exactly this for AAK model reduction in Fuhrmann's book [1]. These algorithms form the original motivation and inspiration for the present article. Its purpose is to present an algorithm for the construction of a balanced state representation directly from the differential equation (or the transfer function) that governs the system. For simplicity of exposition, we restrict attention in this paper to single-input/single-output systems.

A few words about the notation. We use the standard notation $\mathbb{R}$, $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{n}_1\times\mathtt{n}_2}$, for the reals, the set of $\mathtt{n}$-dimensional real vectors, the set of $\mathtt{n}_1\times\mathtt{n}_2$-dimensional real matrices. $M=[m(\mathtt{i},\mathtt{j})]_{\mathtt{i}=1,\ldots,\mathtt{n}_1}^{\mathtt{j}=1,\ldots,\mathtt{n}_2}$, denotes the $\mathtt{n}_1\times\mathtt{n}_2$-matrix whose $(\mathtt{i},\mathtt{j})-$th element is $m(\mathtt{i},\mathtt{j})$, with an analogous notation $[m(\mathtt{j})]^{\mathtt{j}=1,\ldots,\mathtt{n}}$ for row and $[m(\mathtt{i})]_{\mathtt{i}=1,\ldots,\mathtt{n}}$ for column vectors. The ring of real one-variable polynomials in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$, and the set of real two-variable polynomials in the indeterminates ζ,η is denoted by $\mathbb{R}[\zeta,\eta]$. $\mathbb{R}_{\mathtt{n}}[\xi]$ denotes the $(\mathtt{n}+1)$-dimensional real vector space consisting of the real polynomials of degree less that or equal to $\mathtt{n}$. $\mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ denotes the set of maps $f:\mathbb{R}\rightarrow\mathbb{R}$ that are locally square integrable, i.e., such that $\int_{t_1}^{t_2}|f(t)|^2\,dt<\infty$ for all $t_1,t_2\in\mathbb{R}$; $\mathcal{L}_2(A,\mathbb{R})$ denotes the set of maps $f:A\rightarrow\mathbb{R}$ such that $||f||^2_{\mathcal{L}_2(A,\mathbb{R})}:=\int_A|f(t)|^2\,dt<\infty$. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ denotes the set of infinitely differentiable maps from $\mathbb{R}$ to $\mathbb{R}$, $\mathfrak{C}^+(\mathbb{R},\mathbb{R}):=\{w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R})\mid w|_{(-\infty,0]}$ has compact support$\}$, and $\mathfrak{D}(\mathbb{R},\mathbb{R})$ denotes the set of real distributions on $\mathbb{R}$. Analogous notation is used for $\mathbb{R}$ replaced by the field of complex numbers $\mathbb{C}$. * denotes complex conjugation for elements of $\mathbb{C}$, Hermitian conjugate (conjugate transpose) for complex matrices, or, more generally, 'dual'.

3 The system equations

Our starting point is the continuous-time single-input/single-output finite-dimensional linear time-invariant system described by the differential equation

$$p(\frac{d}{dt})y=q(\frac{d}{dt})u, \tag{2}$$

relating the input $u:\mathbb{R}\to\mathbb{R}$ to the output $y:\mathbb{R}\to\mathbb{R}$. The polynomials $p,q\in\mathbb{R}[\xi]$ parametrize the system behavior, formally defined as

$$\mathfrak{B}_{(p,q)} := \{(u,y)\in\mathfrak{L}_2^{\rm loc}(\mathbb{R},\mathbb{R})\times\mathfrak{L}_2^{\rm loc}(\mathbb{R},\mathbb{R}) \\ \mid \text{(2) holds in the sense of distributions}\}.$$

In the sequel, we will identify the system (2) with its behavior $\mathfrak{B}_{(p,q)}$.

The system $\mathfrak{B}_{(p,q)}$ is said to be *controllable* if for all $(u_1,y_1),(u_2,y_2)\in\mathfrak{B}_{(p,q)}$ there exists $T>0$ and $(u,y)\in\mathfrak{B}_{(p,q)}$ such that $(u_1,y_1)(t)=(u,y)(t)$ for $t\leq 0$ and that $(u_2,y_2)(t)=(u,y)(t+T)$ for $t>0$. It is well-known (see [5]) that the system $\mathfrak{B}_{(p,q)}$ is controllable if and only if the polynomials p and q are co-prime (i.e., they have no common roots). It turns out that controllability is also equivalent to the existence of an *image representation* for $\mathfrak{B}_{(p,q)}$, meaning that the *manifest behavior* of the *latent variable system*

$$u=p(\frac{d}{dt})\ell, y=q(\frac{d}{dt})\ell, \tag{3}$$

formally defined as

$$\mathfrak{Im}_{(p,q)} := \{(u,y)\in\mathfrak{L}_2^{\rm loc}(\mathbb{R},\mathbb{R})\times\mathfrak{L}_2^{\rm loc}(\mathbb{R},\mathbb{R}) \mid \text{ there exists } \ell\in\mathfrak{D}(\mathbb{R},\mathbb{R}) \\ \text{such that (3) holds in the sense of distributions}\}$$

is *exactly* equal to $\mathfrak{B}_{(p,q)}$. In (3), we refer to ℓ as the *latent variable.*

