

Linear Systems in Discrete Time

Jan C. Willems

Abstract Representations of linear time-invariant discrete-time systems are discussed. A system is defined as a behavior, that is, as a family of trajectories mapping the time axis into the signal space. The following characterizations are equivalent: (i) the system is linear, time-invariant, and complete, (ii) the behavior is linear, shift-invariant, and closed, (iii) the behavior is kernel of a linear difference operator with a polynomial symbol, (iv) the system allows a linear input/output representation in terms of polynomial matrices, (v) the system allows a linear constant coefficient input/state/output representation, and (vi) the behavior is kernel of a linear difference operator with a rational symbol. If the system is controllable, then the system also allows (vii) an image representation with a polynomial symbol, and an image representation with a rational symbol.

1 Introduction

It is a pleasure to contribute an article to this Festschrift dedicated to Professor Okko Bosgra on the occasion of his 'emeritaat'.

The aim of this presentation is to discuss representations of discrete-time linear time-invariant systems described by difference equations. We discuss systems from the behavioral point of view. Details of this approach may be found in [1, 2, 3, 4, 5].

We view a model as a subset \mathcal{B} of a universum \mathcal{U} of a priori possibilities. This subset $\mathcal{B} \subseteq \mathcal{U}$ is called the *behavior* of the model. Thus, before the phenomenon was captured in a model, all outcomes from \mathcal{U} were in principle possible. But after we accept \mathcal{B} as the model, we declare that only outcomes from \mathcal{B} are possible.

In the case of *dynamical* systems, the phenomenon which is modeled produces functions that map the set of time instances relevant to the model to the *signal space*.

Jan C. Willems
ESAT-SISTA, K.U. Leuven, B-3001 Leuven, Belgium e-mail: Jan.Willems@esat.kuleuven.be

This is the space in which these functions take on their values. In this article we assume that the set of relevant time instances is $\mathbb{N} := \{1, 2, 3, \dots\}$ (the theory is analogous for \mathbb{Z} , \mathbb{R} , and \mathbb{R}_+). We assume also that the signal space is a finite-dimensional real vector space, typically \mathbb{R}^w .

Following our idea of a model, the behavior of the dynamical systems which we consider is therefore a collection \mathcal{B} of functions mapping the time set \mathbb{N} into the signal space \mathbb{R}^w . A dynamical model can therefore be identified with its *behavior* $\mathcal{B} \subseteq (\mathbb{R}^w)^\mathbb{N}$. The behavior is hence a family of maps from \mathbb{N} to \mathbb{R}^w . Of course, also for dynamical systems the behavior \mathcal{B} is usually specified as the set of solutions of equations, for the case at hand typically difference equations. As dynamical models, difference equations thus merely serve as a representation of their solution set. Note that this immediately leads to a notion of equivalence and to canonical forms for difference equations. These are particularly relevant in the context of dynamical systems, because of the multitude of, usually over-parameterized, representations of the behavior of a dynamical system.

2 Linear dynamical systems

The most widely studied model class in systems theory, control, and signal processing consists of dynamical systems that are (i) linear, (ii) time-invariant, and (iii) that satisfy a third property, related to the finite dimensionality of the underlying state space, or to the rationality of a transfer function. It is, however, clearer and advantageous to approach this situation in a more intrinsic way, by imposing this third property directly on the behavior, and not on a representation of it. The purpose of this presentation is to discuss various representations of this model class.

A behavior $\mathcal{B} \subseteq (\mathbb{R}^w)^\mathbb{N}$ is said to be *linear* if $w \in \mathcal{B}$, $w' \in \mathcal{B}$, and $\alpha \in \mathbb{R}$ imply $w + w' \in \mathcal{B}$ and $\alpha w \in \mathcal{B}$, and *time-invariant* if $\sigma\mathcal{B} \subseteq \mathcal{B}$. The *shift* σ is defined by $(\sigma f)(t) := f(t + 1)$. The third property that enters into the specification of the model class is completeness. \mathcal{B} is called *complete* if it has the following property:

$$[[w : \mathbb{N} \rightarrow \mathbb{R}^w \text{ belongs to } \mathcal{B}] \Leftrightarrow [[w|_{[1,t]} \in \mathcal{B}|_{[1,t]} \text{ for all } t \in \mathbb{N}].$$

