Methodological Advances and Perspectives in Nonparametric Frontier Analysis

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Léopold Simar
Institut de Statistique, Biostatistique et Sciences Actuarielles
Université Catholique de Louvain, Belgium

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I. Frontier Models and Efficiency Measures

The Frontier Model -1-

- **Economic Theory** Koopmans (1951), Debreu (1951): "Activity Analysis"
  - $x \in \mathbb{R}_+^p$ vector of **inputs**
  - $y \in \mathbb{R}_+^q$ vector of **outputs**
  - **Production set** Ψ of physically attainable points (x, y):

$$\boldsymbol{\Psi} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_+^{\boldsymbol{p}+\boldsymbol{q}} \mid \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}.$$

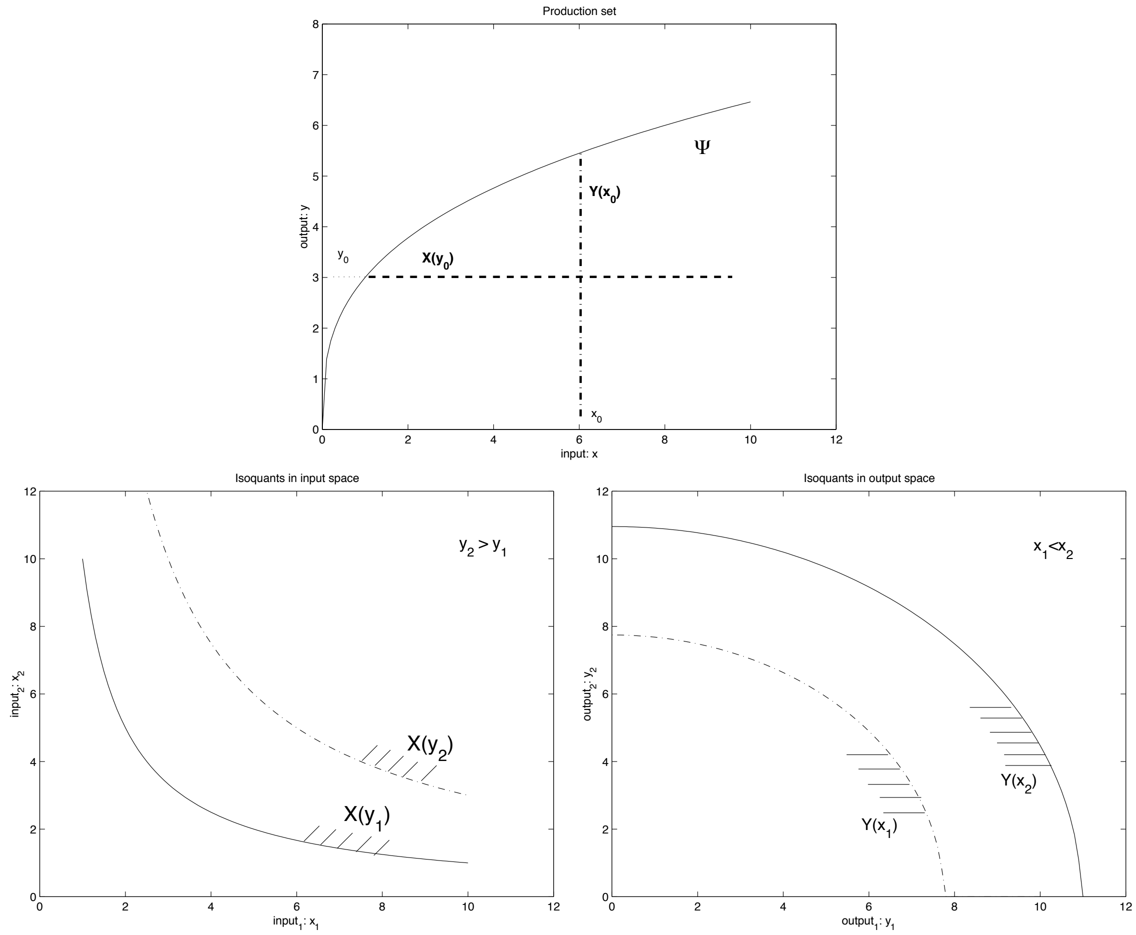
- **The input (output) correspondence sets**
  - Ψ can be described by its sections:

$$\forall\, \boldsymbol{y} \in \boldsymbol{\Psi}, \quad \boldsymbol{X}(\boldsymbol{y}) = \{\boldsymbol{x} \in \mathbb{R}_+^{\boldsymbol{p}} \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\Psi}\}$$

$$\forall\, \boldsymbol{x} \in \boldsymbol{\Psi}, \quad \boldsymbol{Y}(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{R}_+^{\boldsymbol{q}} \mid (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\Psi}\}.$$

  - We have

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{\Psi}\,,\ \boldsymbol{x} \in \boldsymbol{X}(\boldsymbol{y}) \Leftrightarrow \boldsymbol{y} \in \boldsymbol{Y}(\boldsymbol{x}).$$



- **Top panel**: **Production set $\boldsymbol{\Psi}$** for $p = q = 1$.
- **Bottom Panels**: **Correspondence sets $\boldsymbol{X(y)}$ and $\boldsymbol{Y(x)}$** for $p = 2$ and $q = 2$

The Frontier Model -2-

- **Usual Assumptions (a.o.):** (Shephard, 1970)
  - Free Disposability of inputs and outputs

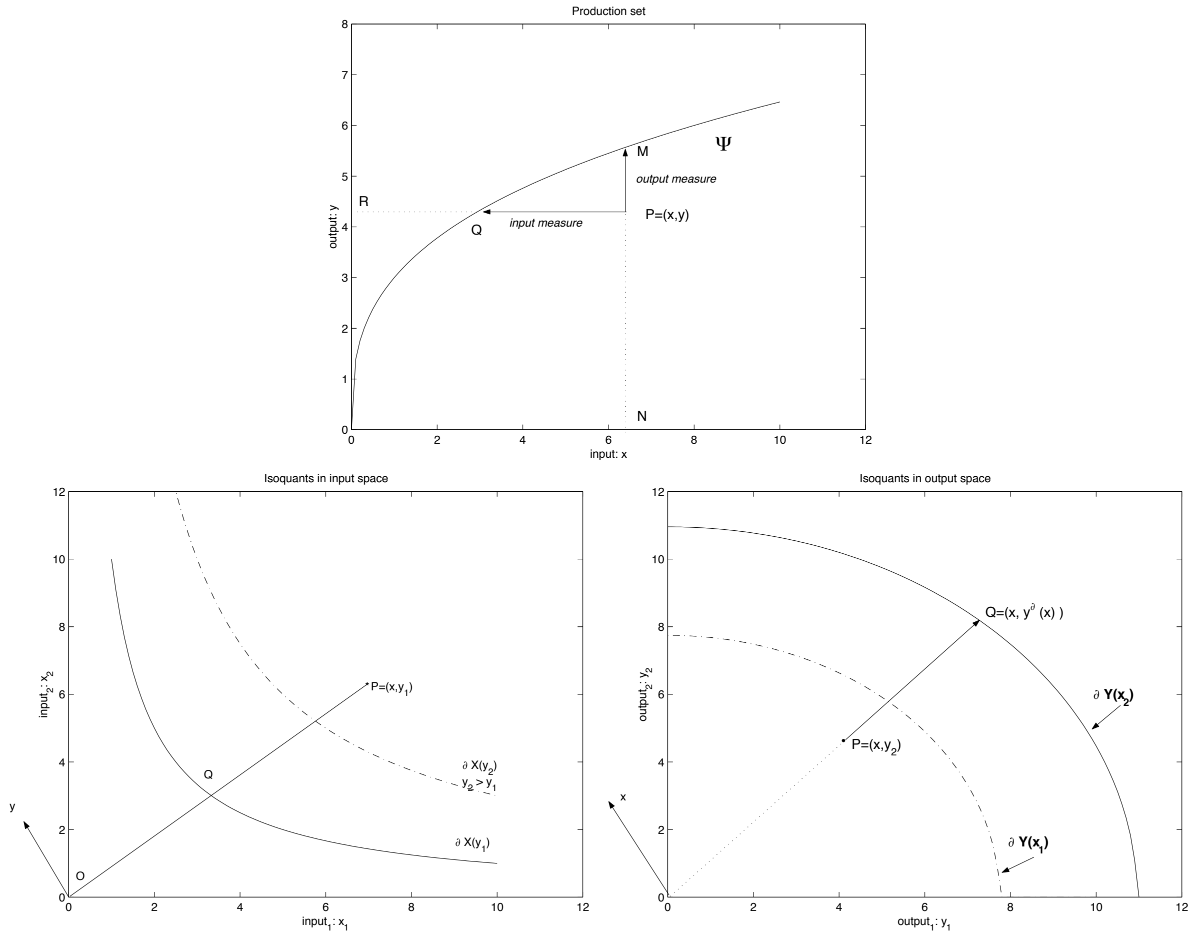
$$\forall (x,y) \in \Psi,\ \text{then if } x' \geq x, y' \leq y,\ (x',y') \in \Psi$$