We assume throughout that $p,q\in\mathbb{R}[\xi]$ are co-prime polynomials, with degree(q) $\leq$ degree(p) $=:\mathtt{n}$. Co-primeness of p and q ensures, in addition to controllability of $\mathfrak{B}_{(p,q)}$, *observability* of the image representation $\mathfrak{Im}_{(p,q)}$, meaning that, for every $(u,y)\in\mathfrak{Im}_{(p,q)}=\mathfrak{B}_{(p,q)}$, the $\ell\in\mathfrak{D}(\mathbb{R},\mathbb{R})$ such that $u=p(\frac{d}{dt})\ell, y=q(\frac{d}{dt})\ell$, is unique.

In addition to expressing controllability, image representations are also useful for state construction (see [6] for an in-depth discussion). For the case at hand, it turns out that any set of polynomials $\{x_1,x_2,\ldots,x_{\mathtt{n}'}\}$ that span $\mathbb{R}_{\mathtt{n}-1}[\xi]$ defines a state representation of $\mathfrak{B}_{(p,q)}$ with state

$$x=(x_1(\frac{d}{dt})\ell, x_2(\frac{d}{dt})\ell,\ldots,x_{\mathtt{n}'-1}(\frac{d}{dt})\ell),$$

i.e., the manifest behavior of

$$u=p(\frac{d}{dt})\ell, y=q(\frac{d}{dt})\ell, \quad x=\mathrm{col}(x_1(\frac{d}{dt})\ell, x_2(\frac{d}{dt})\ell,\ldots,x_{\mathtt{n}'-1}(\frac{d}{dt})\ell) \tag{4}$$

satisfies the axiom of state (see [6] for a formal definition of the axiom of state). The associated system matrices (1) are then obtained as a solution matrix $\begin{bmatrix}A & B\\ C & D\end{bmatrix}$

of the following system of linear equations in $\mathbb{R}_{\mathtt{n}}[\xi]$:

$$\begin{bmatrix} \xi x_1(\xi) \\ \xi x_2(\xi) \\ \vdots \\ \xi x_{\mathtt{n}'}(\xi) \\ q(\xi) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1(\xi) \\ x_2(\xi) \\ \vdots \\ x_{\mathtt{n}'}(\xi) \\ p(\xi) \end{bmatrix}. \tag{5}$$

This state representation is minimal if and only if $\mathtt{n}' = \mathtt{n}$ and hence the polynomials $x_1, x_2, \ldots, x_{\mathtt{n}}$ form a basis for $\mathbb{R}_{\mathtt{n}-1}[\xi]$. Henceforth, we will concentrate on the minimal case, and put $\mathtt{n} = \mathtt{n}'$. Note that in this case the solution of (5) is unique.

The $\mathtt{n}$-th order system (1), assumed minimal (i.e., controllable and observable) and stable, is called *balanced* if there exist real numbers

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{n}} > 0,$$

called the *Hankel singular values*, such that, with

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\mathtt{n}}),$$

there holds

$$\begin{aligned} A\Sigma + \Sigma A^\top + BB^\top &= 0 \\ A^\top \Sigma + \Sigma A + C^\top C &= 0. \end{aligned}$$

Of course, in the context of the state construction through an image representation as explained above, being balanced becomes a property of the polynomials $x_1, x_2, \ldots, x_{\mathtt{n}}$. The central problem of this paper is:

> Choose the polynomials $x_1, x_2, \ldots, x_{\mathtt{n}}$ so that (5) defines a balanced state space system.

4 The controllability and observability gramians

In order to solve this problem, we need polynomial expressions for the controllability and observability gramians. These are actually quadratic differential forms (QDF's) (see [7] for an in-depth study of QDF's). The real two-variable polynomial

$$\Phi(\zeta, \eta) = \Sigma_{\mathtt{i},\mathtt{j}} \Phi_{\mathtt{i},\mathtt{j}} \zeta^{\mathtt{i}} \eta^{\mathtt{j}}$$

induces the map

$$w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \mapsto \Sigma_{\mathtt{i},\mathtt{j}} \frac{d^{\mathtt{i}}}{dt^{\mathtt{i}}} w \; \Phi_{\mathtt{i},\mathtt{j}} \; \frac{d^{\mathtt{j}}}{dt^{\mathtt{j}}} w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}).$$

This map is called a *a quadratic differential form* (QDF). Denote it as Q_Φ. In view of the quadratic nature of this map, we will always assume that $\Phi_{\mathtt{i},\mathtt{j}} = \Phi_{\mathtt{j},\mathtt{i}}$, i.e., that Φ is *symmetric*, i.e., $\Phi = \Phi^*$, with $\Phi^*(\zeta,\eta) := \Phi(\eta,\zeta)$.