In words, \mathcal{B} is complete if we can decide that $w : \mathbb{N} \rightarrow \mathbb{R}^w$ is ‘legal’ (i.e. belongs to \mathcal{B}) by verifying that each of its ‘prefixes’ $(w(1), w(2), \dots, w(t))$ is ‘legal’ (i.e. belongs to $\mathcal{B}|_{[1,t]}$). So, roughly speaking, \mathcal{B} is complete iff the laws of \mathcal{B} do not involve what happens at $+\infty$. Requirements as $w \in \ell_2(\mathbb{N}, \mathbb{R}^w)$, w has compact support, or $\lim_{t \rightarrow \infty} w(t)$ exists, risk at obstructing completeness. However, often crucial information about a complete \mathcal{B} can be obtained by considering its intersection with $\ell_2(\mathbb{N}, \mathbb{R}^w)$, or its compact support elements, etc.

Recall the following standard notation. $\mathbb{R}[\xi]$ denotes the polynomials with real coefficients in the indeterminate ξ , $\mathbb{R}(\xi)$ the real rational functions, and $\mathbb{R}^{n_1 \times n_2}[\xi]$ the polynomial matrices with real $n_1 \times n_2$ matrices as coefficients. When the number of rows is irrelevant and the number of columns is n , the notation $\mathbb{R}^{\bullet \times n}[\xi]$ is used.

So, in effect, $\mathbb{R}^{\bullet \times n}[\xi] = \cup_{k \in \mathbb{N}} \mathbb{R}^{k \times n}[\xi]$. A similar notation is used for polynomial vectors, or when the number of rows and/or columns is irrelevant. The degree of $P \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ equals the largest degree of its entries, and is denoted by $\text{degree}(P)$.

Given a time-series $w : \mathbb{N} \rightarrow \mathbb{R}^w$ and a polynomial matrix $R \in \mathbb{R}^{v \times w}[\xi]$, say $R(\xi) = R_0 + R_1\xi + \dots + R_L\xi^L$, we can form the new v -dimensional time-series

$$R(\sigma)w = R_0w + R_1\sigma w + \dots + R_L\sigma^L w.$$

Hence $R(\sigma) : (\mathbb{R}^w)^\mathbb{N} \rightarrow (\mathbb{R}^v)^\mathbb{N}$, with $R(\sigma)w : t \in \mathbb{N} \mapsto R_0w(t) + R_1w(t+1) + \dots + R_Lw(t+L) \in \mathbb{R}^v$.

The combination of linearity, time-invariance, and completeness can be expressed in many equivalent ways. In particular, the following are equivalent:

1. $\mathcal{B} \subseteq (\mathbb{R}^w)^\mathbb{N}$ is linear, time-invariant, and complete;
2. \mathcal{B} is a linear, shift invariant ($:\Leftrightarrow \sigma\mathcal{B} \subseteq \mathcal{B}$), closed subset of $(\mathbb{R}^w)^\mathbb{N}$, with ‘closed’ understood in the topology of pointwise convergence;
3. $\exists R \in \mathbb{R}^{\bullet \times w}[\xi]$ such that \mathcal{B} consists of the solutions $w : \mathbb{N} \rightarrow \mathbb{R}^w$ of

$$R(\sigma)w = 0. \tag{1}$$

The set of behaviors $\mathcal{B} \subseteq (\mathbb{R}^w)^\mathbb{N}$ that satisfy the equivalent conditions 1. to 3. is denoted by \mathcal{L}^w , or, when the number of variables is unspecified, by \mathcal{L}^\bullet . Thus, in effect, $\mathcal{L}^\bullet = \cup_{w \in \mathbb{N}} \mathcal{L}^w$. Since $\mathcal{B} = \text{kernel}(R(\sigma))$ in (1), we call (1) a *kernel representation* of the behavior \mathcal{B} .