  - Convexity: if $(x_1,y_1),(x_2,y_2) \in \Psi$, then for all $\alpha \in [0,1]$ we have:

$$(x,y) = \alpha(x_1,y_1) + (1-\alpha)(x_2,y_2) \in \Psi$$

  - No Free Lunches: $(x,y) \notin \Psi$ if $x = 0$ and $y \geq 0, y \neq 0$.

- **Farrell-Debreu Efficiency scores**
  radial measures of distance to the boundary of Ψ
  - **Input oriented**: $\boldsymbol{\theta(x,y) = \inf\{\theta \mid (\theta x, y) \in \Psi\} \leq 1}$
  - **Output oriented**: $\boldsymbol{\lambda(x,y) = \sup\{\lambda \mid (x, \lambda y) \in \Psi\} \geq 1}$



- **Top panel**: $\theta_P = |RQ|/|RP| \leq 1$ and $\lambda_P = |NM|/|NP| \geq 1$.
- **Bottom panels**: $\theta_P = |OQ|/|OP| \leq 1$ and $\lambda_P = |OQ|/|OP| \geq 1$

The Frontier Model -3-

- **Extensions**
  - **Hyperbolic Distances**: adjusts simultaneously input and output levels (Färe et al., 1985, Färe and Grosskopf, 2004).

$$\gamma(x, y|\Psi) = \sup\{\gamma > 0|(\gamma^{-1}x, \gamma y) \in \Psi\}.$$

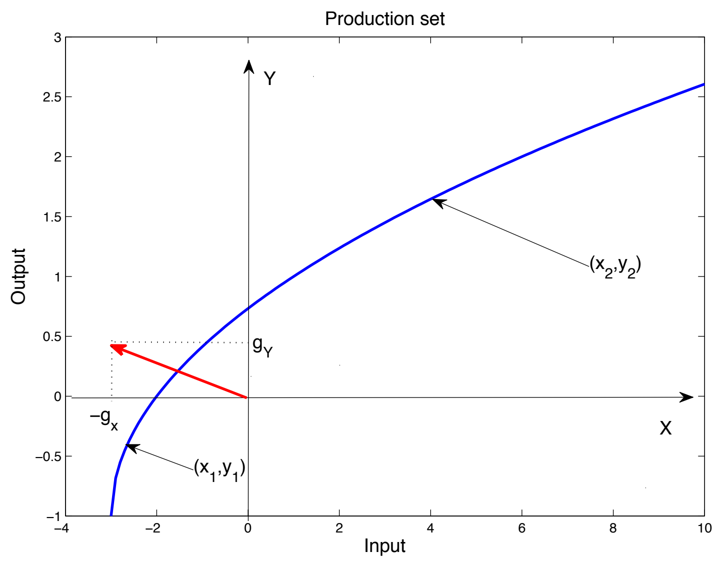
  - **Directional Distances**: Projection of (x, y) onto the technology frontier in a direction $d = (-d_x, d_y)$. (Chambers et al., 1998, Färe and Grosskopf, 2000).

