PATH INTEGRALS AND STABILITY *

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Abstract

A path integral associated with a dynamical system is an integral of a memoryless function of the system variables which, when integrated along trajectories of the system, depends only on the value of the trajectory and its derivatives at the endpoints of the integration interval. In this paper we study path independence for linear systems and integrals of quadratic differential forms. These notions and the results are subsequently applied to stability questions. This leads to Lyapunov stability theory for autonomous systems described by high-order differential equations, and to more general stability concepts for systems in interaction with their environment. The latter stability issues are intimately related to the theory of dissipative systems.

Keywords: Path integrals, path independence, differential systems, stability, Lyapunov theory, dissipative systems, quadratic differential forms, behaviors.

1 Introduction

Contributing this paper to the Festschrift for Roger Brockett on the occasion of his 60-th birthday is a special privilege for me. Roger Brockett influenced my scientific work in a deep way, directly and indirectly. Directly, as my thesis supervisor and teacher in the mid-sixties when the field of systems and 'modern' (as it was called then) control theory was relatively new and undeveloped, and as a frequent co-author. Indirectly, as the person who in many ways helped shape my own scientific taste and attitudes.

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The second half of the twentieth century has witnessed an intensive development in the use of mathematical ideas in engineering. This development has, in some ways, been a successful one, but in other ways, it has shown to be a rocky road. Roger Brockett's scientific career coincides with this era and he has lived and, in his environment, led this uneasy symbiosis between mathematics and engineering in a very successful way. As this volume is testimony of, Roger Brockett's work has had many different facets and touched most aspects of systems and control in one way or another. In the sixties, including the period during which I was his graduate student at MIT, his interests centered on stability problems and optimal control. In this period he developed a very effective technique for constructing Lyapunov functions that allowed to prove stability and instability of feedback systems [1, 2, 4, 5]. These ideas enabled to obtain Lyapunov proofs of the circle criterion (including the instability part), the Popov theorem, and many other results involving various classes of nonlinear systems and multipliers. This method was based on treating systems described by high-order differential equations, instead of the representations by state space equations or input-output operators which were in vogue at that time. Considerations involving the path independence of certain integrals of quadratic expressions involving the system variables and their derivatives played a central role in this work. Later on, he applied these ideas also to find elegant spectral factorization algorithms for linear-quadratic problems [3, 5].

Unfortunately, these methods did not become part of the mainstream of the field, and few researchers seem to have mastered these techniques, to the point that they seem somewhat forgotten. The purpose of this paper is to explain the ideas underlying path independence, apply them to the construction of Lyapunov functions, and show the implication of these methods in stability analysis. An effective new tool which we have recently introduced in this context is the use of two-variable polynomial matrices [24]. These play the same role for dealing with quadratic functionals in the system variables and their derivatives that one-variable polynomial matrices play in representing linear differential systems. Because of length considerations, we will treat in this paper only stability questions. However, the techniques can also be applied very effectively in optimal control problems (see [18]). The material discussed in this paper shows some overlap with related papers [24, 18] where, however, the theory is carried much further.

In order to make the paper reasonably self-contained and easy to follow, we have added some background material in the appendix. The notation is explained in appendix A; appendix B contains a brief introduction on behavioral systems; and the proofs of the propositions and the theorems are given in appendix C.

2 Path independence

In this section we study when an integral of a quadratic expression in a set of variables and their derivatives is independent of path. We refer to appendix A for the notation used, in particular for polynomial matrices, and for the definition of the operators $\star, *, ^\bullet$, and ∂, which play an important role throughout this paper.

Consider the symmetric two-variable polynomial matrix $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$, written out in terms of its coefficient matrices $\Phi_{k\ell} = \Phi_{\ell k}^T \in \mathbb{R}^{q\times q}$ as $\Phi(\zeta, \eta) = \sum\limits_{k,\ell} \Phi_{k,\ell}\zeta^k\eta^\ell$. The map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{k,\ell} \left(\frac{d^k}{dt^k}w\right)^T \Phi_{k\ell} \left(\frac{d^\ell}{dt^\ell}w\right)$$

is called the *quadratic differential form* (QDF) *induced by* Φ.

The integral expression

$$\int_{t_1}^{t_2} Q_\Phi(w)dt \tag{1}$$

(or, briefly, $\int Q_\Phi$) is said to be *independent of path*, or a *path integral*, if it depends only on the values taken on by w and its derivatives at t_1 and t_2. More precisely, if for any $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ and $t_1', t_1'', t_2', t_2'' \in \mathbb{R}$ such that

$$\frac{d^k}{dt^k}w_1(t_1') = \frac{d^k}{dt^k}w_2(t_1'') \text{ and } \frac{d^k}{dt^k}w_1(t_2') = \frac{d^k}{dt^k}w_2(t_2''), \tag{2}$$

for $k = 0, 1, 2, \ldots$, the equality

$$\int_{t_1'}^{t_2'} Q_\Phi(w_1)dt = \int_{t_1''}^{t_2''} Q_\Phi(w_2)dt$$

holds.

In order to fix the ideas, consider the following two trivial examples. Observe that $Q_\Phi(w) = w^T w$ leads to an integral that is not independent of path, while $Q_\Phi(w) = w^T \frac{d}{dt}w$ does. Indeed, in the latter case the integral (1) equals $\frac{1}{2}(w(t_2)^T w(t_2) - w(t_1)^T w(t_1))$, whence path independence is evident.

The question arises what conditions on the two-variable polynomial matrix Φ lead to path independence. The most direct test on Φ is singled out in the following proposition.

Proposition 1 : *$\int Q_\Phi$ is a path integral if and only if $\partial\Phi = 0$, i.e., if and only if $\Phi(-\xi, \xi) = 0$.*

It turns out that path independence is equivalent to many other useful conditions, some of which are collected in the following theorem.

Theorem 2 : *Let $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$. The following conditions are equivalent:*

(i) $\int Q_\Phi$ *is a path integral;*

(ii) $\partial\Phi = 0$*;*

(iii) $\int_{-\infty}^{+\infty} Q_\Phi(w)dt = 0$ *for all* $w \in \mathfrak{D}(\mathbb{R},\mathbb{R}^q)$*;*

(iv) the expression

$$\frac{\Phi(\zeta,\eta)}{\zeta+\eta}$$

is a polynomial matrix, i.e., there exists a two-variable polynomial matrix $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ *such that* $\overset{\bullet}{\Psi} = \Phi$*;*

(v) there exists a two-variable polynomial matrix $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ *such that*

$$\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) \tag{3}$$

for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$*. Note this equality implies that the integral (1) equals* $Q_\Psi(w)(t_2) - Q_\Psi(w)(t_1)$*, which puts path independence into evidence.*

The above theorem tells us when path independence holds for smooth but otherwise unconstrained trajectories $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$. Of course, a casual examination of the arguments used in the proofs shows that $\mathfrak{C}^\infty$ is much more smoothness than is needed: only the derivatives appearing in Q_Φ are required to exist (note that this also holds for the number of derivatives for which equality is required in (2)). We are, however, interested in a more meaningful avenue of generalization. In many applications, we need path independence only for trajectories that are generated by a given dynamical system, i.e., for the w's that satisfy certain dynamical equations.

Consider the dynamical system $\mathfrak{B} \in \mathfrak{L}^q$ (see appendix B for an introduction to linear differential systems), and the QDF induced by $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$. The integral (1) is said to be *independent of path* or a *path integral along* $\mathfrak{B}$ if the path independence condition explained earlier holds for all $w_1, w_2 \in \mathfrak{B}$. Let $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, with $R \in \mathbb{R}^{\bullet\times q}[\xi]$. The question now is to find conditions on Φ and R for path independence along $\ker(R(\frac{d}{dt}))$. The key to this is given by the following polynomial equation relating the one-variable polynomial matrices $X \in \mathbb{R}^{\bullet\times q}[\xi]$ and $Z = Z^\star \in \mathbb{R}^{q\times q}[\xi]$ to $R \in \mathbb{R}^{\bullet\times q}[\xi]$:

$$X^\star R + R^\star X = Z. \tag{4}$$

It is appropriate to call this equation the *polynomial Lyapunov equation.* It is a generalization of the matrix Lyapunov equation in the square matrices $A, X, Z = Z^T \in \mathbb{R}^{q\times q}$ related by $XA + A^T X = Z; X = X^T$, (which may be written as $X^\star(I\xi + A) + (I\xi + A)^\star X = Z$, in order to make it look more like (4)). This equation is studied extensively in matrix theory and first-order state equations, and the polynomial Lyapunov equation plays an analogous role in applications of systems described by high-order differential equations.

Proposition 3 : *$\int Q_\Phi$ is a path integral along $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ if there exists a $X \in \mathbb{R}^{\bullet\times q}[\xi]$ such that (4) holds with $Z = \partial\Phi$. This condition is also necessary if $\mathfrak{B}$ is controllable.*

Note that in the unconstrained case ($R = 0$), this result specializes to proposition 1. As is the case in the unconstrained case, path independence is again equivalent to a number of other insightful conditions, which we state here for controllable systems only (see appendix B for the notion of controllability in the context of behaviors).

Theorem 4 : *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a controllable system, $\mathfrak{B} = \ker(R(\frac{d}{dt})) = im(M(\frac{d}{dt}))$, and $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$. The following conditions are equivalent:*

(i) $\int Q_\Phi$ *is a path integral along* $\mathfrak{B}$*;*

(ii) *the polynomial Lyapunov equation (4) with* $Z = \partial\Phi$ *has a solution* $X \in \mathbb{R}^{\bullet\times q}[\xi]$*;*

(iii) *there exist* $Y \in \mathbb{R}^{\bullet\times q}[\zeta, \eta]$ *and* $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ *such that*

$$\overset{\bullet}{\Psi}(\zeta, \eta) = \Phi(\zeta, \eta) + Y^*(\zeta, \eta)R(\eta) + R^T(\zeta)Y(\zeta, \eta), \tag{5}$$

(iv) *there exists* $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ *such that*

$$\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) \tag{6}$$

for all $w \in \mathfrak{B}$*;*

(v) $\int Q_{\Phi'}$ *is a path integral, where* Φ' *is defined by* $\Phi'(\zeta, \eta) := M^T(\zeta)\Phi(\zeta, \eta)M(\eta)$*;*

(vi) $\partial\Phi' = 0$;

(vii) there exists $\Psi' = \Psi'^* \in \mathbb{R}^{q\times q}(\zeta,\eta)$ *such that*

$$\frac{d}{dt}Q_{\Psi'}(\ell) = Q_{\Phi'}(\ell)$$

for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{coldim(M)})$, *i.e.,* $\overset{\bullet}{\Psi'} = \Phi'$.

The immediate conclusion that may be drawn from theorems 2 and 4 is the equivalence — for controllable linear differential systems — of path independence and the fact that the QDF Q_Φ is a perfect differential, expressed by equations (3) and (6). The polynomial matrix Ψ which expresses this property plays an important role in applications. It may be computed using the polynomial Lyapunov equation (4) as follows. Let $X \in \mathbb{R}^{\bullet\times q}[\xi]$ and $Y \in \mathbb{R}^{\bullet\times q}[\zeta,\eta]$ be related by $\partial Y = -X$. Then Y satisfies (5) for some Ψ if and only if X satisfies (4) with $Z = \partial\Phi$.

Hence, in order to compute Ψ, solve first (4) for X with $Z = \partial\Phi$. From the solution X obtained, use $\partial Y = -X$, to deduce a Y. Then Ψ obtained by

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) + Y^*(\zeta,\eta)R(\eta) + R^T(\zeta)Y(\zeta,\eta)}{\zeta+\eta} \tag{7}$$

is a two-variable polynomial matrix that leads to a QDF satisfying $\frac{d}{dt}Q_\Psi(w) = Q_\Psi(w)$ for $w \in \ker(R(\frac{d}{dt}))$.