3 Polynomial annihilators

We now introduce a characterization that is mathematically more abstract. It identifies a behavior $\mathcal{B} \in \mathcal{L}^\bullet$ with an $\mathbb{R}[\xi]$ -module.

Consider $\mathcal{B} \in \mathcal{L}^w$. The polynomial vector $n \in \mathbb{R}^{1 \times w}[\xi]$ is called an *annihilator* (or a *consequence*) of \mathcal{B} if $n(\sigma)\mathcal{B} = 0$, i.e. if $n(\sigma)w = 0$ for all $w \in \mathcal{B}$. Denote by $\mathcal{N}_{\mathcal{B}}$ the set of annihilators of \mathcal{B} . Observe that $\mathcal{N}_{\mathcal{B}}$ is an $\mathbb{R}[\xi]$ -module. Indeed, $n \in \mathcal{N}_{\mathcal{B}}, n' \in \mathcal{N}_{\mathcal{B}}$, and $\alpha \in \mathbb{R}[\xi]$ imply $n + n' \in \mathcal{N}_{\mathcal{B}}$ and $\alpha n \in \mathcal{N}_{\mathcal{B}}$. Hence the map $\mathcal{B} \mapsto \mathcal{N}_{\mathcal{B}}$ associates with each $\mathcal{B} \in \mathcal{L}^w$ a submodule of $\mathbb{R}^{1 \times w}[\xi]$. It turns out that this map is actually a bijection, i.e. to each submodule of $\mathbb{R}^{1 \times w}[\xi]$, there corresponds exactly one element of \mathcal{L}^w . It is easy to see what the inverse map is. Let \mathcal{K} be a submodule of $\mathbb{R}^{1 \times w}[\xi]$. Submodules of $\mathbb{R}^{1 \times w}[\xi]$ have nice properties. In particular, they are *finitely generated*, meaning that there exist elements (‘generators’) $g_1, g_2, \dots, g_g \in \mathcal{K}$ such that \mathcal{K} consists precisely of the linear combinations $\alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_g g_g$ where the α_k ’s range over $\mathbb{R}[\xi]$. Now consider the system (1) with $R = \text{col}(g_1, g_2, \dots, g_g)$ and prove that

$$\mathcal{N}_{\text{kernel}(\text{col}(g_1, g_2, \dots, g_g)(\sigma))} = \mathcal{K}$$

(\supseteq is obvious, \subseteq requires a little bit of analysis). In terms of (1), we obtain the characterization

$$\llbracket \text{kernel}(R(\sigma)) = \mathcal{B} \rrbracket \Leftrightarrow \llbracket \mathcal{N}_{\mathcal{B}} = \langle R \rangle \rrbracket$$

where $\langle R \rangle$ denotes the $\mathbb{R}[\xi]$ -module generated by the rows of R .

The observation that there is a bijective correspondence between \mathcal{L}^w and the $\mathbb{R}[\xi]$ -submodules of $\mathbb{R}^{1 \times w}[\xi]$ is not altogether trivial. For instance, the surjectivity of the map

$$\mathcal{B} = \text{kernel}(R(\sigma)) \in \mathcal{L}^w \mapsto \mathcal{N}_{\mathcal{B}} = \langle R \rangle$$

onto the $\mathbb{R}[\xi]$ -submodules of $\mathbb{R}^{1 \times w}[\xi]$ depends on the solution concept used in (1). If we would have considered only solutions with compact support, or that are square integrable, this bijective correspondence is lost. Equations, in particular difference or differential equations, all by themselves, without a clear solution concept, i.e. without a definition of the corresponding behavior, are an inadequate specification of a mathematical model. Studying linear time-invariant difference (and certainly differential) equations is not just algebra, through the solution concept, it also requires analysis.

The characterization of \mathcal{B} in terms of its module of annihilators shows precisely what we are looking for in order to identify a system in the model class \mathcal{L}^\bullet : (a set of generators of) the submodule $\mathcal{N}_{\mathcal{B}}$.