The derivative $\frac{d}{dt}Q_\Phi(w)$ of the QDF Q_Φ is again a QDF, $Q_\Psi(w)$, with $\Psi(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$. It readily follows that a given QDF Q_Φ is the derivative of another QDF if and only if $\partial(\Phi) = 0$, where $\partial : \mathbb{R}[\zeta,\eta] \to \mathbb{R}[\xi]$ is defined by $\partial(\Phi)(\xi) := \Phi(\xi,-\xi)$, in which case the QDF Q_Ψ such that $Q_\Phi(w) = \frac{d}{dt}Q_\Phi(w)$ is induced by $\Psi(\zeta,\eta) = \dfrac{\Phi(\zeta,\eta)}{\zeta+\eta}$. Note that this is a polynomial since $\partial(\Phi) = 0$.

Every QDF Q_Φ can de written as the sum and difference of squares, i.e., there exist $f_1^+, f_2^+, \ldots, f_{\mathtt{n}_+}^+, f_1^-, f_2^-, \ldots, f_{\mathtt{n}_-}^- \in \mathbb{R}[\xi]$ such that

$$Q_\Phi(w) = \Sigma_{\mathtt{k}=1}^{\mathtt{n}_+} |f_{\mathtt{k}}^+(\frac{d}{dt})w|^2 - \Sigma_{\mathtt{k}=1}^{\mathtt{n}_-} |f_{\mathtt{k}}^-(\frac{d}{dt})w|^2.$$

Equivalently,

$$\Phi(\zeta,\eta) = \Sigma_{\mathtt{k}=1}^{\mathtt{n}_+} f_{\mathtt{k}}^+(\zeta) f_{\mathtt{k}}^+(\eta) - \Sigma_{\mathtt{k}=1}^{\mathtt{n}_-} f_{\mathtt{k}}^-(\zeta) f_{\mathtt{k}}^-(\eta).$$

If $f_1^+, f_2^+, \ldots, f_{\mathtt{n}_+}^+, f_1^-, f_2^-, \ldots, f_{\mathtt{n}_-}^- \in \mathbb{R}[\xi]$ are linearly independent over $\mathbb{R}$, then $\mathtt{n}_+ + \mathtt{n}_-$ is the *rank* and $\mathtt{n}_+ - \mathtt{n}_-$ the *signature* of Q_Φ (or Φ). The QDF Q_Φ is said to be *non-negative* if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$. Equivalently, if and only if the rank of Φ is equal to its signature.

While it would be natural to consider the controllability and observability gramians as QDF's on $\mathfrak{B}_{(p,q)}$, we will consider them as QDF's acting on the latent variable ℓ of the image representation (3). This entails no loss of generality, since there is a one-to-one relation between ℓ in (3) and $(u,y) \in \mathfrak{B}_{(p,q)}$.

The *controllability gramian* Q_K (equivalently, K) is defined as follows. Let $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ and define $Q_K(\ell)$ by

$$Q_K(\ell)(0) := \text{infimum} \int_{-\infty}^{0} |p(\frac{d}{dt})\ell'(t)|^2 \, dt,$$

where the infimum is taken over all $\ell' \in \mathfrak{C}^+(\mathbb{R},\mathbb{R})$ that join ℓ at $t = 0$, i.e., such that $\ell(t) = \ell'(t)$ for $t \geq 0$.

The *observability gramian* Q_W (equivalently, W) is defined as follows. Let $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$ and define $Q_W(\ell)$ by

$$Q_W(\ell)(0) := \int_0^\infty |q(\frac{d}{dt})\ell'(t)|^2 \, dt,$$

where $\ell' \in \mathfrak{D}(\mathbb{R},\mathbb{R})$ is such that

(i) $\ell|_{(-\infty,0)} = \ell'|_{-\infty,0)}$,

(ii) $(p(\frac{d}{dt})\ell', q(\frac{d}{dt})\ell') \in \mathfrak{B}_{(p,q)}$,

(iii) $p(\frac{d}{dt})\ell'(t)|_{(0,\infty)} = 0$.

Thus ℓ' is a latent variable trajectory that continues ℓ at $t = 0$ with an ℓ' such that $u|_{(0,\infty)} = p(\frac{d}{dt})\ell'|_{(0,\infty)} = 0$. This continuation must be sufficiently smooth so that the resulting $(u, y) = (p(\frac{d}{dt})\ell', q(\frac{d}{dt})\ell')$ belongs to $\mathfrak{B}_{(p,q)}$, thus in particular to $\mathfrak{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R})$. This actually means that the $(\mathtt{n} - 1)$-th derivative of ℓ' must be in $\mathfrak{L}_2^{\text{loc}}(\mathbb{R}, \mathbb{R})$.

The computation of the two-variable polynomials K and W is one of the central results of this paper.