The solvability of the polynomial Lyapunov equation (4) with $Z = \partial\Phi$ appears to be the key to path independence. Its solvability is always a sufficient condition for path independence and in the controllable case, it is necessary as well. Moreover, we have seen how a solution X to (4) with $\partial\Phi = Z$ leads, via Y such that $\partial Y = -X$, and (7) to a Ψ that puts path independence into evidence. Unfortunately, controllability is not a superfluous condition. In order to illustrate this, consider the autonomous (see appendix B for the definition of this notion) — hence non-controllable — system described by $w - \frac{d^2}{dt^2}w = 0$. Its behavior equals all linear combinations of e^t and e^{-t}. Consider furthermore the QDF defined by w^2, i.e., $\Phi = 1$. Then path independence holds trivially since there is at most one path of this dynamical system connecting any set of initial and terminal conditions $w(t_1) = a_0, \frac{d}{dt}w(t_1) = a_1, w(t_2) = b_0, \frac{d}{dt}w(t_2) = b_1$. In fact, it is possible to give an explicit expression for the path integral $\int w^2$ in terms of the endpoints. Indeed,

$$\int_{t_1}^{t_2} w^2 dt = \frac{1}{2}(w(t_2)\frac{d}{dt}w(t_2) - w(t_1)\frac{d}{dt}w(t_1)) + \alpha((w+\frac{d}{dt}w)(t_1), (w+\frac{d}{dt}w)(t_2))$$

where $\alpha(y,z) = 0$ if $y = 0$, and $= \frac{1}{2}\log\frac{z}{y}$ if $y \neq 0$. In the case at hand, (4) with $\partial\Phi = 1$ becomes $(1-\xi^2)(X(\xi) + X(-\xi)) = 1$, which obviously is not solvable for $X \in \mathbb{R}[\xi]$. This example shows that path independence along $\mathfrak{B} \in \mathfrak{L}^q$ is in the non-controllable case unfortunately simply not equivalent to solvability of the polynomial Lyapunov equation (4) with $Z = \partial\Phi$. There are various ways in which equivalence can nevertheless be enforced. For example, by requiring that the integral (1) depends quadratically on w and its derivatives at the end-points. This idea is obviously restricted to linear systems and quadratic differential forms. Another way is to extend the set of solutions of $R(\frac{d}{dt})w = 0$ so as to allow trajectories that pass through infinity. In fact, it is the presence of the time-reversible non-periodic solutions that cause the difficulty.

3 Positivity of quadratic differential forms

Positivity of QDF's plays, as is to be expected, an important role in applications, for example as Lyapunov functions in stability analysis, for determining the sign of the second variation in optimal control, etc.

Let $\Phi_1^*, \Phi_2^* \in \mathbb{R}^{q\times q}[\zeta,\eta], \Phi_1 = \Phi_1^*, \Phi_2 = \Phi_2^*$, and $\mathfrak{B} \in \mathfrak{L}^q$. We call Q_{Φ_1} and Q_{Φ_2} (or Φ_1 and Φ_2) $\mathfrak{B}$-*equivalent* (denoted $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$) if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for all $w \in \mathfrak{B}$. Thus $\Phi_1 \overset{\mathfrak{B}}{=} \Phi_2$ if and only if $Q_{\Phi_1 - \Phi_2} \overset{\mathfrak{B}}{=} 0$.

The QDF induced by $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ is said to be *nonnegative* (denoted $\Phi \geq 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q)$, and *positive* (denoted $\Phi > 0$) if it is nonnegative and if $Q(w) = 0$ implies $w = 0$. It is easy to see that Φ is nonnegative if and only if there exists $D \in \mathbb{R}^{\bullet\times q}[\xi]$ such that $\Phi(\zeta,\eta) = D^T(\zeta)D(\eta)$, and positive if and only if in addition $\text{rank}(D(\lambda)) = q$ for all $\lambda \in \mathbb{C}$. This last condition may be interpreted as stating that in the system $d = D(\frac{d}{dt})w, w$ must be observable from d (see appendix B for the behavioral definition of observability). It is important to have also positivity concepts when QDF's are evaluated along trajectories of a system. Let $\mathfrak{B} \in \mathfrak{L}^q$. The QDF induced by $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ is said to be $\mathfrak{B}$-*nonnegative* (denoted $\Phi \overset{\mathfrak{B}}{\geq} 0$) if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$ and $\mathfrak{B}$-*positive* (denoted $\Phi \overset{\mathfrak{B}}{>} 0$) if it is $\mathfrak{B}$-nonnegative and if $Q_\Phi(w) = 0$ and $w \in \mathfrak{B}$ imply $w = 0$.

Let $\mathfrak{B} = \ker(R(\frac{d}{dt}))$. The following proposition gives conditions of $\mathfrak{B}$-positivity in terms of Φ and R.

Proposition 5 : *Let $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ and $\mathfrak{B} = \ker(R(\frac{d}{dt})) \in \mathfrak{L}^q$.*

(i) $\Phi \overset{\mathfrak{B}}{=} 0$ *if and only if there exists $Y \in \mathbb{R}^{\bullet\times q}[\zeta,\eta]$ such that*

$$\Phi(\zeta,\eta) = Y^*(\zeta,\eta)R(\eta) + R^T(\zeta)Y(\zeta,\eta);$$

(ii) Φ *is* $\mathfrak{B}$*-nonnegative if and only if there exists* $D \in \mathbb{R}^{\bullet \times q}[\xi]$ *such that* $\Phi(\zeta, \eta) \stackrel{\mathfrak{B}}{=} D^T(\zeta)D(\eta)$*;*

(iii) Φ *is* $\mathfrak{B}$*-positive if and only if in addition* $\mathrm{rank}(\mathrm{col}[R(\lambda), D(\lambda)]) = q$ *for all* $\lambda \in \mathbb{C}$.

The rank condition in this proposition admits again an interpretation in terms of observability. Indeed, it is equivalent to the requirement that in the system $R(\frac{d}{dt})w = 0, d = D(\frac{d}{dt})w$, w is observable from d (i.e., if (R, D) is an observable pair — see appendix B for what it means that a pair of polynomial matrices is observable); equivalently if in this system $d = 0$ implies $w = 0$.

Quadratic differential forms that are $\mathfrak{B}$-positive can only be zero when the trajectory along which it is evaluated is zero. However, when the system $\mathfrak{B}$ is autonomous, which is the case of interest in Lyapunov theory, it is useful to consider also a stronger concept of positivity. Let $\mathfrak{B} \in \mathfrak{L}^q$ and $\Phi = \Phi^* \in \mathbb{R}^{q \times q}[\zeta, \eta]$; we call Φ *strongly* $\mathfrak{B}$*-positive* (denoted $\Phi \stackrel{\mathfrak{B}}{\gg} 0$) if $\Phi \stackrel{\mathfrak{B}}{\geq} 0$ and if $w \in \mathfrak{B}$ and $Q_\Phi(w)(0) = 0$ imply $w = 0$. Thus in this case $Q_\Phi(w) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ is zero if and only if it is zero at one point. Obviously, $\Phi \stackrel{\mathfrak{B}}{\gg} 0$ implies $\Phi \stackrel{\mathfrak{B}}{>} 0$. It is possible to prove that in order for Φ to be strongly $\mathfrak{B}$-positive, $\mathfrak{B}$ must necessarily be autonomous.

4 Lyapunov theory for high-order differential equations.

State-of-the-art Lyapunov theory pertains to systems described by explicit first-order differential equations. However, first-principles-models are seldom in this form, they may contain high-order derivatives, they usually involve auxiliary (latent) variables, and invariably there will be implicit (e.g., algebraic) equations. Writing them in explicit first-order form may not be easy, nor desirable: the transformations required to bring them in explicit first-order form will upset the parametric integrity of the equations, and may introduce spurious solutions. However, stability analysis does not require systems to be in explicit first-order form. Historically, in fact, the very first stability questions and results, as Maxwell's stability analysis of governors for steam engines and the Routh-Hurwitz conditions which it led to, pertain to high-order differential equations. It is only since the work of Lyapunov and Poincaré that stability analysis has been preoccupied by explicit first-order models. Under the influence of the work of Kalman, systems and control theory has also succumbed to this fashion, indeed with great conviction, one may say. Oddly enough, to our knowledge, no attempts seem to have been made to establish Lyapunov theory for high-order differential equations. The purpose of this section is to set up such a theory. We limit attention to linear differential systems and to Lyapunov functions that are quadratic differential forms.

First, we introduce the notion of stability. In this section, we restrict attention to autonomous differential systems in $\mathfrak{L}$. More general notions of stability will be considered in section 7. An autonomous $\mathfrak{B} \in \mathfrak{L}^q$ is said to be *asymptotically stable* if $w \in \mathfrak{B}$ implies $w(t) \overset{t\to\infty}{\longrightarrow} 0$. Let $R \in \mathbb{R}^{\bullet\times q}$. The complex number $\lambda \in \mathbb{C}$ is said to be a *singularity* of R if $\text{rank}(R(\lambda)) < \text{rank}(R)$; R is said to be *Hurwitz* if $\text{rank}(R) = q$ and if R has all its singularities in the open left half of the complex plane. Thus a square polynomial matrix $R \in \mathbb{R}^{q\times q}[\xi]$ is Hurwitz if and only if $\det(R)$ is a *Hurwitz polynomial*, i.e., a non-zero polynomial with its roots in the open left half plane. From the elementary theory of differential equations, we conclude that $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ is asymptotically stable if and only if R is Hurwitz.

We now examine how asymptotic stability may be deduced from the behavior of QDF's along the trajectories of $\mathfrak{B}$, and what asymptotic stability implies regarding the existence of suitable QDF's that may serve as Lyapunov functions. Our most basic Lyapunov theorem regarding high-order systems is the following. In the remainder of this section, we will use the notation Ψ for the QDF that serves as the Lyapunov function, and Φ for its derivative.

Theorem 6 : *Let $\mathfrak{B} \in \mathfrak{L}^q$. Then $\mathfrak{B}$ is asymptotically stable if and only if there exists $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ such that $\Psi \overset{\mathfrak{B}}{\geq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{<} 0$.*

The point of theorem 6 is two-fold: it avoids the state construction which algorithmically (and conceptually) is not always easy in the multi-variable case, and it has the usual Lyapunov theory as a special case, by applying it to systems in first-order form and using memoryless QDF's.

As an illustration, consider the multi-variable system described by

$$Kw + D\frac{d}{dt}w + M\frac{d^2}{dt^2}w = 0,$$

with $K, D, M \in \mathbb{R}^{q\times q}$, $K = K^T \geq 0$, $D + D^T \geq 0$, and $M = M^T \geq 0$. Second order equations of this type occur frequently as models of mechanical systems, with elasticity (the K-term), viscous damping (the D-term), and inertial effects (the M-term), all of which may be negligible. In order to analyze the stability of this system, consider $\Psi(\zeta, \eta) = K + M\zeta\eta$. Then $\overset{\bullet}{\Psi}(\zeta, \eta) = K(\zeta + \eta) + M(\zeta^2\eta + \zeta\eta^2)$ which is obviously $\mathfrak{B}$-equivalent to $-(D+D^T)\zeta\eta$. Thus asymptotic stability follows if $\text{col}[K + D\lambda + M\lambda^2, \sqrt{(D + D^T)}\lambda]$ has full column rank for all $\lambda \in \mathbb{C}$. This is the case, for example, if $\{0\} = \ker(K) \subseteq \ker(D + D^T) \subseteq \ker(M)$.

In our basic Lyapunov theorem, theorem 6, we only need nonnegativity of the Lyapunov function. However, positivity, in fact, strong positivity, can be concluded, as is shown in the following proposition.

Proposition 7 : *Let* $\mathfrak{B} \in \mathfrak{L}^q$, *and* $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$. *If* $\Psi \overset{\mathfrak{B}}{\geq} 0$ *and* $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{<} 0$, *then* $\mathfrak{B}$ *is asymptotically stable and* $\Psi \overset{\mathfrak{B}}{\gg} 0$.

It is well-known that in linear asymptotically stable state space systems, one can always choose the derivative of the Lyapunov function, and construct the Lyapunov function accordingly. The same holds for the case at hand, leading to the following stronger version of the 'only if' part of theorem 6.

Theorem 8 : *Assume that* $\mathfrak{B} \in \mathfrak{L}^q$ *is asymptotically stable. Then for any* $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ *there exists a* $\Psi = \Psi^\star \in \mathbb{R}^{q\times q}[\zeta,\eta]$ *such that* $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{=} \Phi$. *If* $\Phi \overset{\mathfrak{B}}{\leq} 0$, *then* $\Psi \overset{\mathfrak{B}}{\geq} 0$, *and if* $\Phi \overset{\mathfrak{B}}{<} 0$, *then* $\Psi \overset{\mathfrak{B}}{\gg} 0$. *The following algorithm allows to compute* Ψ *from* Φ. *Let* $\mathfrak{B} = \ker(R(\frac{d}{dt}))$. *Then the polynomial Lyapunov equation (4) with* $Z = \partial\Phi$ *has a solution* $X \in \mathbb{R}^{q\times q}[\xi]$. *Take any* $Y \in \mathbb{R}^{\bullet\times q}[\zeta,\eta]$ *such that* $\partial Y = -X$ *and compute* $\Psi(\zeta,\eta)$ *using equation (7). Then* $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{=} \Phi$.