4 Input/output representations

Behaviors in \mathcal{L}^\bullet admit many other representations. The following two are exceedingly familiar to system theorists. In fact,

- 4) $\llbracket \mathcal{B} \in \mathcal{L}^w \rrbracket \Leftrightarrow \llbracket \exists \text{ integers } m, p \in \mathbb{Z}_+, \text{ with } m + p = w, \text{ polynomial matrices } P \in \mathbb{R}^{p \times p}[\xi], Q \in \mathbb{R}^{p \times m}[\xi], \text{ with } \det(P) \neq 0, \text{ and a permutation matrix } \Pi \in \mathbb{R}^{w \times w} \text{ such that } \mathcal{B} \text{ consists of all } w : \mathbb{N} \rightarrow \mathbb{R}^w \text{ for which there exist } u : \mathbb{N} \rightarrow \mathbb{R}^m \text{ and } y : \mathbb{N} \rightarrow \mathbb{R}^p \text{ such that}$

$$P(\sigma)y = Q(\sigma)u \quad (2)$$

and $w = \Pi \begin{bmatrix} u \\ y \end{bmatrix}$. The matrix of rational functions $G = P^{-1}Q \in (\mathbb{R}(\xi))^{p \times m}$ is called the *transfer function* of (2). Actually, for a given $\mathcal{B} \in \mathcal{L}^w$, it is always possible to choose Π such that G is proper. If we would allow a basis change in \mathbb{R}^w , i.e. allow any non-singular matrix for Π (instead of only a permutation matrix), then we could always take G to be strictly proper.

- 5) $\llbracket \mathcal{B} \in \mathcal{L}^w \rrbracket \Leftrightarrow \llbracket \exists \text{ integers } m, p, n \in \mathbb{Z}_+ \text{ with } m + p = w, \text{ matrices } A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}, \text{ and a permutation matrix } \Pi \in \mathbb{R}^{w \times w} \text{ such that } \mathcal{B} \text{ consists of all } w : \mathbb{N} \rightarrow \mathbb{R}^w \text{ for which there exist } u : \mathbb{N} \rightarrow \mathbb{R}^m, x : \mathbb{N} \rightarrow \mathbb{R}^n, \text{ and } y : \mathbb{N} \rightarrow \mathbb{R}^p \text{ such that}$

$$\sigma x = Ax + Bu, \quad y = Cx + Du, \quad w = \Pi \begin{bmatrix} u \\ y \end{bmatrix}. \quad (3)$$

If we allow also a basis change in \mathbb{R}^w , i.e. allow any non-singular matrix for Π , then we can also take $D = 0$.

(2) is called an *input/output* (i/o) and (3) an *input/state/output* (i/s/o) representation of the corresponding behavior $\mathcal{B} \in \mathcal{L}^w$.

Why, if any element $\mathcal{B} \in \mathcal{L}^\bullet$ indeed admits a representation (2) or (3), should one not use one of these familiar representations *ab initio*? There are many good reasons for not doing so. To begin with, and most importantly, first principles models aim at describing a behavior, but are seldom in the form (2) or (3). Consequently, one must have a theory that supersedes (2) or (3) in order to have a clear idea what transformations are allowed in bringing a first principles model into the form (2) or (3). Secondly, as a rule, physical systems are simply not endowed with a signal flow direction. Adding a signal flow direction is often a figment of one's imagination, and when something is not real, it will turn out to be cumbersome sooner or later. A third reason, very much related to the second, is that the input/output framework is totally inappropriate for dealing with all but the most special system interconnections. We are surrounded by interconnected systems, but only very sparingly can these be viewed as input-to-output connections. The second and third reason are valid, in an amplified way, for continuous-time systems. Fourthly, the structure implied by (2) or (3) often needlessly complicates matters, mathematically and conceptually.