Theorem 1 *Consider the system $\mathfrak{B}_{(p,q)}$ with $p, q \in \mathbb{R}[\xi]$, p, q are co-prime, degree(q) $\leq$ degree(p) $=: \mathtt{n}$, and p Hurwitz (meaning that all its roots have negative real part). The controllability gramian and the observability gramian are QDF's. Denote them by Q_K and Q_W respectively, with $K \in \mathbb{R}[\zeta, \eta]$ and $W \in \mathbb{R}[\zeta, \eta]$. They can be computed as follows*

$$\boxed{K(\zeta, \eta) = \frac{p(\zeta)p(\eta) - p(-\zeta)p(-\eta)}{\zeta + \eta},} \tag{6}$$

$$\boxed{W(\zeta, \eta) = \frac{p(\zeta)f(\eta) + f(\zeta)p(\eta) - q(\zeta)q(\eta)}{\zeta + \eta},} \tag{7}$$

with $f \in \mathbb{R}_{\mathtt{n}-1}[\xi]$ the (unique) solution of the Bezout-type equation

$$\boxed{p(\xi)f(-\xi) + f(\xi)p(-\xi) - q(\xi)q(-\xi) = 0.} \tag{8}$$

Moreover, both Q_K and Q_W are nonnegative and have rank $\mathtt{n}$. Finally, $Q_K(\ell)$ and $Q_W(\ell)$ only contain the derivatives $\ell, \frac{d}{dt}\ell, \ldots, \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}\ell$.

The proof of this theorem is given in the appendix (section 7.1).

Note that the equation for f has a unique solution in $\mathbb{R}_{\mathtt{n}-1}[\xi]$ since $p(\xi)$ and $p(-\xi)$ are co-prime, a consequence of the fact that p is Hurwitz.

What we call the controllability gramian measures the *difficulty* it takes to join the latent variable trajectory ℓ at $t = 0$ by a trajectory ℓ' that is zero in the far past, as measured by

$$\int_{-\infty}^{0} |u(t)|^2 \, dt = \int_{-\infty}^{0} |p(\frac{d}{dt})\ell'(t)|^2 \, dt.$$

The observability gramian on the other hand measures the *ease* with which it is possible to observe the effect of the latent variable trajectory ℓ as measured by

$$\int_{0}^{+\infty} |y(t)|^2 \, dt = \int_{0}^{+\infty} |q(\frac{d}{dt})\ell'(t)|^2 \, dt,$$

assuming that the input $u = p(\frac{d}{dt})\ell'(t)$ is zero for $t \geq 0$. Note the slight difference with the classical terminology where the controllability gramian corresponds to the 'inverse' of the QDF Q_K.

5 Balanced state representation

The minimal state representation (4) with state polynomials $(x_1, x_2, \ldots, x_{\mathtt{n}})$ is *balanced* if

1. for $\ell_{\mathtt{i}} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ such that $x_{\mathtt{j}}(\frac{d}{dt})\ell_{\mathtt{i}}(0) = \delta_{\mathtt{ij}}$ ($\delta_{\mathtt{ij}}$ denotes the Kronecker delta), we have
$$Q_W(\ell_{\mathtt{i}})(0) = \frac{1}{Q_K(\ell_{\mathtt{i}})(0)},$$
i.e., the state components that are difficult to reach are also difficult to observe, and

2. the state components are ordered so that
$$0 < Q_K(\ell_1)(0) \leq Q_K(\ell_2)(0) \leq \cdots \leq Q_K(\ell_{\mathtt{n}})(0),$$
and hence
$$Q_W(\ell_1)(0) \geq Q_W(\ell_2)(0) \geq \cdots \geq Q_W(\ell_{\mathtt{n}})(0) > 0.$$

The general state construction (4) and a suitable factorization of the controllability and observability gramians readily lead to a balanced state representation.

It is a standard result from linear algebra (see [2], chapter 9) that theorem 1 implies that there exist polynomials

$$(x_1^{\mathrm{bal}}, x_2^{\mathrm{bal}}, \ldots, x_{\mathtt{n}}^{\mathrm{bal}})$$

that form a basis for $\mathbb{R}_{\mathtt{n}-1}[\xi]$, and real numbers

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{n}} > 0 \tag{9}$$

(the $\sigma_{\mathtt{k}}$'s are uniquely defined by K and W) such that

$$K(\zeta, \eta) = \Sigma_{k=1}^{\mathtt{n}} \sigma_{\mathtt{k}}^{-1} x_{\mathtt{k}}^{\mathrm{bal}}(\zeta) x_{\mathtt{k}}^{\mathrm{bal}}(\eta), \tag{10}$$

$$W(\zeta, \eta) = \Sigma_{\mathtt{k}=1}^{\mathtt{n}} \sigma_{\mathtt{k}} \;\; x_{\mathtt{k}}^{\mathrm{bal}}(\zeta) x_{\mathtt{k}}^{\mathrm{bal}}(\eta). \tag{11}$$

This leads to the main result of this paper.