The above results allow generalizations in various directions, in particular to unstable systems. Let us briefly mention a few. We have seen that $\mathfrak{B}$ is asymptotically stable if and only if there exists $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\Psi \overset{\mathfrak{B}}{\geq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{<} 0$. There also holds that $\mathfrak{B}$ is stable (in the sense that all solutions are bounded on the half line $[0,\infty)$) if and only if there exists $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\Psi \overset{\mathfrak{B}}{>} 0$ and $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{\leq} 0$. Furthermore an autonomous $\mathfrak{B}$ is not stable if there exists $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\Psi \overset{\mathfrak{B}}{\not\geq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{<} 0$. However, the result that a Lyapunov function Q_Ψ can be constructed so that it has a given derivative Q_Φ is of course not always true for systems that are not asymptotically stable. It is the case when the singularities of R and $R^\star$ are distinct. More generally, the derivative can be taken to be Q_Φ if and only if the polynomial Lyapunov equation (4) with $Z = \partial\Phi$ has a solution X. As such the construction of Lyapunov functions via equation (4) extends to a large class of unstable systems and Φ's.

The above results are related to path integrals. When $\mathfrak{B} \in \mathfrak{L}^q$ is autonomous, then an element $w \in \mathfrak{B}$ is uniquely specified by its initial conditions $w(0)$, $\frac{d}{dt}w(0)$, $\frac{d^2}{dt^2}w(0), \ldots$, in fact, by a finite subset of these. From there, it follows in the case of asymptotically stable systems (but not for general autonomous systems) that any QDF is independent of path. Indeed,

$$\int_{t'}^{\infty} Q_\Phi(w)dt$$

obviously depends only on $w(t'), \frac{d}{dt}w(t'), \frac{d^2}{dt^2}w(t'), \ldots$. Hence since

$$\int_{t_1}^{t_2} Q_\Phi(w)dt = \int_{t_1}^{\infty} Q_\Phi(w)dt - \int_{t_2}^{\infty} Q_\Phi(w)dt,$$

path independence of $\int Q_\Phi$ along $\mathfrak{B}$ follows. The construction of a Lyapunov function can therefore be carried out by integrating $Q_\Phi(w)$ and computing the QDF Q_Ψ such that $Q_\Psi(w)(0) = -\int_0^\infty Q_\Phi(w)dt$. This obviously comes down to finding $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\overset{\bullet}{\Psi} = \Phi$. Imposing suitable positivity and observability conditions on Φ leads to $\mathfrak{B}$-positivity Ψ.

Theorem (8) shows how the polynomial Lyapunov equation (4) may be used for constructing a Lyapunov function for asymptotically stable systems $\mathfrak{B} = \ker(R(\frac{d}{dt})) \in \mathfrak{L}^q$. There are two basic ways of proceeding. In the first one, choose $Z = Z^\star \in \mathbb{R}^{q\times q}[\xi]$ such that $Z(i\omega) < 0$ for all $\omega \in \mathbb{R}$. It is well-known from the theory of spectral factorization that Z may then be factored as $Z = D^\star D$ with $D \in \mathbb{R}^{q\times q}[\xi]$. Hence taking $\Phi(\zeta,\eta) = -D^T(\zeta)D(\eta)$ leads to a $\Phi \leq 0$ such that $\partial\Phi = Z$. Then solve (4) for X, take any Y such that $\partial Y = -X$, and use (7) to compute a $\Psi \overset{\mathfrak{B}}{\geq} 0$ such that $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{=} \Phi$. If Z is chosen such that (R,D) is an observable pair, then $\Phi \overset{\mathfrak{B}}{<} 0$ and $\Psi \overset{\mathfrak{B}}{\gg} 0$.

In the second method, the polynomial Lyapunov equation (4) is used by choosing an X such that Z has the required properties. Assume $R \in \mathbb{R}^{q\times q}[\xi]$. Then asymptotic stability implies that $R^{-1} \in \mathbb{R}^{q\times q}(\xi)$ is analytic in the closed right half plane. Now choose $X \in \mathbb{R}^{q\times q}[\xi]$ such that $-XR^{-1}$ is positive real (meaning $XR^{-1}(i\omega)+(XR^{-1})^T(-i\omega) \leq 0$ for all $\omega \in \mathbb{R}$). Then $X^\star(i\omega)R(i\omega)+R^\star(i\omega)X(i\omega) \leq 0)$ for all $\omega \in \mathbb{R}$, and hence we are in the situation of the previous paragraph, leading to $\Phi \leq 0$ such that $\Psi \overset{\mathfrak{B}}{\geq} 0$ and $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{=} \Phi$. If we choose X such that $-XR^{-1}$ is strictly positive real (meaning $X^\star R + R^\star X = -D^\star D$ with (R,D) observable — this is but one of many possible definitions of strict positive realness of $-XR^{-1}$), then $\Phi \overset{\mathfrak{B}}{<} 0$ and $\Psi \overset{\mathfrak{B}}{\gg} 0$.

Thus the polynomial Lyapunov equation (4) leads to two alternative (but related) procedures for constructing Lyapunov functions: choosing $Z = Z^\star$ such that $Z(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$ and solving for X, or choosing X such that $-XR^{-1}$ is strictly positive real.

5 The Bezoutian

We will now use the results of the previous section in order to obtain the Bezoutian as a universal Lyapunov function for scalar systems, and deduce Lyapunov proofs of the Routh test for the stability of scalar differential equations and of the Kharitonov theorem for interval families of polynomials. None of the results that will be obtained are novel. However, we believe that the proofs are of some pedagogical interest, in the spirit and elegance of the mathematical arguments in [5].

Since we consider in this section only scalar systems, we use, in keeping with tradition, the notation $p \in \mathbb{R}[\xi]$ for the polynomial that defines the differential system whose stability is at issue. Consider the n-th order scalar system

$$p_0 w + p_1 \frac{d}{dt} w + \cdots + p_{n-1} \frac{d^{n-1}}{dt^{n-1}} w + p_n \frac{d^n}{dt^n} w = 0, \tag{8}$$

with $p_0, p_1, \cdots, p_{n-1}, p_n \in \mathbb{R}$, with $p_n \neq 0$. Let $p(\xi) = p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + p_n\xi^n$ denote the polynomial associated with (8). A classical problem going back to Maxwell, Routh, and Hurwitz is to find conditions on the coefficients $(p_0, p_1, \cdots, p_{n-1}, p_n)$ of p for it to be Hurwitz, i.e., for its roots to lie in the open left half of the complex plane. An effective way to obtain such conditions is by considering the following two-variable polynomial

$$B_p(\zeta, \eta) = \frac{p(\zeta)p(\eta) - p(-\zeta)p(-\eta)}{\zeta + \eta}. \tag{9}$$

Note that since the numerator of the right hand side of (9) is zero when evaluated at $\zeta = -\xi$ and $\eta = \xi$, B_p is indeed a two-variable polynomial. Expressed in the even and odd part of $p, p = E_p + O_p$, with $E_p = \frac{1}{2}(p + p^\star)$ and $O_p = \frac{1}{2}(p - p^\star)$, B_p becomes

$$B_p(\zeta, \eta) = 2\frac{E_p(\zeta)O_p(\eta) + O_p(\zeta)E_p(\eta)}{\zeta + \eta}. \tag{10}$$

Write $B_p(\zeta, \eta) = \sum_{k,\ell=0}^{n-1} b_{k,\ell}\zeta^k\eta^\ell$. Obviously, $B_p = B_p^*$, whence $b_{k,\ell} = b_{\ell,k}$. Define the rank of the QDF Q_{B_p} to be that of the associated symmetric $(n \times n)$ matrix

$$\tilde{B}_p = \begin{bmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} \\ b_{1,0} & b_{1,1} & \cdots & b_{1,n-1} \\ \vdots & \vdots & & \vdots \\ b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-1} \end{bmatrix}.$$

The two-variable polynomial B_p (or the matrix $\tilde{B}_p$) is called the *Bezoutian* of p. It is one of the very classical objects [7, 8] studied in the interaction of polynomials/matrices/linear systems. Note that $\overset{\bullet}{Q}_{B_p}(w)$ evaluated along solutions of (8) equals $-|p(-d/dt)w|^2$, which shows that Q_{B_p} is suitable as a Lyapunov function for (8).

Theorem 9 : *The following conditions are equivalent:*

(i) the system defined by (8) is asymptotically stable;

(ii) p is Hurwitz;

(iii) $Q_{B_p} \geq 0$ and p and $p^\star$ are co-prime polynomials;

(iv) $\tilde{B}_p$ is positive definite.

The proof of the above theorem shows that the QDF defined by the Bezoutian is a universal Lyapunov function for scalar autonomous differential systems: it can be written down directly from the system equation, and a system is asymptotically stable if and only if the QDF induced by the Bezoutian as a Lyapunov function shows it to be. Actually, this generalizes to the multi-variable case, with the Bezoutian associated with $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ defined by

$$B_R(\zeta,\eta) = \frac{R^T(\zeta)R(\eta) - R^T(-\eta)R(-\zeta)}{\zeta+\eta}.$$

However, it appears much more difficult to obtain concrete results from this multi-variable version.

The Bezoutian as a Lyapunov function can be deduced from the procedure explained in the previous section: take in the Lyapunov equation $X = -p/2, Z = -p^\star p, \Phi(\zeta,\eta) = -p(-\zeta)p(-\eta)$ and $Y(\zeta,\eta) = p(\eta)/2$. Using (7), this results in $\Psi = B_p$. There are other ways to obtain this Lyapunov function. For example, take $X = -E_p$, $Z = -2E_p^2$, $\Phi(\zeta,\eta) = -2E_p(\zeta)E_p(\eta)$, and $Y(\zeta,\eta) = E_p(\zeta)$; alternatively, $X = O_p$, $Z = -2O_p^2, \Phi(\zeta,\eta) = -2O_p(\zeta)O_p(\eta)$, and $Y(\zeta,\eta) = O_p(\eta)$. These also lead to $\Psi = B_p$. Of course, the three Φ's obtained are all $\ker(p(\frac{d}{dt}))$-equivalent.

5.1 The Routh test

Consider the polynomial p. Write it in terms of its even $E_p(\xi) = E_1(\xi^2)$, and odd part as $O_p(\xi) = \xi E_2(\xi^2)$, as $p(\xi) = E_0(\xi^2) + \xi E_1(\xi^2)$. Now use E_0 and E_1 to define recursively the polynomials

$$E_k(\xi) = \frac{E_{k-2}(0)E_{k-1}(\xi) - E_{k-1}(0)E_{k-2}(\xi)}{\xi} \tag{11}$$

for $k = 2, 3, \cdots, n$. The polynomials $(E_0, E_1, \cdots, E_{n-1}, E_n)$ obtained this way play an important role in stability tests. In fact, it is easy to see that the coefficients of $(E_0, E_1, \cdots, E_{n-1}, E_n)$ listed in increasing order underneath each other form the *Routh table*. Their leading elements $(E_0(0), E_1(0), \cdots, E_{n-1}(0), E_n(0))$ form the

Routh array. We now show the classical result that (8) is asymptotically stable if and only if the elements of the Routh array are all positive.