A good theory of systems takes the behavior as the basic notion and the reference point for concepts and definitions, and switches back and forth between a wide variety of convenient representations. (2) or (3) have useful properties, but for many purposes other representations may be more convenient. For example, a kernel representation (1) is very relevant in system identification. It suggests that we should look for (approximate) annihilators. On the other hand, when it comes to constructing trajectories, (3) is very convenient. It shows how trajectories are parameterized and generated : by the initial state $x(1) \in \mathbb{R}^n$ and the input $u : \mathbb{N} \rightarrow \mathbb{R}^m$.

5 Representations with rational symbols

Our next representation involves rational functions and is a bit more 'tricky'. Let $G \in (\mathbb{R}(\xi))^{\bullet \times w}$ and consider the system of 'difference equations'

$$G(\sigma)w = 0. \quad (4)$$

What is meant by the behavior of (4)? Since G is a matrix of rational functions, it is not evident how to define solutions. This may be done in terms of co-prime factorizations, as follows. G can be factored $G = P^{-1}Q$ with $P \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ square, $\det(P) \neq 0$, $Q \in \mathbb{R}^{\bullet \times w}[\xi]$ and (P, Q) left co-prime (meaning that $F = [P \ Q]$ is left prime, i.e.

$$\llbracket (U, F' \in \mathbb{R}^{\bullet \times \bullet}[\xi]) \wedge (F = UF') \rrbracket \Rightarrow \llbracket U \text{ is square and unimodular} \rrbracket,$$

equivalently $\exists H \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that $FH = I$. We *define* the behavior of (4) as that of

$$Q(\sigma)w = 0, \quad \text{i.e. as } \ker(Q(\sigma))$$

Hence (4) defines a behavior $\in \mathcal{L}^w$. It is easy to see that this definition is independent of which co-prime factorization is taken. There are other reasonable ways of approaching the problem of defining the behavior of (4), but they all turn out to be equivalent to the definition given. Rational representations are studied in [6]. Note that, in a trivial way, since (1) is a special case of (4), every element of \mathcal{L}^w admits a representation (4).

- 6) $\llbracket \mathcal{B} \in \mathcal{L}^w \rrbracket \Leftrightarrow \llbracket \text{there exists } G \in \mathbb{R}(\xi)^{\bullet \times w} \text{ such that it admits a kernel representation (4)} \rrbracket$.

6 Integer invariants

Certain integer ‘invariants’ (meaning maps from \mathcal{L}^\bullet to \mathbb{Z}_+) associated with systems in \mathcal{L}^\bullet are important. One is the *lag*, denoted by $L(\mathcal{B})$, defined as the smallest $L \in \mathbb{Z}_+$ such that $\llbracket w|_{[t, t+L]} \in \mathcal{B}|_{[1, t+1]} \text{ for all } t \in \mathbb{N} \rrbracket \Rightarrow \llbracket w \in \mathcal{B} \rrbracket$. Equivalently, the smallest degree over the polynomial matrices R such that $\mathcal{B} = \ker(R(\sigma))$. A second integer invariant that is important is the *input cardinality*, denoted by $m(\mathcal{B})$, defined as m , the number of input variables in any (2) representation of \mathcal{B} . It turns out that m is an invariant (while the input/output partition, i.e. the permutation matrix Π in (2), is not). The number of output variables, p , yields the *output cardinality* $p(\mathcal{B})$. A third important integer invariant is the *state cardinality*, $n(\mathcal{B})$, defined as the smallest number n of state variables over all i/s/o representations (3) of \mathcal{B} . The three integer invariants $m(\mathcal{B})$, $n(\mathcal{B})$, and $L(\mathcal{B})$ can be nicely captured in one single formula, involving the growth as a function of t of the dimension of the subspace $\mathcal{B}|_{[1, t]}$. Indeed, there holds

$$\dim(\mathcal{B}|_{[1, t]}) \leq m(\mathcal{B})t + n(\mathcal{B}) \text{ with equality iff } t \geq L(\mathcal{B}).$$

7 Latent variables

State models (3) are an example of the more general, but very useful, class of latent variable models. Such models involve, in addition to the *manifest* variables (denoted by w in (5)), the variables which the model aims at, also auxiliary, *latent* variables (denoted by ℓ in (5)). For the case at hand this leads to behaviors $\mathcal{B}_{\text{full}} \in \mathcal{L}^{w+1}$ described by

$$R(\sigma)w = M(\sigma)\ell, \tag{5}$$

with $R \in \mathbb{R}^{\bullet \times w}[\xi]$ and $M \in \mathbb{R}^{\bullet \times 1}[\xi]$.