Theorem 2 *Define the polynomials* $(x_1^{\mathrm{bal}}, x_2^{\mathrm{bal}}, \ldots, x_{\mathtt{n}}^{\mathrm{bal}}) \in \mathbb{R}_{\mathtt{n}-1}[\xi]$ *and the real numbers* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{n}} > 0$ *by equations (9, 10, 11). Then the* $\sigma_{\mathtt{k}}$*'s are the Hankel singular values of the system* $\mathfrak{B}_{(p,q)}$ *and*

$$u = p(\frac{d}{dt})\ell, y = q(\frac{d}{dt})\ell, x^{\mathrm{bal}} = (x_1^{\mathrm{bal}}(\frac{d}{dt})\ell, x_2^{\mathrm{bal}}(\frac{d}{dt})\ell, \ldots, x_{\mathtt{n}}^{\mathrm{bal}}(\frac{d}{dt})\ell)$$

is a balanced state space representation of $\mathfrak{B}_{(p,q)}$*. The associated balanced system matrices are obtained as the solution matrix* $\begin{bmatrix} A^{\text{bal}} & B^{\text{bal}} \\ C^{\text{bal}} & D^{\text{bal}} \end{bmatrix}$ *of the following system of linear equations in* $\mathbb{R}_{\mathtt{n}}[\xi]$*:*

$$\begin{bmatrix} \xi x_1^{\text{bal}}(\xi) \\ \xi x_2^{\text{bal}}(\xi) \\ \vdots \\ \xi x_{\mathtt{n}}^{\text{bal}}(\xi) \\ q(\xi) \end{bmatrix} = \begin{bmatrix} A^{\text{bal}} & B^{\text{bal}} \\ C^{\text{bal}} & D^{\text{bal}} \end{bmatrix} \begin{bmatrix} x_1^{\text{bal}}(\xi) \\ x_2^{\text{bal}}(\xi) \\ \vdots \\ x_{\mathtt{n}}^{\text{bal}}(\xi) \\ p(\xi) \end{bmatrix}. \tag{12}$$

The proof of this theorem is given in the appendix (section 7.2).

We summarize this algorithm:

DATA: $p, q \in \mathbb{R}[\xi]$, co-prime, degree$(q) \leq$ degree$(p) := \mathtt{n}$, p Hurwitz.

COMPUTE:

1. $K \in \mathbb{R}[\zeta, \eta]$ by (6),
2. $f \in \mathbb{R}_{\mathtt{n}-1}[\xi]$ by (8) and $W \in \mathbb{R}[\zeta, \eta]$ by (7),
3. $(x_1^{\text{bal}}, x_2^{\text{bal}}, \ldots, x_{\mathtt{n}}^{\text{bal}})$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\mathtt{n}} > 0$ by the expansions (9, 10, 11):
$$K(\zeta, \eta) = \Sigma_{k=1}^{\mathtt{n}} \sigma_{\mathtt{k}}^{-1} x_{\mathtt{k}}^{\text{bal}}(\zeta) x_{\mathtt{k}}^{\text{bal}}(\eta), W(\zeta, \eta) = \Sigma_{\mathtt{k}=1}^{\mathtt{n}} \sigma_{\mathtt{k}} \;\; x_{\mathtt{k}}^{\text{bal}}(\zeta) x_{\mathtt{k}}^{\text{bal}}(\eta),$$
4. the balanced system matrices $\begin{bmatrix} A^{\text{bal}} & B^{\text{bal}} \\ C^{\text{bal}} & D^{\text{bal}} \end{bmatrix}$ by solving (12).

The result is a balanced state representation of $\mathfrak{B}_{(p,q)}$**.**

The above algorithm shows how to obtain a balancing-reduced model. Assume that we wish to keep the significant states

$$x_1^{\text{bal}}(\frac{d}{dt})\ell, x_2^{\text{bal}}(\frac{d}{dt})\ell, \ldots, x_{\mathtt{n}_{\text{red}}}^{\text{bal}}(\frac{d}{dt})\ell,$$

and neglect the insignificant ones

$$x_{\mathtt{n}_{\text{red}}+1}^{\text{bal}}(\frac{d}{dt})\ell, x_{\mathtt{n}_{\text{red}}+2}^{\text{bal}}(\frac{d}{dt})\ell, \ldots, x_{\mathtt{n}}^{\text{bal}}(\frac{d}{dt})\ell.$$

Now, solve the following linear equations in the components of the matrices

$$\left[A_{\mathtt{i},\mathtt{j}}^{\text{balred}}\right]_{\mathtt{i}=1,\ldots,\mathtt{n}_{\text{red}}}^{\mathtt{j}=1,\ldots,\mathtt{n}_{\text{red}}}, \left[B_{\mathtt{i}}^{\text{balred}}\right]_{\mathtt{i}=1,\ldots,\mathtt{n}_{\text{red}}}, \left[C_j^{\text{balred}}\right]^{\mathtt{j}=1,\ldots,\mathtt{n}_{\text{red}}}, D^{\text{balred}}$$