In order to prove this, define the polynomials $p_k(\xi) = E_{k-1}(\xi^2) + \xi E_k(\xi^2)$ for $k = 1, 2, \cdots, n$. Note that $p_1 = p$, and that the degree of p_k is less than or equal to $n - k + 1$. Now use the definition of the Bezoutian to verify that

$$B_{p_k}(\zeta, \eta) = \frac{\zeta E_{k-1}(\zeta^2) E_k(\eta^2) + \eta E_{k-1}(\zeta^2) E_k(\eta^2)}{\zeta + \eta}.$$

Combined with (11) this yields the following backwards recursion for the B_{p_k}'s:

$$E_k(0) B_{p_k}(\zeta, \eta) = \zeta\eta B_{p_{k+1}}(\zeta, \eta) + E_{k-1}(0) E_k(\zeta^2) E_k(\eta^2)$$

This equation combined with theorem 9 shows that $p = p_1$ is Hurwitz if and only if p_2 is Hurwitz, $E_0(0) > 0$, and $E_1(0) > 0$. Recursively, this yields the Routh test for the asymptotic stability of p. The important point is that

$$B_p(\zeta, \eta) = \sum_{k=1}^{n} \frac{E_{k-1}(0)}{\prod_{\ell=1}^{k} E_\ell(0)} \zeta^{k-1} E_k(\zeta^2) \eta^{k-1} E_k(\eta^2) \tag{12}$$

generates a QDF satisfying

$$\frac{d}{dt} Q_{B_p}(w) = - \mid p(-\frac{d}{dt}) w \mid^2$$

along solutions of (8). The positivity of (12) is what allows to conclude asymptotic stability. This reasoning yields a fully self-contained Lyapunov proof of the Routh test. This proof is readily generalized to unstable systems for which the $E_k(0)$'s are nonzero.

5.2 The Kharitonov theorem

As a second application of the Bezoutian, consider the problem of the stability of interval polynomials. It is well-known that the set of Hurwitz polynomials is not convex. However, there are interesting subsets that have this property. Using the Bezoutian, it is possible to obtain a nice example of such a convex family.

Consider

$$p(\xi) = \sum_k \alpha_k E_k(\xi^2) + \sum_{k'} \beta_{k'} \xi E_{k'}(\xi^2) \tag{13}$$

i.e., the even and odd parts of p are finite linear combinations of certain given even and odd polynomials. We now show that if all the α_k's and $\beta_{k'}$'s are nonnegative, and if all the polynomials $p_{k,k'}(\xi) = E_k(\xi^2) + \xi E_{k'}(\xi^2)$, obtained by combining the even and odd polynomials appearing in p, are Hurwitz, then p is also Hurwitz. This follows readily from theorem 9. Indeed, use (10) and (13) to show that the Bezoutian B_p is related to the Bezoutians $B_{p_{k,k'}}$ by

$$B_p(\zeta, \eta) = \sum_{k,k'} \alpha_k \beta_{k'} B_{p_{k,k'}}(\zeta, \eta),$$

and apply theorem 9.

This result implies for example that if $p(\xi)$ and $q(\xi) = p(\xi) + \alpha\xi^d$ are both Hurwitz, so will be $\beta p(\xi) + (1-\beta)q(\xi)$ for $0 \leq \beta \leq 1$. From this we immediately deduce the *weak Kharitonov theorem* which states that all the polynomials $p_0 + p_1\xi + \cdots + p_{n-1}\xi^{n-1} + p_n\xi^n$ in the interval family defined by $0 \leq a_k \leq p_k \leq A_k$ for $k = 0, 1, \cdots, n$ are Hurwitz if and only if the 2^{n+1} *extreme polynomials* obtained by taking $p_k = a_k$ or A_k for each k are all Hurwitz. In this case $p(\xi)$ can indeed be written as a convex combination of the extreme polynomials, which yields $p(\xi) = \sum_k \alpha_k E_k(\xi^2) + \sum_{k'} \beta_{k'} \xi E'_{k'}(\xi^2)$ with $0 \leq \alpha_k, \beta_{k'}$, and the $E_k(\xi^2)$'s and the $\xi E'_{k'}(\xi^2)$'s the even and odd part of the extreme polynomials. Equation (10) shows that the Bezoutian $B_p(\zeta, \eta)$ can be written as

$$B_p(\zeta, \eta) = \sum_{\ell=1}^{2^{n+1}} \gamma_\ell B_\ell(\zeta, \eta),$$

with $0 \leq \gamma_\ell$, and the B_ℓ's the Bezoutians associated with the extreme polynomials.

It is well-known that his result can be strengthened to lead to the remarkable *strong Kharitonov theorem* [9] which states that the interval polynomials are all Hurwitz if the following four specimens, the *Kharitonov polynomials,* are:

$$\begin{aligned}
k_1(\xi) &= a_0 + a_1\xi + A_2\xi^2 + A_3\xi^3 + a_4\xi^4 + \cdots, \\
k_2(\xi) &= a_0 + A_1\xi + A_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \cdots, \\
k_3(\xi) &= A_0 + A_1\xi + a_2\xi^2 + a_3\xi^3 + A_4\xi^4 + \cdots, \\
k_4(\xi) &= A_0 + a_1\xi + a_2\xi^2 + A_3\xi^3 + A_4\xi^4 + \cdots.
\end{aligned}$$

This may be proven as follows. First, note that any convex combination k of k_1, k_2, k_3, k_4 is Hurwitz. In order to see this, define $E_1 = E_2, E_3 = E_4$ and $O_1 = O_4, O_2 = O_3$ to be the even and odd parts of $k_1 = E_1 + O_1, k_2 = E_1 + O_3, k_3 = E_3 + O_3, k_4 = E_3 + O_1$. This shows that the even and odd parts of k are convex combinations of E_1, E_3 and O_1, O_3, respectively. It follows that k is Hurwitz if k_1, k_2, k_3,k_4 are. Next, use the crucial observation from [11] that, since all the coefficients of k_1, k_2, k_3, k_4 are positive, there holds for all $\omega \in [0, \infty)$:

$$
\begin{aligned}
E_1(i\omega) = \mathrm{Re}(k_1(i\omega) = \\
\mathrm{Re}(k_2(i\omega)) \leq \mathrm{Re}(p(i\omega)) \leq \mathrm{Re}(k_3(i\omega)) \\
= \mathrm{Re}(k_4(i\omega)) = E_3(i\omega),
\end{aligned}
$$

$$
\begin{aligned}
O_1(i\omega) = \mathrm{Im}(k_1(i\omega) = \\
\mathrm{Im}(k_4(i\omega)) \leq \mathrm{Im}(p(i\omega)) \leq \mathrm{Im}(k_2(i\omega)) \\
= \mathrm{Im}(k_3(i\omega)) = O_3(i\omega).
\end{aligned}
$$

It follows that for all $\omega \in \mathbb{R}, p(i\omega)$ is a convex combination of $\{k_1(i\omega),\ k_2(i\omega),\ k_3(i\omega),\ k_4(i\omega)\}$. Since if k_1, k_2, k_3, k_4 are Hurwitz, then every convex combination of k_1, k_2, k_3, k_4 is Hurwitz (and therefore has no roots on the imaginary axis), this implies that also p cannot have roots on the imaginary axis.

This leads to Kharitonov's result. Observe that the degree of p equals the degree of one of the Kharitonov polynomials. Now consider the convex combinations of p and this particular Kharitonov polynomial. By the reasoning that we have just used, none of these convex combinations can have roots on the imaginary axis. Furthermore, they have all the same degree, and the Kharitonov polynomial is Hurwitz. Hence p is Hurwitz as well. Note that this proof yields the strong Kharitonov theorem, without having to impose the usual assumption that all the interval polynomials have the same degree. More details concerning this difficulty can be found in [25].

The Kharitonov theorem implies that the Bezoutian B_p is positive definite if the Bezoutians $B_{k_1}, B_{k_2}, B_{k_3}, B_{k_4}$ are. It would be interesting to give a direct proof of this. It seems reasonable to conjecture that B_p must somehow be a nonnegative sum of $B_{k_1}, B_{k_2}, B_{k_3}, B_{k_4}$. This conjecture, if true, would yield a simple and elegant proof of the Kharitonov theorem that is completely based on Bezoutians and Lyapunov arguments.

6 Dissipative systems

One of the concepts that was put forward as a result of the stability work of the sixties and early seventies is the notion of a dissipative system [20] (the influence in these papers of Roger Brockett's work is too obvious to mention). This notion came to play an important role in the development of control, especially in H_∞-theory and robust control. In this section we put forward our most recent viewpoint on this. For simplicity of exposition, we restrict attention to linear differential systems and quadratic differential forms.

The notion of a dissipative system involves:

- *dynamics:* expressed by a dynamical system $\mathfrak{B} \in \mathfrak{L}^q$;
- a *supply rate:* expressed by a QDF Q_Φ with $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$;
- a *storage function:* expressed by a QDF Q_Ψ with $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$;
- a *dissipation rate:* expressed by a QDF Q_Δ with $\Delta = \Delta^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$.

The triple (Φ, Ψ, Δ) is called $\mathfrak{B}$-*matched* if

$$\frac{d}{dt} Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w) \tag{14}$$

for all $w \in \mathfrak{B}$, i.e., if $\overset{\bullet}{\Psi} \overset{\mathfrak{B}}{=} \Phi - \Delta$.

The system $\mathfrak{B} \in \mathfrak{L}^q$ is said to be *dissipative* with respect to the triple of QDF's (Φ, Ψ, Δ) if (Φ, Ψ, Δ) is $\mathfrak{B}$-matched, and if $\Delta \overset{\mathfrak{B}}{\geq} 0$. A dissipative system with $\Delta = 0$ is said to be *conservative.*

The intuition behind these definitions is rather apparent: $Q_\Phi(w)$ models the rate at which supply is delivered to the system, $Q_\Psi(w)$ models the amount stored, and $Q_\Delta(w) = Q_\Phi(w) - \frac{d}{dt}Q_\Psi(w)$ models the dissipation rate, which in a dissipative system must be nonnegative, and in a conservative system must be zero. Physical systems where this definition is appropriate are electrical circuits, with as supply rate the power delivered to the circuit via the external terminals, as storage function the energy stored in the inductors and the capacitors, and as dissipation rate the heat produced in the resistors; or mechanical systems with as supply rate the mechanical power supplied through external forces acting on the system, as storage the potential plus kinetic energy, and as dissipation the heat produced by friction.

An important question is whether one can deduce dissipativity from observing for a *black-box* how supply flows in and out of a system, i.e., can one conclude dissipativity by observing $Q_\Phi(w)$ for $w \in \mathfrak{B}$? The question is then to deduce, from $\mathfrak{B}$ and $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$, the existence of an appropriate $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ and $\Delta = \Delta^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ such that $\Delta \overset{\mathfrak{B}}{\geq} 0$ and the matching condition (14) holds. For example, in thermodynamics, $Q_\Psi(w)$ corresponds to the rate of heat supplied to the environment divided by the temperature, and the storage function corresponds to minus the entropy. However, the entropy is hardly an 'observable', but for the second law, in first instance only its existence matters. The dissipation, in this case the rate of entropy production, is then defined by the matching condition (14).

Note that by redefining Δ, if need be, we may as well assume in this question that $\Delta \geq 0$, instead of $\Delta \overset{\mathfrak{B}}{\geq} 0$. Second, instead of looking for both Ψ and Δ, we may as well look only for a Ψ such that $\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w)$ for all $w \in \mathfrak{B}$. Once Ψ is found, Δ can be deduced from $\Delta \stackrel{\mathfrak{B}}{=} \stackrel{\bullet}{\Psi} - \Phi$. The issue is then how to deduce Ψ from observing $Q_\Phi(w)$ for $w \in \mathfrak{B}$. The existence of such a Ψ can indeed be solved very nicely for controllable systems.

Theorem 10 : *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a controllable system, $\mathfrak{B} = \ker(R(\frac{d}{dt})) = im(M(\frac{d}{dt}))$, and $\Phi = \Phi^* \in \mathbb{R}^{q \times q}[\zeta, \eta]$. The following conditions are equivalent:*

(i) there exist $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ and $\Delta = \Delta^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\mathfrak{B}$ is dissipative with respect to (Φ, Ψ, Δ);

(ii) $\int_{-\infty}^{+\infty} Q_\Phi(w)dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$;

(iii) there exists $\Delta = \Delta^* \in \mathbb{R}^{q\times q}[\zeta,\eta] \geq 0$ such that

$$\int_{-\infty}^{+\infty} Q_\Phi(w) = \int_{-\infty}^{+\infty} Q_\Delta(w)$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$,

(iv) there exists $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that

$$\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w) \tag{15}$$

for all $w \in \mathfrak{B}$;

(v) $\Phi'(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, where $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$;

(vi) there exists $X \in \mathbb{R}^{\bullet \times q}$ such that

$$\Phi(-i\omega, i\omega) + X^T(-i\omega)R(i\omega) + R^T(-i\omega)X(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$.