Although the notion of observability applies more generally, we use it here for latent variable models only. We call $\mathcal{B}_{\text{full}} \in \mathcal{L}^{w+1}$ *observable* if

$$\llbracket (w, \ell_1) \in \mathcal{B}_{\text{full}} \text{ and } (w, \ell_2) \in \mathcal{B}_{\text{full}} \rrbracket \Rightarrow \llbracket \ell_1 = \ell_2 \rrbracket.$$

(5) defines an observable latent variable system iff $M(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. For state systems (with x the latent variable), this corresponds to the usual observability of the pair (A, C) .

An important result, the *elimination theorem*, states that \mathcal{L}^\bullet is closed under projection. Hence $\mathcal{B}_{\text{full}} \in \mathcal{L}^{w+1}$ implies that the *manifest* behavior

$$\mathcal{B} = \text{projection}(\mathcal{B}) = \{w : \mathbb{N} \rightarrow \mathbb{R}^w \mid \exists \ell : \mathbb{N} \rightarrow \mathbb{R}^1 \text{ such that (5) holds}\}$$

belongs to \mathcal{L}^w , and therefore admits a kernel representation (1) of its own. So, in a trivial sense, (5) is yet another representation of \mathcal{L}^w .

Latent variable representations (also unobservable ones) are very useful in all kinds of applications. This, notwithstanding the elimination theorem. They are the end result of modeling interconnected systems by *tearing, zooming, and linking* [5], with the interconnection variables viewed as latent variables. Many physical models (for example, in mechanics) express basic laws using latent variables.

8 Controllability

In many areas of system theory, controllability enters as a regularizing assumption. In the behavioral theory, an appealing notion of controllability has been put forward. It expresses what is needed intuitively, it applies to any dynamical system, regardless of its representation, it has the classical state transfer definition as a special case, and it is readily generalized, for instance to distributed systems. It is somewhat strange that this definition has not been generally adopted. Adapted to the case at hand, it reads as follows. The time-invariant behavior $\mathcal{B} \subseteq (\mathbb{R}^\bullet)^{\mathbb{N}}$ is said to be *controllable* if for any $w_1 \in \mathcal{B}$, $w_2 \in \mathcal{B}$, and $\mathfrak{t}_1 \in \mathbb{N}$, there exists a $\mathfrak{t}_2 \in \mathbb{N}$ and a $w \in \mathcal{B}$ such that $w(\mathfrak{t}) = w_1(\mathfrak{t})$ for $1 \leq \mathfrak{t} \leq \mathfrak{t}_1$, and $w(\mathfrak{t}) = w_2(\mathfrak{t} - \mathfrak{t}_1 - \mathfrak{t}_2)$ for $\mathfrak{t} > \mathfrak{t}_1 + \mathfrak{t}_2$. For $\mathcal{B} \in \mathcal{L}^\bullet$, one can take without loss of generality $w_1 = 0$ in the above definition. Denote the controllable elements of \mathcal{L}^\bullet by $\mathcal{L}_{\text{cont}}^\bullet$ and of \mathcal{L}^w by $\mathcal{L}_{\text{cont}}^w$.

The kernel representation (1) defines a controllable system iff $R(\lambda)$ has the same rank for each $\lambda \in \mathbb{C}$. There is a very nice representation result that characterizes controllability: it is equivalent to the existence of an image representation. More precisely, $\mathcal{B} \in \mathcal{L}_{\text{cont}}^\bullet$ iff there exists $M \in \mathbb{R}^{\bullet \times \bullet}[\xi]$ such that \mathcal{B} equals the manifest behavior of the latent variable system

$$w = M(\sigma)\ell. \tag{6}$$

In other words,

7) $[\mathcal{B} \in \mathcal{L}_{\text{cont}}^\bullet] \Leftrightarrow [\mathcal{B} = \text{im}(M(\sigma))]$.