$$\xi x_{\mathtt{i}}^{\mathrm{bal}}(\xi) = \Sigma_{\mathtt{j}=1}^{\mathtt{n}_{\mathrm{red}}} A_{\mathtt{i},\mathtt{j}}^{\mathrm{balred}} x_{\mathtt{j}}^{\mathrm{bal}}(\xi) + B_{\mathtt{i}}^{\mathrm{balred}} p(\xi) \qquad \mathrm{modulo}(\mathrm{x}_{\mathtt{n}_{\mathrm{red}}+1}^{\mathrm{bal}}(\xi), \mathrm{x}_{\mathtt{n}_{\mathrm{red}}+2}^{\mathrm{bal}}(\xi), \ldots, \mathrm{x}_{\mathtt{n}}^{\mathrm{bal}}(\xi))$$

$$q(\xi) = \Sigma_{\mathtt{j}=1}^{\mathtt{n}_{\mathrm{red}}} C_{\mathtt{j}}^{\mathrm{balred}} x_{\mathtt{j}}^{\mathrm{bal}}(\xi) + D^{\mathrm{balred}} p(\xi) \qquad \mathrm{modulo}(\mathrm{x}_{\mathtt{n}_{\mathrm{red}}+1}^{\mathrm{bal}}(\xi), \mathrm{x}_{\mathtt{n}_{\mathrm{red}}+2}^{\mathrm{bal}}(\xi), \ldots, \mathrm{x}_{\mathtt{n}}^{\mathrm{bal}}(\xi)).$$

Then $\begin{bmatrix} A^{\mathrm{balred}} & B^{\mathrm{balred}} \\ C^{\mathrm{balred}} & D^{\mathrm{balred}} \end{bmatrix}$ is an $\mathtt{n}_{\mathrm{red}}$-th order balancing-reduced state space model for $\mathfrak{B}_{(p,q)}$.

6 Comments

6.1

Our algorithms for obtaining the controllability and observability gramians and balanced state representations, being polynomial based, offer a number of advantages over the classical matrix based algorithms. In particular, they open up the possibility to involve the know-how on Bezoutians, Bezout and Sylvester matrices and equations, and bring 'fast' polynomial computations to bear on the problem of model reduction.

6.2

Instead of computing the $\sigma_{\mathtt{k}}$'s and the $x_{\mathtt{k}}^{\mathrm{bal}}$'s by the factorization of K, W given by (9, 10, 11), we can also obtain the balanced state representation by evaluating K and W at $\mathtt{n}$ points of the complex plane.

Let $\lambda_1, \lambda_2, \cdots, \lambda_{\mathtt{n}} \in \mathbb{C}$ be distinct points of the complex plane. Organize them into the diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{\mathtt{n}})$, and define

$$\begin{aligned} K_\Lambda &= \left[K(\lambda_{\mathtt{i}}^*, \lambda_{\mathtt{j}})\right]_{\mathtt{i}=1,\ldots,\mathtt{n}}^{\mathtt{j}=1,\ldots,\mathtt{n}} \\ W_\Lambda &= \left[W(\lambda_{\mathtt{i}}^*, \lambda_{\mathtt{j}})\right]_{\mathtt{i}=1,\ldots,\mathtt{n}}^{\mathtt{j}=1,\ldots,\mathtt{n}} \end{aligned}$$

Define further

$$\begin{aligned} X_\Lambda &= \left[x_{\mathtt{i}}^{\mathrm{bal}}(\lambda_{\mathtt{j}})\right]_{\mathtt{i}=1,\ldots,\mathtt{n}}^{\mathtt{j}=1,\ldots,\mathtt{n}} \\ \Sigma &= \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{\mathtt{n}}). \end{aligned}$$

There holds

$$K_\Lambda = X_\Lambda^* \Sigma^{-1} X_\Lambda, W_\Lambda = X_\Lambda^* \Sigma X_\Lambda.$$

It is easy to show that, since the $\lambda_{\mathtt{k}}$'s are distinct and the $x_{\mathtt{k}}^{\mathrm{bal}}$'s form a basis for $\mathbb{R}_{\mathtt{n}-1}[\xi]$, X_Λ is non-singular. This implies that X_Λ and Σ can be computed by analyzing the regular pencil formed by the Hermitian matrices K_Λ, W_Λ.

Once X_Λ is known, the balanced state representation is readily computed. However, in order to do so, we need to evaluate K (or W) at one more set of points of the complex plane. Let $\lambda_{n+1} \in \mathbb{C}$ be distinct from the λ_k's. Define $x^{bal}(\lambda_{n+1}) = \left[x_i^{bal}(\lambda_{n+1})\right]_{i=1,\dots,n}$. The vector $x^{bal}(\lambda_{n+1}) \in \mathbb{C}^n$ can be computed by solving the linear equation

$$K(\lambda_i^*, \lambda_{n+1})) = X_\Lambda^* \Sigma^{-1} x^{bal}(\lambda_{n+1}).$$