The first of the statements in the above theorem gives a black-box-type condition for dissipativity: if a system always absorbs some external supply when it starts at rest and is driven back to rest, then it can be viewed as being externally dissipative, in the sense that an appropriate storage function and dissipation rate exists. The required non-negativity of this integral along any compact support trajectory in $\mathfrak{B}$ can be replaced by non-negativity along any periodic trajectory in $\mathfrak{B}$ when the integral is evaluated over one period.

An important issue which is taken up in detail in [24] is the study of the set of storage functions satisfying (15). This set consists in general of many elements, it is convex, and attains its upper and lower bound. These properties are of importance in applications, for example in H_∞-control [18].

In many applications, the storage function is nonnegative, as, for example, the stored energy in electrical circuits. Actually, since the physical energy is only defined up to a constant, non-negativity can be interpreted as bounded from below (but, when considering QDF's, this yields non-negativity). The question thus occurs when a nonnegative storage function exists. This is treated in the following theorem.

Theorem 11 : *Let $\mathfrak{B} \in \mathfrak{L}^q$ be a controllable system and $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Then there exist $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta] \geq 0$ and $\Delta = \Delta^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ such that $\mathfrak{B}$ is dissipative with respect to (Φ, Ψ, Δ) if and only if*

$$\int_{-\infty}^{0} Q_\Phi(w) \geq 0$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$.

A number of other equivalent statements in addition to those given in the above theorem can be stated, so as to make it look more like an analogue of theorem 10. Both theorems 10 and 11 allow refinements in a number of directions, in particular by imposing appropriate strict dissipativity and positivity conditions. We will not dwell on these ramifications here, but mention only the following point. The main deficiency of theorem 11 is that the condition given for the existence of a nonnegative storage function is not particularly explicit in $\mathfrak{B}$ and Φ. However, recently [24] necessary and sufficient conditions have been obtained — under some strictness assumptions — for the existence of a nonnegative storage function in terms of a Pick matrix derived from $\partial\Phi$. For details of these results, we refer to the source given.

The theory of dissipative systems is closely connected to path integrals and Lyapunov theory. Theorem 2 shows that a system is externally conservative with respect to the QDF induced by Φ if and only if $\int Q_\Phi$ is a path integral along $\mathfrak{B}$. The storage function is then simply the QDF Q_Ψ that expresses the path independence via (3) and (6). In a dissipative system, the dissipation rate is a nonnegative QDF that makes $\int Q_{\Phi-\Delta}$ into a path integral along $\mathfrak{B}$. Equation (15) shows that a storage function for an externally dissipative system plays the role for general systems in interaction with their environment that a Lyapunov plays for autonomous systems (in which case it is natural to take the supply rate to be zero). Thus, given the importance of Lyapunov theory, and given the isolated position for autonomous systems, it stands to reason that dissipative systems should play an important role in the theory of dynamical systems.

The notion of a storage function refers to accumulation of a physical quantity as energy, or entropy, or mass. The notion of the state of a system, on the other hand, refers to the memory of a system, intuitively, to the accumulation of information. It is an intriguing idea to question whether the two are related. They are, indeed: the ('physical') storage function of a dissipative system is necessarily a memoryless function of the ('information') state. Note that, following [20], it is usually *assumed* in the theory of dissipative systems that the storage is a state function, but now it is a fact that can actually be *proven* [17] (at least for linear differential systems and quadratic supply rates). However, the relevant system of which we have to consider the state, involves both the dynamics of the system in combination with the differentiations in the supply function.

Let $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Then Φ can be factored as $\Phi(\zeta,\eta) = M^T(\zeta)\Sigma M(\eta)$ with $M \in \mathbb{R}^{\bullet\times q}[\xi]$ and $\Sigma = \Sigma^T \in \mathbb{R}^{\bullet\times\bullet}$. Let Σ be of dimension $q' \times q'$. Consider now the system $\mathfrak{B} \in \mathfrak{L}^q$. This induces the behavior $\mathfrak{B}' \in \mathfrak{L}^{q'}$ defined by

$$\mathfrak{B}' := \{w' \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q) \mid w' = M(\frac{d}{dt})w, w \in \mathfrak{B}\}.$$

This way we obtain the dynamical system $\Sigma' = (\mathbb{R},\mathbb{R}^q,\mathfrak{B}')$. It is of this system that the storage function is a memoryless function of the state (at least for controllable systems). We refer to the appendix B for a precise definition of a *state map*.

Theorem 12 : *Assume that $\mathfrak{B} \in \mathfrak{L}^q$ is controllable, and $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$. Factor Φ as $\Phi(\zeta,\eta) = M^T(\zeta)\Sigma M(\eta)$ with $M \in \mathbb{R}^{q'\times q}[\xi]$ and $\Sigma = \Sigma^T \in \mathbb{R}^{q'\times q'}$. Define $\mathfrak{B}' := M(\frac{d}{dt})\mathfrak{B}$ and assume that $X \in \mathbb{R}^{n\times q}[\xi]$ induces by $x = X(\frac{d}{dt})w'$ a state map for $\mathfrak{B}'$. Assume that $\mathfrak{B}$ is dissipative with respect to (Φ,Ψ,Δ). Then $Q_\Phi(w)$ is a memoryless function of the state of $\mathfrak{B}'$, i.e., there exists $K = K^T \in \mathbb{R}^{n\times n}$ such that $Q_\Psi(w') = ||X(\frac{d}{dt})w'||_K^2$ for $w' \in \mathfrak{B}'$, equivalently, $\Psi(\zeta,\eta) = M^T(\zeta)X^T(\zeta)KX(\eta)M(\eta)$, and Δ is a memoryless function of the state and the manifest variable of $\mathfrak{B}'$, i.e., there exists $D = D^T \geq 0$ such that $Q_\Delta(w') = ||\mathrm{col}[X(\frac{d}{dt})w', w']||_D^2$ for $w' \in \mathfrak{B}'$.*

The interesting part of this theorem is the fact that the storage is a memoryless function of the state of the system that combines in a precise way the dynamics and the supply rate.

7 Stability of non-autonomous systems

How should one formulate stability of dynamical systems that are not autonomous?

This question has preoccupied me for a long time. For autonomous systems, it is natural to define stability in terms of boundedness or convergence of the manifest trajectories in the behavior (or, which leads to very similar conditions, in terms of the convergence of the underlying state trajectory). However, a reasonable model for a physical (or economic) system considers a system in interaction with its environment, and the preoccupation in mathematics and physics with autonomous systems must be considered as somewhat of an abstract anomaly: it forces one in the absurd situation that in order to model a system, one ends up being forced to model also its environment. The question thus occurs: *When is a physical system that interacts with its environment, to be called stable?*

A possible approach is the following. Assume that $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is a dynamical system (see appendix B). In order to define stability, consider two additional behaviors, $\mathfrak{B}_a, \mathfrak{B}_g \subseteq \mathbb{W}^{\mathbb{T}}$ (a suggests 'admissible', g suggests 'good'); $\mathfrak{B}_a$ are the signals that can be imposed by the environment and against which stability has to be tested, and $\mathfrak{B}_g$ are the signals that are indicative of stable, of non-explosive behavior. Thus $\mathfrak{B} \cap \mathfrak{B}_a$ are all the signals that are physically possible when the system has an admissible interaction with its environment. The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be $\mathfrak{B}_a/\mathfrak{B}_g$-stable if $\mathfrak{B} \cap \mathfrak{B}_a \subseteq \mathfrak{B}_g$, i.e., if all signals that are physically possible and admissible are non-explosive in character.

Consider the following examples of this type of stability. For simplicity, we assume $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B}) \in \mathfrak{L}^q$.

- Take $\mathfrak{B}_a = (\mathbb{R}^q)^{\mathbb{R}}$, and $\mathfrak{B}_g = \{w : \mathbb{R} \to \mathbb{R}^q \mid w \overset{t\to\infty}{\longrightarrow} 0)\}$. In this case stability corresponds to what in section 4 we called asymptotic stability. In order to have this type of stability, Σ must be autonomous, and, with $\mathfrak{B} = \ker(R(\frac{d}{dt}))$, R must be Hurwitz.

- Assume $\mathbb{R}^q = \mathbb{R}^m \times \mathbb{R}^p$ corresponding to a partition of w into (u, y) with u the input, and y the output (see appendix B for what this means it a behavioral context). Take

$$\mathfrak{B}_a = \{w = (u, y) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^p \mid \int_0^\infty ||u||^2 dt < \infty\},$$

and

$$\mathfrak{B}_g = \{w = (u, y) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^p \mid \int_0^\infty ||w||^2 dt < \infty\}.$$

This type of stability corresponds to $\mathfrak{L}^2$-*input/output stability.* If $\mathfrak{B}$ is described by

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u,$$

with $P, Q \in \mathbb{R}^{\bullet\times\bullet}[\xi], P$ square, $\det(P) \neq 0$, and $P^{-1}Q \in \mathbb{R}^{p\times n}(\xi)$ proper, then for this type of stability requires P to be Hurwitz.

- Continuing the previous example, take

$$\mathfrak{B}_a = \{w = (u, y) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^p \mid u = 0\},$$

and

$$\mathfrak{B}_g = \{w = (u, y) : \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^p \mid w(t) \overset{t\to\infty}{\longrightarrow} 0)\}.$$

This type of stability again requires P to be Hurwitz.

- The next type of stability uses QDF's. Let $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ and define

$$\mathfrak{B}_a = \{w : \mathbb{R} \to \mathbb{R}^q \mid \int_0^\infty Q_\Phi(w)dt < +\infty\},$$

and

$$\mathfrak{B}_g = \{w :\to \mathbb{R}^q \mid \int_0^\infty ||w||^2 dt < \infty\}.$$

This last type of stability is physically certainly the most pleasing one, and we will pursue it now. Intuitively, it significance is as follows. Imagine that our (physical) system $\mathfrak{B}$ exchanges supply, say energy, with its environment. The instantaneous power delivered to the system by the environment is $Q_\Phi(w)$. To test stability, we assume that the energy delivered to the system by the environment during $[0, \infty)$ is finite. We call the system stable if the manifest variable w has then a non-explosive character, modeled as requiring it to be square integrable. This definition suggests that the dissipation present in the system avoids unstable explosive behavior. However, as we shall see, we also have to assume non-negativity of the storage function for dissipation to lead to this type of stability. Note that replacing the square integrability condition $w \in \mathfrak{L}^2(\mathbb{R}, \mathbb{R}^q)$ on $\mathfrak{B}_g$ by boundedness of w on $[0, \infty)$, or by $w(t) \overset{t\to\infty}{\longrightarrow} 0$, are very reasonable alternatives. The former is reminiscent of $\mathfrak{L}^2$-input/output stability, the second of ordinary stability, the third of asymptotic stability.

We now formulate a few results using this type of stability. Let $\mathfrak{B} \in \mathfrak{L}^q$ be controllable and $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$. We call $\mathfrak{B}$ – *strictly dissipative* on $\mathbb{R}_-$ with respect to Φ if there exists $\varepsilon > 0$ such that

$$\int_{-\infty}^0 Q_\Phi(w)dt \geq \varepsilon \int_{-\infty}^0 ||w||^2 dt$$

for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Using theorem 11 it is easy to show that strict dissipativity on $\mathbb{R}_-$ implies the existence of a $\mathfrak{B}$-positive storage function $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$. In fact, there exists a storage function that is positive definite in the state of $\mathfrak{B}$. More precisely, let $X \in \mathbb{R}^{n\times q}[\xi]$ be such that $x = X(\frac{d}{dt})w$ induces a state for $\mathfrak{B}$. Then, if $\mathfrak{B} \in \mathfrak{L}^q$ is controllable and strictly dissipative with respect to $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$, there exists $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ and $\varepsilon_1 > 0, \varepsilon_2 > 0$ such that for $w \in \mathfrak{B}$, there holds

$$\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w) - \varepsilon_1 ||w||^2, \tag{16}$$

$$Q_\Psi(w) \geq \varepsilon_2 ||X(\frac{d}{dt})w||^2. \tag{17}$$

These inequalities are proven in appendix C. Using these inequalities we readily obtain the following result.

Theorem 13 : *Let $\mathfrak{B} \in \mathfrak{L}^q$ be controllable and $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$, and assume that $\mathfrak{B}$ is strictly dissipative with respect to Φ. Let $\mathfrak{B}_a$ and $\mathfrak{B}_g$ be defined as:*

$$\mathfrak{B}_a = \{w : \mathbb{R} \to \mathbb{R}^q \mid \int_0^\infty Q_\Phi(w)dt < +\infty\},$$

$$\mathfrak{B}_g = \{w : \to \mathbb{R}^q \mid \int_0^\infty ||w||^2 dt < \infty\}.$$

Then $\mathfrak{B}$ is $\mathfrak{B}_a/\mathfrak{B}_g$–stable.