So, images, contrary to kernels, are always controllable. This image representation of a controllable system can always be taken to be observable.

For $\mathcal{B} \in \mathcal{L}^\bullet$, we define its *controllable part*, denoted by $\mathcal{B}_{\text{controllable}}$, as

$$\mathcal{B}_{\text{controllable}} := \{w \in \mathcal{B} \mid \forall t' \in \mathbb{N}, \exists t'' \in \mathbb{Z}_+, \text{ and } w' \in \mathcal{B} \text{ such that } w'(t) = 0 \text{ for } 1 \leq t \leq t' \text{ and } w'(t) = w(t - t' - t'') \text{ for } t > t' + t''\}.$$

Equivalently, $\mathcal{B}_{\text{controllable}}$ is the largest controllable subsystem contained in \mathcal{B} . It turns out that two systems of the form (2) (with the same input/output partition) have the same transfer function iff they have the same controllable part.

9 Rational annihilators

Consider $\mathcal{B} \in \mathcal{L}^w$. The vector of rational functions $n \in \mathbb{R}^{1 \times w}(\xi)$ is called a *rational annihilator* of \mathcal{B} if $n(\sigma)\mathcal{B} = 0$ (note that, since we gave a meaning to (4), this is well defined). Denote by $\mathcal{N}_{\mathcal{B}}^{\text{rational}}$ the set of rational annihilators of \mathcal{B} . Observe that $\mathcal{N}_{\mathcal{B}}^{\text{rational}}$ is a $\mathbb{R}(\xi)$ -subspace of $\mathbb{R}^{1 \times w}(\xi)$. The map $\mathcal{B} \mapsto \mathcal{N}_{\mathcal{B}}^{\text{rational}}$ is not a bijection from \mathcal{L}^w to the $\mathbb{R}(\xi)$ -subspaces of $\mathbb{R}^{1 \times w}(\xi)$. Indeed,

$$[\mathcal{N}_{\mathcal{B}'}^{\text{rational}} = \mathcal{N}_{\mathcal{B}''}^{\text{rational}}] \Leftrightarrow [\mathcal{B}'_{\text{controllable}} = \mathcal{B}''_{\text{controllable}}].$$

However, there exists a bijective correspondence between $\mathcal{L}_{\text{cont}}^w$ and the $\mathbb{R}(\xi)$ -subspaces of $\mathbb{R}^{1 \times w}(\xi)$. Summarizing, $\mathbb{R}[\xi]$ -submodules of $\mathbb{R}^{1 \times w}[\xi]$ stand in bijective correspondence with \mathcal{L}^w , with each submodule corresponding to the set of polynomial annihilators, while $\mathbb{R}(\xi)$ -subspaces of $\mathbb{R}^{1 \times w}(\xi)$ stand in bijective correspondence with $\mathcal{L}_{\text{cont}}^w$, with each subspace corresponding to the set of rational annihilators.

Controllability enters in a subtle way whenever a system is identified with its transfer function. Indeed, it is easy to prove that the system described by

$$w_2 = G(\sigma)w_1, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad (7)$$

a special case of (4), is automatically controllable. This again shows the limitation of identifying a system with its transfer function. Two input/output systems (2) with the same transfer function are the same iff they are both controllable. In the end, transfer function thinking can deal with non-controllable systems only in contorted ways.