Define consecutively

$$\begin{aligned}
\Lambda_{ext} &= \text{diag}(\Lambda, \lambda_{n+1}),\\
X_{\Lambda_{ext}} &= \left[X_\Lambda \quad x^{bal}(\lambda_{n+1})\right],\\
p_{\Lambda_{ext}} &= \left[p_\Lambda \quad p(\lambda_{n+1})\right],\\
q_{\Lambda_{ext}} &= \left[q_\Lambda \quad q(\lambda_{n+1})\right].
\end{aligned}$$

Since the λ_k's are distinct, and $\{x_1^{bal}, x_2^{bal}, \cdots, x_n^{bal}, p\}$ forms a basis for $\mathbb{R}_n[\xi]$, $\begin{bmatrix} X_{\Lambda_{ext}} \\ p_{\Lambda_{ext}} \end{bmatrix}$ is also non-singular. The balanced state representation then follows by solving

$$\begin{bmatrix} X_{\Lambda_{ext}} \Lambda_{ext} \\ q_{\Lambda_{ext}} \end{bmatrix} = \begin{bmatrix} A^{bal} & B^{bal} \\ C^{bal} & D^{bal} \end{bmatrix} \begin{bmatrix} X_{\Lambda_{ext}} \\ p_{\Lambda_{ext}} \end{bmatrix}. \tag{13}$$

Note that the entries K_Λ follow immediately from (6). However, in order to compute the elements of W_Λ from (7) it seems unavoidable to have to solve (8) for f, at least, it is not clear if it is possible to evaluate the $f(\lambda_k)$'s directly from the $p(\lambda_k)$'s and $q(\lambda_k)$'s.

6.3

When we take for the λ_k's, the roots of p, assumed distinct, then f is not needed, and a very explicit expression for K and W is obtained. In this case

$$\begin{aligned}
K_\Lambda &= -\left[\frac{p(-\lambda_i^*)p(-\lambda_j)}{\lambda_i^* + \lambda_j}\right]_{i=1,\dots,n}^{j=1,\dots,n}\\
W_\Lambda &= -\left[\frac{q(\lambda_i^*)q(\lambda_j)}{\lambda_i^* + \lambda_j}\right]_{i=1,\dots,n}^{j=1,\dots,n}
\end{aligned}$$

Further, $x^{bal}(\lambda_{n+1})$ is then obtained from the linear equation

$$-\left[\tfrac{p(-\lambda_i^*)p(-\lambda_{n+1})}{\lambda_i^* + \lambda_{n+1}}\right]_{i=1,\dots,n} = X_\Lambda^* \Sigma^{-1} x^{bal}(\lambda_{n+1}).$$

Equation (13) yields

$$\begin{bmatrix} A^{bal} = X_\Lambda \Lambda X_\Lambda^{-1} & B^{bal} = \dfrac{(\lambda_{n+1} I - A^{bal}) x^{bal}(\lambda_{n+1})}{p(\lambda_{n+1})} \\ C^{bal} = q_\Lambda X_\Lambda^{-1} & D^{bal} = \dfrac{p_n}{q_n} \end{bmatrix}$$

with p_n and q_n the coefficients of ξ^n of p and q.

6.4

The balancing-reduced model is usually obtained by simply truncating the matrices of the balanced model. That is in fact what we also did in our discussion of the reduced model. However, in our algorithm, the system matrices of the balanced model are obtained by solving linear equations in $\mathbb{R}_{\mathtt{n}}[\xi]$. This suggests other possibilities for obtaining the reduced system matrices. For example, rather that solving equations (12) modulo $(x^{\rm bal}_{\mathtt{n}_{\rm red}+1}, x^{\rm bal}_{\rm nred+2}, \ldots, x^{\rm bal}_{\mathtt{n}})$, one could obtain the best least squares solution of these equations, perhaps subject to constraints, etc.

Further, by combining these least squares ideas with those of section 6.3, it may be possible to obtain balanced reductions that pay special attention to the fit of the reduced order transfer function with the original transfer function at certain privileged frequencies or selected points of the complex plane.

6.5

The algorithms discussed have obvious counterparts for discrete-time systems. It is interesting to compare our algorithm for obtaining a balanced state representation with the classical SVD-based algorithm of Kung [4]. Kung's algorithm starts from the Hankel matrix formed by the impulse response and requires the computation of the SVD of an *infinite* matrix. In contrast, our algorithm requires first finding (a least squares approximation of) the governing difference equation, followed by finite polynomial algebra.