The stability concept used in theorem 13 only requires to prove that $\int_0^\infty ||w||^2 dt < \infty$ if $\int_0^\infty Q_\Psi(w)dt < 0$. However, further analysis allows to conclude other types of stable behavior as well. Indeed, the proof of theorem 13 shows that under the conditions of the theorem, $\int_0^\infty Q_\Psi(w) < 0$ implies that the state $X(\frac{d}{dt})w$ of $\mathfrak{B}$ is bounded on $[0,\infty)$ and in fact approaches zero as $t \to \infty$. The significance of theorem 13 is as follows. Whenever a strictly dissipative dynamical system is interconnected with an (uncertain) system that can deliver only a finite total supply to the system, then the system itself will behave in a stable way: its manifest variables will be square integrable and its internal state will go to zero. Intuitively, a lossy passive system to which only a finite amount of energy is delivered, will dissipate this energy and have a stable behavior. It is this principle that lies at the root of the small gain and positive operator theorem of Zames [26] and the many related stability results obtained in the sixties. In theorem 13, these results are presented in both a very general and a very intuitive way. The proof relies on the construction of storage functions for dissipative systems. This construction uses path integrals and in this sense formalizes Brockett's path integral/even part approach to stability as developed in [1, 3].

Examples:

1. Assume that $\mathfrak{B} \in \mathfrak{L}^q$ is controllable. Then, up to re-ordering of its components, $w = \text{col}[w_1, w_2]$ admits kernel and image representations [13]

$$P(\frac{d}{dt})w_2 = Q(\frac{d}{dt})w_1$$

and

$$w_1 = D(\frac{d}{dt})\ell; \quad w_2 = N(\frac{d}{dt})\ell$$

respectively, with P and N square, $\det P \neq 0, \det N \neq 0$ and $P^{-1}Q = ND^{-1} \in \mathbb{R}^{\bullet \times \bullet}(\xi)$ respectively left and right co-prime factorizations of $G = P^{-1}Q = ND^{-1}$. The H_∞–norm of G, $||G||_\infty$, satisfies $||G||_\infty < 1$, if and only if D (and hence P) is Hurwitz and there exists an $\varepsilon > 0$ such that

$$D^T(-i\omega)D(i\omega) - N^T(-i\omega)N(i\omega) \geq \varepsilon(D^T(-i\omega)D(i\omega) + N^T(-i\omega)N(i\omega)) \quad (18)$$

for all $\omega \in \mathbb{R}$. This inequality implies that

$$\int_{-\infty}^{0} (||w_1||^2 - ||w_2||^2)dt \geq \varepsilon \int_{-\infty}^{0} ||w_1||^2 dt \quad (19)$$

for all $w \in \mathfrak{B}$ with compact support. Thus $||G||_\infty < 1$ implies that $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}_-$ with respect to $\Phi(\zeta, \eta) = W = \begin{bmatrix} I_m & 0 \\ 0 & -I_p \end{bmatrix}$ where I_m and I_p denote the identity matrix of dimensions m = the number of components of w_1, and p = the number of components of w_2.

Assume now that $\mathfrak{B}$ is interconnected to a (possible nonlinear and/or time-varying) system that restricts w to satisfy $\int_0^\infty (||w_1||^2 - ||w_2||^2)dt < +\infty$. This is the case, for example, if the interconnection implies that $||w_1(t)|| \leq ||w_2(t)||$ holds point-wise for $t \in \mathbb{R}$. This will be the case, for instance, if the interconnection is defined by $w_1(t) = K(t)w_2(t)$ with $K(\cdot)$ a time-varying matrix whose Euclidean induced norm satisfies $||K(t)|| \leq 1$ for all $t \in \mathbb{R}$. Theorem 13 then implies that in the interconnected system $\int_0^\infty (||w||^2 dt < \infty$, and it can moreover be shown that the state of $\mathfrak{B}$ goes to zero as $t \to \infty$.

Of course, the above result is the small loop gain theorem. Replacing W by another symmetric matrix and changing the weighting matrix (18) and (19) appropriately, yields the positive operator ([26, 19]) and conicity ([26, 16]) based stability results.

2. Consider the controllable system $\mathfrak{B} \in \mathfrak{L}^2$ with kernel representation

$$p(\frac{d}{dt})w_2 = q(\frac{d}{dt})w_1,$$

and equivalent observable image representation

$$w_1 = p(\frac{d}{dt})\ell; \quad w_2 = q(\frac{d}{dt})\ell.$$

Assume that $\mathfrak{B}$ is interconnected to a memoryless time-invariant nonlinearity $w_1 = -f(w_2)$ with $f : \mathbb{R} \to \mathbb{R}$ a measurable map satisfying $0 \leq \sigma f(\sigma)$ for all $\sigma \in \mathbb{R}$, and such that $F(\sigma) := \int_0^\sigma f(\sigma')d\sigma'$ is a well-defined map $F : \mathbb{R} \to \mathbb{R}$. Obviously, $F \geq 0$.

Define $\Phi(\zeta, \eta) = 1 + \alpha(\zeta + \eta)$ with $\alpha \geq 0$. Note that

$$\int_0^\infty Q_\Phi(w)dt = \int_0^\infty (w_1 w_2 + \alpha w_1 \frac{dw_2}{dt})dt.$$

Consider the right-hand side of this equality. The first term equals $-\int_0^\infty f(w_2)w_2 dt$ and is obviously non-positive. To compute the second term, observe that

$$\alpha \int_0^t w_1 \frac{dw_2}{dt} dt = -F(w_2(t)) + F(w_2(0)),$$

whence

$$\alpha \int_0^\infty w_1 \frac{dw_2}{dt} dt \leq F(w_2(0)) < +\infty.$$

Therefore

$$\alpha \int_0^\infty Q_\Phi(w)dt < \infty.$$

It follows from theorem 13 that in the interconnected system $\text{col}[w_1, w_2]$ will be square integrable if p is Hurwitz and if there exists an $\varepsilon > 0$ such that

$$Re\left(\frac{(1 + \alpha i\omega)q(i\omega)}{p(i\omega)}\right) \geq \varepsilon > 0$$

for all $\omega \in \mathbb{R}$. This criterion is the celebrated Popov criterion [14]. The interpretation given here leading to this criterion, is that when a system is interconnected to a memoryless nonlinearity, it will acquire from its environment only a finite amount of supply corresponding to a well-chosen supply rate. When the system itself is strictly dissipative with respect to this supply rate, stability results. For the system under consideration, other kinds of stability can again be deduced. In particular, it can be shown that the internal state of the system will converge to zero as $t \to \infty$. The method explained will also yield many other related well-known stability criteria. For example, criteria with f bounded by $K_1\sigma^2 \leq \sigma f(\sigma) \leq K_2\sigma^2$.

In closing, we remark the rather obvious connection between the choice of the supply rate, i.e., of the two-variable polynomial (matrix) Φ, and the resulting stability conditions. This is the idea behind the use of multipliers [19] in stability analysis. However, the implied restriction to the use of QDF's (rather than more general operators) is at first sight somewhat restrictive. It would be interesting indeed to cast the results on multipliers using LMI's [12], IQC's [10] and their application to μ-analysis and -synthesis [6] in our framework of dissipative systems.

8 Conclusions

The main technical idea that we have put forward in this paper is the calculus of two-variable polynomial matrices as a means of analyzing quadratic differential forms. We have shown how this allows to construct path integrals which in turn yield storage functions for dissipative systems and very effective Lyapunov functions for stability analysis. We believe that (quadratic) differential forms are a very natural and wanting concept in systems theory, paralleling the use of polynomial matrices for system descriptions, and (quadratic) Lyapunov functions for stability analysis. Roger Brockett's path integrals and 'even part' calculations which he developed over two decades ago, contained more than the germ of these ideas. In this sense this paper is a fitting tribute to him at the occasion of his 60-th birthday.

9 Appendix

9.1 Appendix A: Notation

As usual we denote by $\mathbb{R}$ the reals, by $\mathbb{C}$ the complex numbers, and by $\mathbb{R}^{n_1 \times n_2}$ the $n_1 \times n_2$ real matrices. The integers are denoted by $\mathbb{Z}$, and the nonnegative ones by $\mathbb{Z}_+$. By $\mathbb{R}^{\bullet \times n}$ we denote the real matrices with n columns (this notation leaves the number of rows free, but it is, of course, finite); the meaning of $\mathbb{R}^{n \times \bullet}$, $\mathbb{R}^{\bullet \times \bullet}$, $\mathbb{C}^{n_1 \times n_2}$, etc., follows. The notation $\text{col}[A_1, A_2, \dots, A_n]$ denotes the matrix formed by stacking the matrices $A_1, A_2, \dots A_n$ underneath each other (this saves space in printing — of course the A_k's must all have the same number of columns). For $A \in \mathbb{R}^{n_1 \times n_2}$, $\text{rank}(A)$, $\det(A)$ are defined in the usual way; $\text{rowdim}(A) = n_1$, $\text{coldim}(A) = n_2$, and $\dim(A) = (n_1, n_2)$. This notation obviously carries over to complex and polynomial matrices. Finally, $\ker(L)$ and $\text{im}(L)$ denote the kernel and the image of L.

As usual we denote by $\mathfrak{C}(\mathbb{R}, \mathbb{R}^n)$, $\mathfrak{C}^k(\mathbb{R}, \mathbb{R}^n)$, or $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ the maps from $\mathbb{R}$ to $\mathbb{R}^q$ that are once, k times, or infinitely often continuously differentiable. Further, we denote by $\mathfrak{D}(\mathbb{R}, \mathbb{R}^n)$ the elements of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ that have compact support; $\mathfrak{L}^2(\mathbb{R}, \mathbb{R}^n)$ denotes the square integrable maps from $\mathbb{R}$ to $\mathbb{R}^n$.

We denote one-variable polynomials with real coefficients by $\mathbb{R}[\xi]$, and two-variable polynomials by $\mathbb{R}[\zeta, \eta]$ (thus ξ is the indeterminate in the one-variable case, and ζ, η are the indeterminates in the two-variable case). Polynomial matrices are denoted by $\mathbb{R}^{\bullet \times \bullet}[\xi], \mathbb{R}^{\bullet \times n}[\xi]$ (when the number of columns is free), etc., with similar notation in the two-variable case.

We use some important special operators acting on polynomial matrices: $\star, *, ^\bullet$ and ∂ :

$\star$ maps $\mathbb{R}^{\bullet\times\bullet}[\xi]$ into itself: if $P \in \mathbb{R}^{n_1\times n_2}[\xi]$, then $P^\star \in \mathbb{R}^{n_2\times n_1}[\xi]$ is defined as $P^\star(\xi) := P^T(-\xi)$ (^{T} denotes transposition);

$*$ maps $\mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ into itself: if $P \in \mathbb{R}^{n_1\times n_2}[\zeta,\eta]$, then $P^* \in \mathbb{R}^{n_2\times n_1}[\zeta,\eta]$ is defined as $P^*(\zeta,\eta) := P^T(\eta,\zeta)$;

$\bullet$ maps $\mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ into itself: if $P \in \mathbb{R}^{n_1\times n_2}[\zeta,\eta]$, then $\overset{\bullet}{P} \in \mathbb{R}^{n_1\times n_2}[\zeta,\eta]$ is defined as $\overset{\bullet}{P}(\zeta,\eta) := (\zeta+\eta)P(\zeta,\eta)$;

∂ maps $\mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ into $\mathbb{R}^{\bullet\times\bullet}[\xi]$: if $P \in \mathbb{R}^{n_1\times n_2}[\zeta,\eta]$, then $\partial P \in \mathbb{R}^{n_1\times n_2}[\xi]$ is defined as $\partial P(\xi) := P(-\xi,\xi)$.

There is an important relation among the $^\bullet$ and the ∂ operators. Indeed, the image of the $^\bullet$ operator equals the kernel of the ∂ operator. In other words, for $\Phi \in \mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$,

$$\frac{\Phi(\zeta,\eta)}{\zeta+\eta}$$

is polynomial if and only if $\Phi(-\xi,\xi) = 0$.