10 Stabilizability

A property related to controllability is stabilizability. The behavior $\mathcal{B} \subseteq (\mathbb{R}^\bullet)^\mathbb{N}$ is said to be *stabilizable* if for any $w \in \mathcal{B}$ and $\mathfrak{t} \in \mathbb{N}$, there exists a $w' \in \mathcal{B}$ such that $w'(\mathfrak{t}') = w(\mathfrak{t}')$ for $1 \leq \mathfrak{t}' \leq \mathfrak{t}$, and $w'(\mathfrak{t}) \rightarrow 0$ for $t \rightarrow \infty$. (1) defines a stabilizable system iff $R(\lambda)$ has the same rank for each $\lambda \in \mathbb{C}$ with $\text{Real}(\lambda) \geq 0$. An important system theoretic result (leading up to the parametrization of stabilizing controllers) states that $\mathcal{B} \in \mathcal{L}^w$ is stabilizable iff it allows a representation (4) with $G \in (\mathbb{R}(\xi))^{\bullet \times w}$ left prime over the ring \mathbb{RH}_∞ ($:= \{f \in \mathbb{R}(\xi) \mid f \text{ is proper and has no poles in the closed right half of the complex plane}\}$). $\mathcal{B} \in \mathcal{L}^w$ is controllable iff it allows a representation $w = G(\sigma)\ell$ with $G \in (\mathbb{R}(\xi))^{w \times \bullet}$ right prime over the ring \mathbb{RH}_∞ .

11 Autonomous systems

Autonomous systems are on the other extreme of controllable ones. $\mathcal{B} \subseteq (\mathbb{R}^\bullet)^\mathbb{N}$ is said to be *autonomous* if for every $w \in \mathcal{B}$, there exists a $\mathfrak{t} \in \mathbb{N}$ such that $w|_{[1, \mathfrak{t}]}$ uniquely specifies $w|_{[\mathfrak{t}+1, \infty)}$, i.e. such that $w' \in \mathcal{B}$ and $w|_{[1, \mathfrak{t}]} = w'|_{[1, \mathfrak{t}]}$ imply $w' = w$. It can be shown that $\mathcal{B} \in \mathcal{L}^\bullet$ is autonomous iff it is finite dimensional. Autonomous systems and, more generally, uncontrollable systems are of utmost importance in systems theory, in spite of much system theory folklore claiming the contrary. Controllability as a system property is much more restrictive than is generally appreciated.

Acknowledgments

The SISTA-SMC research program is supported by the Research Council KUL: GOA AM-BioRICS, CoE EF/05/006 Optimization in Engineering (OPTEC), IOF-SCORES4CHEM, several PhD/postdoc and fellow grants; by the Flemish Government: FWO: PhD/postdoc grants, projects G.0452.04 (new quantum algorithms), G.0499.04 (Statistics), G.0211.05 (Nonlinear), G.0226.06 (cooperative systems and optimization), G.0321.06 (Tensors), G.0302.07 (SVM/Kernel, research communities (ICCoS, ANMMM, MLDM); and IWT: PhD Grants, McKnow-E, Eureka-Flite; by the Belgian Federal Science Policy Office: IUAP P6/04 (DYSCO, Dynamical systems, control and optimization, 2007-2011) ; and by the EU: ERNSI.

References

1. J. C. Willems, From time series to linear system — Part I. Finite dimensional linear time invariant systems, Part II. Exact modelling, Part III. Approximate modelling, *Automatica*, volume 22, pages 561–580 and 675–694, 1986, volume 23, pages 87–115, 1987.
2. J.C. Willems, Paradigms and puzzles in the theory of dynamical systems, *IEEE Transactions on Automatic Control*, volume 36, pages 259–294, 1991.
3. J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Springer-Verlag, 1998.

4. J.C. Willems, Thoughts on system identification, *Control of Uncertain Systems: Modelling, Approximation and Design* (edited by B.A. Francis, M.C. Smith, and J.C. Willems), Springer Verlag Lecture Notes on Control and Information Systems, volume 329, pages 389–416, 2006.
5. J.C. Willems, The behavioral approach to open and interconnected systems, Modeling by tearing, zooming, and linking, *Control Systems Magazine*, volume 27, pages 49–99, 2007.
6. J.C. Willems and Y. Yamamoto, Behaviors defined by rational functions, *Linear Algebra and Its Applications*, volume 425, pages 226–241, 2007.