$$\delta(x, y|d_x, d_y, \Psi) = \sup\{\delta|(x - \delta d_x, y + \delta d_y) \in \Psi\}.$$

    * **Additive**: allow negative values of x and/or y.
    * **Special cases**:
      - If $d = (-x, 0)$ with $x > 0$: $\delta(x, y|d_x, d_y, \Psi) = 1 - \theta(x, y|\Psi)^{-1}$
      - If $d = (0, y)$ with $y > 0$: $\delta(x, y|d_x, d_y, \Psi) = \lambda(x, y|\Psi)^{-1} - 1$

The Frontier Model -4-

- Under free disposability, characterization of the technology
  - $\delta(x, y|d_x, d_y, \Psi) \geq 0$ if and only if $(x, y) \in \Psi$
  - $\delta(x, y|d_x, d_y, \Psi) = 0$ if (x, y) is on the frontier.



- **Presentation today and below**: Radial cases, but can be extended (Wilson, 2011, Simar and Vanhems, 2012, Simar, Vanhems and Wilson, 2012)

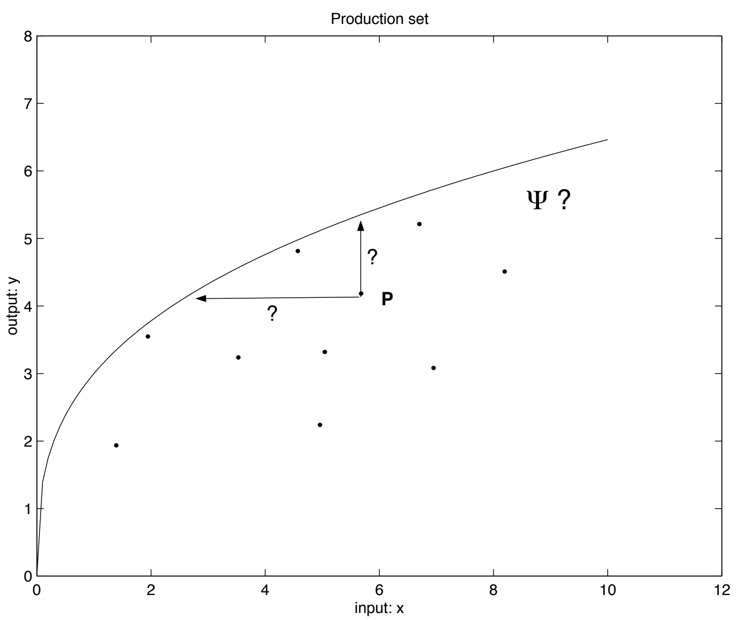
II. The Statistical Paradigm

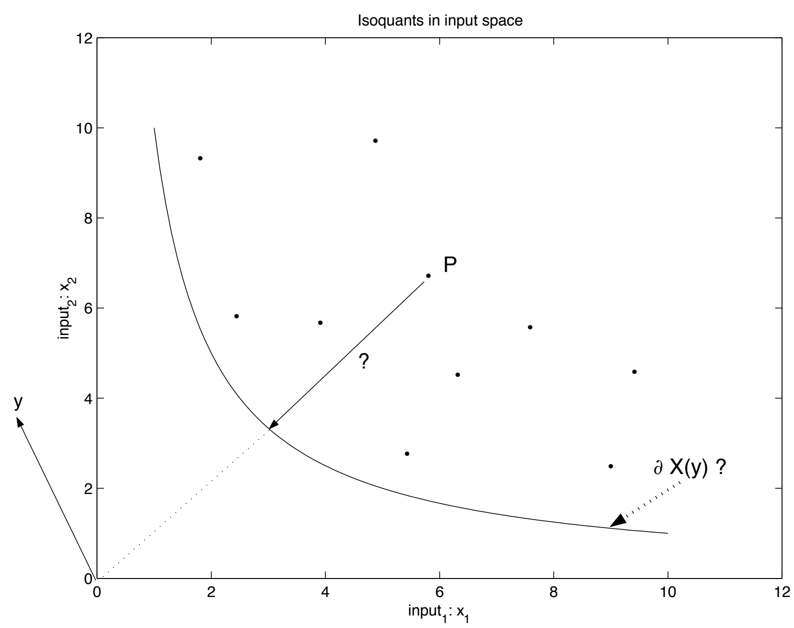
The Statistical Paradigm

- In practice, Ψ is **unknown**

$$\Rightarrow \theta(x,y) \textbf{ and/or } \lambda(x,y) \textbf{ are also unknown}.$$