The field of rational functions over $\mathbb{R}$ is denoted by $\mathbb{R}(\xi)$; $R^{\bullet\times\bullet}(\xi)$ denotes the set of matrices of rational functions. An element $P \in \mathbb{R}(\xi)$ is said to be *proper* if $P = p_1/p_2$, with the degree of $p_1 \in \mathbb{R}[\xi]$ less than or equal to that of $p_2 \in \mathbb{R}[\xi]$, and *strictly proper* if "less than" holds.

9.2 Appendix B: Linear differential systems

In order to make this paper reasonably self-contained, we introduce in this appendix some basic facts from the behavioral approach to linear differential dynamical systems. The ideas follow those introduced in [21, 22, 23, 13], where more details may be found. As also the present paper illustrates, the behavioral approach provides a very suitable point of view for treating dynamical systems, more so than the state space approach of the sixties (see [5]) does.

A dynamical system Σ is defined as a triple $\Sigma = (\mathbb{T},\mathbb{W},\mathfrak{B})$ with$\mathbb{T} \subseteq \mathbb{R}$ the time-axis, $\mathbb{W}$ a set called the signal space, and $\mathfrak{B} \subseteq /mathbbW^{\mathbb{T}}$ the *behavior.* In this paper we deal exclusively with continuous-time systems with $\mathbb{T} = \mathbb{R}$ which are *time-invariant* (meaning $\sigma^t\mathfrak{B} = \mathfrak{B}$ for all $t \in \mathbb{R}$, where σ^t denoted the t-shift) and with signal space $\mathbb{W} = \mathbb{R}^q$. The dynamical system $(\mathbb{R},\mathbb{R}^q,\mathfrak{B})$ is said to be *linear* if $w_1, w_2 \in \mathfrak{B}$ implies that $w_1 + w_2 \in \mathfrak{B}$ and that $\alpha w_1 \in \mathfrak{B}$ for all $\alpha \in \mathbb{R}$.

In this paper, we deal almost exclusively with continuous-time real linear time-invariant differential dynamical systems. *Differential* means that the systems are described by differential equations. Thus the time-axis is $\mathbb{R}$, the signal space is $\mathbb{R}^q$, and the behavior $\mathfrak{B}$ is the solution set of a system of linear constant coefficient differential equations

$$R(\frac{d}{dt})w = 0 \tag{20}$$

in the real variables $w_1, w_2, \cdots, w_q$, arranged as the column vector w; R is a real polynomial matrix with q columns. The number of rows of R depends, as do its coefficients, on the particular dynamical system described. Thus if $R(\xi) = R_0 + R_1\xi + \cdots + R_N\xi^N$, then we are considering the system of differential equations

$$R_0 w + R_1 \frac{dw}{dt} + \cdots + R_N \frac{d^N w}{dt^N} = 0.$$

For the behavior, i.e., for the solution set of this system of differential equations, one usually considers locally integrable w's as candidate solutions, and interprets the differential equation in the sense of distributions. However, in order to avoid mathematical technicalities, we assume in this paper that the solution set consists of infinitely differentiable functions. Hence the behavior of (20) is defined as

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid R(\frac{d}{dt})w = 0\}. \tag{21}$$

We denote the family of dynamical systems obtained this way by $\mathfrak{L}^q$, and $\mathbb{L} = \cup_{q \in \mathbb{Z}_+} \mathfrak{L}^q$. Hence elements of $\mathfrak{L}^q$ are dynamical systems $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with time-axis $\mathbb{R}$, signal space $\mathbb{R}^q$, and behavior $\mathfrak{B}$ described through some $R \in \mathbb{R}^{q \times \bullet}[\xi]$ by (21). Note that instead of writing $\Sigma \in \mathfrak{L}^q$ we may as well write $\mathfrak{B} \in \mathfrak{L}^q$, and we use this notation in this paper. There are many other ways of specifying a given behavior $\mathfrak{B} \in \mathfrak{L}^q$. Note that (21) describes $\mathfrak{B}$ as $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ with $R(\frac{d}{dt})$ viewed as a map from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ into $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{rowdim}(R)})$. For obvious reasons, we hence refer to (20) as *a kernel representation* of $\mathfrak{B}$. We will meet other representations, in particular image, latent variable, input/output, and state representations. These are now briefly introduced.

A time-invariant system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exists a $w \in \mathfrak{B}$ and a $t' \geq 0$ such that $w(t) = w_1(t)$ for $t < 0$ and $w(t) = w_2(t - t')$ for $t \geq t'$. It can be shown that $\mathfrak{B} \in \mathfrak{L}$ is controllable if and only if its kernel representation $\mathfrak{B} = \ker(R(\frac{d}{dt}))$ satisfies $\text{rank}(R(\lambda)) = \text{rank}(R)$ for all $\lambda \in \mathbb{C}$. Here, $\text{rank}(R)$ is defined as the rank of R considered as a matrix with elements in the field $\mathbb{R}(\xi)$ of real rational functions. On the other hand, for a given $\lambda \in \mathbb{C}$, $R(\lambda)$ is a matrix with elements in $\mathbb{C}$. Accordingly, $\text{rank}(R(\lambda))$ denotes the rank of the complex matrix $R(\lambda)$. It is easy to see that $\text{rank}(R) = \max_{\lambda \in \mathbb{C}}(\text{rank}(R(\lambda)))$.

Controllable systems are exactly those that admit an image representation. More concretely, $\mathfrak{B} \in \mathfrak{L}^q$ is controllable if and only if there exists a polynomial matrix $M \in \mathbb{R}^{q \times \bullet}[\xi]$ such that $\mathfrak{B} = \text{im}(M(\frac{d}{dt}))$, with $M(\frac{d}{dt})$ viewed as a mapping from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}}(M))$ into $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. The resulting representation

$$w = M(\frac{d}{dt})\ell \tag{22}$$

is called an *image representation* of $\mathfrak{B}$.

An image representation is a special case of what we call a latent variable representation of $\mathfrak{B}$. The system of differential equations

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell \tag{23}$$

is said to be a *latent variable representation* of $\mathfrak{B} \in \mathfrak{L}^q$ if

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \mid \exists\, \ell \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}(M)}) \text{ such that (23) holds}\}.$$

A latent variable representation is said to be *observable* if $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell_1$ and $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell_2$ implies $\ell_1 = \ell_2$. Observability is equivalent to the condition that $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$. A controllable system, it turns out, allows an observable image representation, i.e., an image representation (22) with $\text{rank}(M(\lambda)) = \text{coldim}(M)$ for all $\lambda \in \mathbb{C}$. Actually, the notion of observability applies to more general situations than the latent variable case. Thus w_2 is said to be *observable* from w_1 in the system with kernel representation $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$ if $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2'$ and $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2''$ implies $w_2' = w_2''$. Of course, for observability conditions analogous to those on M must now hold on R_2: $R_2(\lambda)$ must be of full column rank for all $\lambda \in \mathbb{C}$. Of special interest in this paper is the observability of a system of the form $A(\frac{d}{dt})\ell = 0$, $w = C(\frac{d}{dt})\ell$. If this system is observable, then we call the pair of polynomial matrices (A, C) with the same number of columns an *observable pair*. Hence (A, C) is an observable pair if and only if $\text{col}[A(\lambda) \;\; C(\lambda)]$ is of full column rank for all $\lambda \in \mathbb{C}$.

Systems in $\mathfrak{L}$ admit many other useful representations. We already encountered kernel and image representations. Next, we introduce state representations. In [15] the notion of state models and their construction has been discussed in detail. Here we limit ourselves to the bare essentials. A latent variable representation (with the latent variable denoted by x) of the form (23) is said to be a *state model* if whenever (w_1, x_1) and (w_2, x_2) are $\mathfrak{C}^\infty$-solutions of (23) with $x_1(0) = x_2(0)$, then the concatenation $(w_1, x_1) \wedge (w_2, x_2)$ also satisfies (23) ($f \wedge g$ is the function such that $(f \wedge g)(t)$ equals $f(t)$ for $t < 0$ and $g(t)$ for $t \geq 0$). Since this concatenation need not be in $\mathfrak{C}^\infty$, it need only be a weak solution of (23), that is, a solution in the sense of distributions. State models are governed by equations of the form (23) with special structure. In fact, a state model can be described by $Gw + Fx + E\frac{d}{dt}x = 0$. The important feature of this representation is that it consists of a system of (implicit) differential equations containing derivatives of order at most one in the latent (state) variable x and of order zero in w.

Every system $\mathfrak{B} \in \mathfrak{L}$ also admits an input/output representation. By reordering the components of the vector w, if need be, we can decompose w into $w = \text{col}[u \quad y]$ with, in terms of R, rank(R) components for y and $q - \text{rank}(R)$ components for u, such that $\mathfrak{B}$ admits the special kernel representation

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u \tag{24}$$

with P square, det $P \neq 0$, and $P^{-1}Q$ a matrix of proper rational functions. Thus in (24), u has the usual properties of input (in the sense that it is free) and y those of output (in the sense that it is determined by the input and the initial conditions). Therefore (24) is called an *input/output representation.* It is possible to combine state and input/output representations, leading to the familiar input/state/output representation

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du.$$

Summarizing, given any $w \in \mathfrak{B}$, we may partition the components of w into inputs and outputs. Also, there exists a $X \in \mathbb{R}^{\bullet \times q}$ such that

$$x = X(\frac{d}{dt})w$$

is a state for $\mathfrak{B}$. For a system in image representation this leads to a state representation of the form

$$x = X'(\frac{d}{dt})\ell.$$

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be *autonomous* if $w_1, w_2 \in \mathfrak{B}$ and $w_1(t) = w_2(t)$ for $t < 0$ imply $w_1 = w_2$, i.e., if the past of a trajectory in the behavior determines its future. If $\mathfrak{B} \in \mathfrak{L}^q, \mathfrak{B} = \ker(R(\frac{d}{dt}))$ then $\mathfrak{B}$ is autonomous if and only if rank(R) $= q$. Alternatively, if and only if $\mathfrak{B}$ admits a state representation of the special form $\frac{d}{dt}x = Ax, \ w = Cx$.

9.3 Appendix C: Proofs

We only give the main lines of the proofs, and refer to the literature cited, in particular to [24] for more details.

Proposition 1:

This proposition is part of theorem 2.

Theorem 2:

$(i) \Rightarrow (iii)$ is trivial by taking in (2) t_1 and t_2 outside the support of w.

$(ii) \Leftrightarrow (iii)$ follows by considering the Fourier transform $\hat{w}$ of w. This yields

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{w}^T(-i\omega)\Phi(-i\omega, i\omega)\hat{w}(i\omega)d\omega.$$

Hence $(ii) \Rightarrow (iii)$. The converse follows by assuming $\Phi(-i\omega_0, i\omega_0) \neq 0$ for some $\omega_0 \in \mathbb{R}$, and constructing a suitable $\hat{w}$ (see [24]).

$(ii) \Leftrightarrow (iv)$ follows from observing that a two-variable polynomial $f \in \mathbb{R}[\zeta, \eta]$ has a factor $(\zeta + \eta)$ if and only if $\partial f = 0$. The 'if' part of this is trivial, and for the 'only if' part, we refer again to [24].

$(iv) \Leftrightarrow (v)$ is trivial since $\frac{d}{dt}Q_\Psi = Q_{\overset{\bullet}{\Psi}}$.

Finally note that the implication $(v) \Rightarrow i)$ is trivial.

Theorem 4:

The equivalence of (v), (vi), and (vii) follows from theorem 2.

Next, observe that $w \in \mathfrak{B}$ if and only if there exists $\ell \in \mathfrak{C}^\infty(\mathbb{R}\mathbb{R}^{\text{coldim}(M)})$ such that $w = M(\frac{d}{dt})\ell$. Equivalence of (iv) and (vii) follows.

$(iii) \Leftrightarrow (iv)$ is a consequence of proposition 5, part (i).

$(iii) \Leftrightarrow (ii)$: the direction $(iii) \Rightarrow (ii)$ is trivial: take $X = -\partial Y$. To see the converse, take Y such that $\partial Y = -X$ and observe that this implies

$$\partial(\Phi(\xi, \eta) + Y^*(\xi, \eta)R(\eta) + R^T(\xi)Y(\xi, \eta)) = 0.$$

Now use equivalence of (ii) and (iv) of theorem 2.

The implications $(iv) \Rightarrow (i)$ and $(i) \Rightarrow (vi)$ are obvious, which closes the loop.