7 Appendix

7.1 Proof of theorem 1:

Define K by (6). Note that K it is symmetric ($K = K^*$), and that the highest degree in ζ or η is $\mathtt{n} - 1$. For every $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, there holds

$$\frac{d}{dt} Q_K(w) = |p(\frac{d}{dt})w|^2 - |p(-\frac{d}{dt})w|^2. \tag{14}$$

Let $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ be given. We first prove that

$$\text{minimum} \int_{-\infty}^{0} |p(\frac{d}{dt})\ell'|^2 \, dt = Q_K(\ell)(0),$$

where the minimum is taken over all $\ell' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\ell'(t), \frac{d}{dt}\ell'(t), \ldots, \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}\ell'(t) \to 0 \text{ as } t \to -\infty \tag{15}$$

and

$$\ell'(0) = \ell(0), \frac{d}{dt}\ell'(0) = \frac{d}{dt}\ell(0), \ldots, \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}\ell'(0) = \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}\ell(0). \tag{16}$$

Integrating (14), yields

$$\int_{-\infty}^{0} |p(\frac{d}{dt})\ell'|^2 \, dt = Q_K(\ell)(0) + \int_{-\infty}^{0} |p(-\frac{d}{dt})\ell'|^2 \, dt.$$

Therefore, the minimum is obtained by the solution of $p(-\frac{d}{dt})\ell' = 0$ that satisfies the initial conditions (16). Note that it follows that $Q_K(\ell)(0) \geq 0$. Moreover, $p(-\frac{d}{dt})\ell'(t) = 0, p(\frac{d}{dt})\ell'(t) = 0$ for $t < 0$ implies $\ell'(t) = 0$ for $t < 0$, since $p(\xi)$ and $p(-\xi)$ are co-prime. Therefore the rank of Q_K is $\mathtt{n}$.

Now use a smoothness argument to show that

$$\text{infimum} \int_{-\infty}^{0} |p(\frac{d}{dt})\ell'|^2 \, dt = Q_K(\ell)(0),$$

where the infimum is taken over all $\ell' \in \mathfrak{C}^{+}(\mathbb{R},\mathbb{R})$ (instead of just having the limit conditions (15)) such that $\ell'(t) = \ell(t)$ for $t \geq 0$ (hence ℓ and ℓ' must be glued at $t = 0$ in a $\mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})$ way, instead of just by the initial conditions (16)). This implies that Q_K is indeed the controllability gramian.

Next, consider W, defined by (7, 8). For every $w \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})$ here holds

$$\frac{d}{dt}Q_W(w) = -|q(\frac{d}{dt})w|^2 + 2p(\frac{d}{dt})w \;\; f(\frac{d}{dt})w. \tag{17}$$

Note further that W is symmetric ($W = W^*$), and that the highest degree in ζ or η is $\mathtt{n} - 1$. Therefore, if $\ell' \in \mathfrak{D}(\mathbb{R},\mathbb{R})$ is such that $p(\frac{d}{dt})\ell'(t) = 0$ for $t \geq 0$ there holds, by integrating (17) and using the fact that p is Hurwitz,

$$\int_{0}^{\infty} |q(\frac{d}{dt})\ell'|^2 \, dt = Q_W(\ell')(0).$$

Let $\ell \in \mathfrak{C}^{\infty}(\mathbb{R},\mathbb{R})$ be given, and assume that (i) $\ell_{(-\infty,0)} = \ell'|_{(-\infty,0)}$, and (ii) $(p(\frac{d}{dt})\ell', q(\frac{d}{dt})\ell') \in \mathfrak{B}_{(p,q)}$. Then $\ell'|_{(-\infty,0]}$ is actually a function, with

$$\ell'(0) = \ell(0), \frac{d}{dt}\ell'(0) = \frac{d}{dt}\ell'(0), \ldots, \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}\ell'(0) = \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}\ell'(0).$$

Therefore, if, in addition to (i) and (ii), (iii) holds: $p(\frac{d}{dt})\ell'(t) = 0$ for $t \geq 0$, we obtain

$$\int_{0}^{\infty} |q(\frac{d}{dt})\ell'|^2 \, dt = Q_W(\ell)(0).$$

It follows that Q_W defines the observability gramian. That $Q_K \geq 0$ is immediate, and that its rank is $\mathtt{n}$ follows from the fact that p and q are co-prime.

7.2 Proof of theorem 2:

Using (10,11), we obtain

$$Q_K(\ell) = \Sigma_{k=1}^{\mathtt{n}} \sigma_{\mathtt{k}}^{-1} |x_{\mathtt{k}}^{\mathrm{bal}}(\frac{d}{dt})\ell|^2,$$

and

$$Q_W(\ell) = \Sigma_{\mathtt{k}=1}^{\mathtt{n}} \sigma_{\mathtt{k}} |x_{\mathtt{k}}^{\mathrm{bal}}(\frac{d}{dt})\ell|^2.$$

Hence, if $\ell_{\mathtt{i}} \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ is such that $x_{\mathtt{i}}^{\mathrm{bal}}(\frac{d}{dt})\ell_{\mathtt{i}}(0) = \delta_{\mathtt{ij}}$, then

$$Q_K(\ell_{\mathtt{i}})(0) = \sigma_{\mathtt{i}}^{-1}, \text{ and } Q_W(\ell_{\mathtt{i}})(0) = \sigma_{\mathtt{i}}.$$

This shows that the $x_{\mathtt{k}}^{\mathrm{bal}}$'s define a balanced state representation, as claimed. That the $\sigma_{\mathtt{k}}$'s are the Hankel SV's of $\mathfrak{B}_{(p,q)}$ is a standard consequence of the theory of balanced state representations.

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