Proposition 3:

Verify that controllability was not used in the proof of the implications $(ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i)$ of theorem 4.

Proposition 5:

(i): The 'if' part can be seen from observing that for all $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ there holds that $Q_\Phi(w) = 2B_Y(w, R(\frac{d}{dt})w)$, where B_Y denotes the bilinear differential form defined by

$$B_Y(w_1, w_2) = \Sigma_{k,\ell}\left(\frac{d^k}{dt^k}w_1\right)^T Y_{k,\ell}\left(\frac{d^\ell}{dt^\ell}w_2\right)$$

with $Y(\xi, \eta) = \Sigma_{k,\ell}Y_{k,\ell}\xi^k\eta^\ell$. The 'only if' part may be shown by putting R is Smith form. The proof is easy for controllable systems, but a bit harder in the general case. The details can be found in [24].

(ii): The 'if' part is again obvious since $Q_\Phi(w) \stackrel{\mathfrak{B}}{=} ||D(\frac{d}{dt})w||^2$. The 'only if' part uses again the Smith form and is completely analogous to the proof of the 'only if'-part of (i).

(iii): Follows from the fact that $w \in \mathfrak{B}$ and $D(\frac{d}{dt})w = 0$ imply $w = 0$ if and only if in $R(\frac{d}{dt})w = 0$, $d = D(\frac{d}{dt})w$, w is observable from d. This is equivalent to $\text{rank}(\text{col}[R(\lambda), D(\lambda)]) = q$ for all $\lambda \in \mathbb{C}$.

Theorem 6:

'if': let $\exp_\lambda : t \in \mathbb{R} \mapsto e^{\lambda t} \in \mathbb{C}$ be the exponential function with parameter λ. The system $\mathfrak{B} \in \mathfrak{L}^q$ is asymptotically stable if and only if $\exp_\lambda a \in \mathfrak{B}, a \neq 0$ implies $\text{Re}(\lambda) < 0$. Note that we have silently decided to consider that also complex functions $w : \mathbb{R} \to \mathbb{C}^q$ belong to $\mathfrak{B}$ if $\text{Re}(w)$ and $\text{Im}(w)$ belong to $\mathfrak{B}$. Assume $\exp_\lambda a \in \mathfrak{B}$. Then

$$Q_\Psi(\exp_\lambda a) = \bar{a}^T \Psi(\bar{\lambda}, \lambda) a \exp_{2\text{Re}(\lambda)},$$

and

$$Q_{\overset{\bullet}{\Psi}}(\exp_\lambda a) = 2\text{Re}(\lambda) \bar{a}^T \Psi(\bar{\lambda}, \lambda) a \exp_{2\text{Re}(\lambda)}.$$

Hence $Q_\Psi(w) \stackrel{\mathfrak{B}}{\geq} 0$ implies $\bar{a}^T \Psi(\bar{\lambda}, \lambda) a \leq 0$. Therefore $Q_{\overset{\bullet}{\Psi}}(w) \stackrel{\mathfrak{B}}{<} 0$ implies $\text{Re}(\lambda) < 0$, as desired.

The 'only if' part will be proven as part of theorem 8.

Proposition 7:

Asymptotic stability follows from theorem 6. Let $w \in \mathfrak{B}$. Then

$$Q_\Psi(w)(0) = -\int_0^\infty Q_{\overset{\bullet}{\Psi}}(w) dt.$$

Since $\overset{\bullet}{\Psi} \stackrel{\mathfrak{B}}{\leq} 0$ and $w \neq 0$ imply $Q_{\overset{\bullet}{\Psi}}(w) \neq 0$, it follows that $Q_\Psi(w)(0) > 0$.

Theorem 8:

The key to this proof is that if R is Hurwitz, then the polynomial Lyapunov equation (4) has a solution X for all $Z = Z^\star$. In order to see this, verify first that without loss of generality, we can assume that R is in Smith form. Then in the obvious notation equation (4) reduces to q^2 scalar equations of the form $x^\star_{k,\ell} r_k + r^\star_\ell x_{\ell,k} = z_{k,\ell}$ with r_k and r_ℓ Hurwitz. Since the polynomials r_k and $r^\star_\ell$ are co-prime, these equations are indeed solvable.Now consider (4) with $Z = \partial\Phi$. Let X be a solution. Take $Y \in \mathbb{R}^{\bullet \times q}[\zeta, \eta]$ such that $\partial Y = -X$. Then (7) yields $Q_{\overset{\bullet}{\Psi}} \stackrel{\mathfrak{B}}{=} Q_\Phi$, as desired. Note that Q_Ψ is in fact given by

$$Q_\Psi(w)(0) \stackrel{\mathfrak{B}}{=} -\int_0^\infty Q_\Phi(w) dt$$

This formula shows that $\Phi \overset{\mathfrak{B}}{\leq} 0$ implies $\Psi \overset{\mathfrak{B}}{\geq} 0$. That $\Phi \overset{\mathfrak{B}}{<} 0$ implies $\Psi \overset{\mathfrak{B}}{\gg} 0$ follows from proposition 7 .

Theorem 9:

Let $\mathfrak{B} = \ker(p\frac{d}{dt})$.

$(i) \Leftrightarrow (ii)$ is a (perhaps the most) classical result from the theory of differential equations.

$(i) \Leftrightarrow (iii)$ follows from the fact that $\frac{d}{dt}Q_{B_p}(w) \overset{\mathfrak{B}}{=} -|p(-\frac{d}{dt})w|^2$, and theorem 6.

To see that $(i) \Rightarrow (iv)$, note that for $w \in \mathfrak{B}$, there holds

$$Q_{B_p}(w)(0) = \int_0^\infty |p(-\frac{d}{dt})w|^2 dt.$$

Hence, $Q_{B_p}(w)(0) > 0$ for all $w(0), \frac{d}{dt}w(0), \ldots, \frac{d^{n-1}}{dt^{n-1}}w(0)$. This implies that $\tilde{B}_p$ is positive definite.

To show that $(iv) \Rightarrow (iii)$ we need to show that p and $p^\star$ are co-prime. Assume that $p = rq$ and $p^\star = rq^\star$. Then $B_p(\zeta, \eta) = r(\zeta)B_q(\zeta, \eta)r(\eta)$. From this it follows that $\text{rank}(\tilde{B}_p) \leq \text{rank}(\tilde{B}_q))$. Hence $\text{rank}(\tilde{B}_p) = n$ implies that p and $p^\star$ are co-prime.

Theorem 10:

We will run the circle $(i) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i)$.

The implications $(i) \Rightarrow (iii)$ and $(iv) \Rightarrow (ii)$ are obvious.

$(iii) \Rightarrow (iv)$ follows from theorem 4. Indeed, (iii) implies the existence of $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta, \eta]$ such that $\frac{d}{dt}Q_\Psi \overset{\mathfrak{B}}{=} Q_\Phi - Q_\Delta$, which yields (iv).

The implication $(ii) \Rightarrow (v)$ follows since $\mathfrak{B}$ consists of the w's of the form $w = M(\frac{d}{dt})\ell$. It can be shown that the elements in $\mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ are those generated by the ℓ's which are also of compact support. Since

$$\int_{-\infty}^{+\infty} Q_\Phi(w)dt = \int_{-\infty}^{+\infty} Q_{\Phi'}(\ell)dt$$

the implication follows.

To show $(v) \Rightarrow (vi)$, observe that we can without loss of generality assume that R is in Smith form, $R(\xi) = [I \;\; 0]$. Partition Φ comformably as

$$\Phi = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} \\ \Phi_{1,2}^* & \Phi_{2,2} \end{bmatrix}.$$

Then (v) states that $\int_{-\infty}^{+\infty} Q_{\Phi_{11}}(\ell)dt \geq 0$ for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\text{coldim}(M)})$ of compact support. Using Fourier transforms and arguments analogous to the implication

$(ii) \Leftrightarrow (iii)$ of theorem 2, it is possible to prove that this is the case if and only if $\partial\Phi_{1,1}(i\omega) \geq 0$ for all ω, which yields (vi).

To show that (vi) implies (i), assume

$$\Phi(-i\omega, i\omega) + X^T(-i\omega)R(i\omega) + R^T(-i\omega)X(i\omega) \geq 0$$

for all $\omega \in \mathbb{R}$. Use spectral factorization to obtain a $D \in \mathbb{R}^{\bullet\times q}[\xi]$ such that $\Phi(-i\omega, i\omega) = D^T(-i\omega)D(i\omega)$. Now use $\Delta(\zeta, \eta) = D^T(\zeta)D(\eta)$ and

$$\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) + Y^*(\zeta, \eta)R(\eta) + R^T(\zeta)Y(\zeta, \eta) - \Delta(\zeta, \eta)}{\zeta + \eta}$$

with Y such that $\partial Y = X$. It is easy to verify that $\mathfrak{B}$ is dissipative with respect to (Φ, Ψ, Δ).

Theorem 11:

(only if): $\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w)$ and $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ implies

$$Q_\Psi(w)(0) \leq \int_{-\infty}^0 Q_\Phi(w)dt.$$

Hence $\Psi \overset{\mathfrak{B}}{\geq} 0$ implies $\int_{-\infty}^0 Q_\Phi(w)dt \geq 0$.

(if) Using controllability and image representations, it suffices to give the proof in the case $\mathfrak{B} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$. Note that $\int_{-\infty}^0 Q_\Phi(w)dt \geq 0$ for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$ implies, using theorem 10, $\Phi(-i\omega, i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. We will give the proof under the somewhat stronger assumption that there exists $\varepsilon > 0$ such that $\Phi(-i\omega, i\omega) > \varepsilon I$ for all $\omega \in \mathbb{R}$. In this case, $\Phi(-i\omega, i\omega)$ can be factored as$A^T(-i\omega)A(i\omega)$ with A anti-Hurwitz (i.e., with $A^\star$ Hurwitz). Define

$$\Psi(\zeta, \eta) = \frac{\Phi(\zeta, \eta) - A^T(\zeta)A(\eta)}{\zeta + \eta}.$$

Then

$$Q_\Psi(w)(0) = \int_{-\infty}^0 Q_\Phi(w)dt - \int_{-\infty}^0 ||A(\frac{d}{dt})w||^2 dt$$

for all $w \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^q)$. Now apply this formula for the w's such that $A(\frac{d}{dt})w = 0$ to conclude (using a suitable limit argument to get around the compact support difficulty) that $Q_\Psi(w)(0) \geq 0$ for all w's in the behavior of $A(\frac{d}{dt})w = 0$. To prove that this indeed implies $Q_\Psi \geq 0$ requires an argument for which we refer to [24].

Theorem 12:

See [17].

Equations (16) and (17):

Since $\mathfrak{B}$ is strictly dissipative with respect to $\Phi = \Phi^* \in \mathbb{R}^{q\times q}[\zeta,\eta]$ on $\mathbb{R}_-$, there exists an $\varepsilon > 0$ such that

$$\int_{-\infty}^{0} Q_\Phi(w)dt \geq \varepsilon \int_{-\infty}^{0} ||w||^2 dt$$

for $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R},\mathbb{R}^q)$. Using theorem 11 this yields the existence of $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta] \geq 0$ such that

$$\frac{d}{dt}Q_\Psi(w) \leq Q_\Phi(w) - \varepsilon||w||^2,$$

for $w \in \mathfrak{B}$. This yields (16). Next, observe that there exists $M < \infty$ such that

$$\frac{d}{dt}||X(\frac{d}{dt})w||^2 \leq M||w||^2$$

for $w \in \mathfrak{B}$. Combining the above two inequalities yields (17).

Theorem 13 :

Integrating (16) yields

$$\varepsilon \int_0^t ||w||^2 dt \leq \int_0^t Q - \Phi(w)dt - Q_\Psi(w)(t) + Q_\Psi(w)(0).$$

Since $\Psi = \Psi^* \in \mathbb{R}^{q\times q}[\zeta,\eta] > 0$, this yields

$$\varepsilon \int_0^t ||w||^2 dt \leq \int_0^t Q - \Phi(w)dt + Q_\Psi(w)(0)$$

for $w \in \mathfrak{B}$. Now assume $\int_0^\infty Q - \Phi(w)dt < \infty$, and let $t \to \infty$ in the above inequality to obtain $\int_0^\infty ||w||^2 dt < \infty$. This yields $\mathfrak{B}_a/\mathfrak{B}_g$–stability.

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