- Estimation based on a **sample** $\mathcal{X} = \{(x_i, y_i),\ i = 1, \ldots, n\}$





The Statistical Paradigm -2-

- **Different Approaches**
  - **Deterministic** Frontiers: $\text{Prob}\{(x_i, y_i) \in \Psi\} = 1$, pour tout $i = 1, \ldots, n$.
    - No noise on the data, no random shocks ...
    - Distance to frontier is pure inefficiency.
    - Drawback: **sensitivity** to outliers (superefficient units or errors)
  - **Stochastic** Frontiers
    - Random noise: some observations may $\notin \Psi$.
    - Distance to frontier has 2 components (noise and inefficiency)
    - Drawback: **identification** problems
- **Different Models:** for frontier function and for the law of (X, Y), $F(x, y)$
  - Parametric Models: very restrictive, standard methods (MLE, OLS,...)

    e.g. SFA $\quad Y_i = \beta' X_i + V_i - U_i,$ where $V_i \sim N(0, \sigma_V^2), U_i \sim N^+(0, \sigma_U^2),$ indep.
  - Nonparametric Models: very flexible but more difficult and more challenging.

Choosing a Model: A Summary

Models	**Parametric $\mathcal{P}$**	**Nonparametric $\mathcal{NP}$**
Deterministic $\mathcal{D}$	Analytical models for frontier and for $F(x, y)$	No specific model for frontier and for $F(x, y)$
Stochastic $\mathcal{S}$	Analytical models for frontier for $F(x, y)$ including noise	No specific model for frontier and for $F(x, y)$ including noise (Some structure on noise)

Remarks:

- $\mathcal{D} \subseteq \mathcal{S}$ and $\mathcal{P} \subseteq \mathcal{NP}$
- **Horizontal and Vertical** comparisons are legitimate and may be useful.
- **Semiparametric Models**: combine $\mathcal{P}$ and $\mathcal{NP}$ (see below)

Choosing a Model: Inference

Inference is:	**Parametric $\mathcal{P}$**	**Nonparametric $\mathcal{NP}$**
Deterministic $\mathcal{D}$	**Very Easy** COLS, MOLS, MLE (restrictive) **Two-stages**: $\mathcal{P}$ fit of $\mathcal{NP}$ Bootstrap for efficiency scores	**Easy** FDH: $\widehat{F}_n(x, y) \Rightarrow F(x, y)$ DEA: convexify FDH Bootstrap
Stochastic $\mathcal{S}$	**Easy** MOLS, MLE (restricted models) Identification problems (noise vs inefficency) Sensitivity: Bagging	**Complicated** Identification problems (deconvolution problem) Localizing $\mathcal{P}$ and SFDH/SDEA Semi-(non)parametric models

Bootstrap is needed almost everywhere!

The Statistical Paradigm -3-

- **Statistical Inference**
  - Estimation individual inefficiencies ("rankings")
  - Confidence intervals for these measures
  - Specification tests
    - Aggregation of inputs and/or outputs
    - Relevance of the chosen variables
  - Hypothesis testing on the shape of the efficient frontier ("technology")
    - Convexity
    - Returns to scale (increasing/decreasing/constant)
  - Evolution over time
    - Panel data
    - Gain or loss of productivity?
    - Technical progress or gain of efficiency?

The Literature

- **Parametric deterministic or stochastic frontier models**: hundreds of papers in Econometric literature (*Journal of Econometrics*,...)

  Easier but are the parametric assumptions reasonable ones?

- **Nonparametric deterministic frontier models**: thousands of papers in hundreds of different journals (Management sciences, OR, Econometrics)

  Very popular (flexibility) but some drawbacks (see below).

- **Nonparametric stochastic frontier models**: very recent, very few applications (theoretical econometric literature)

  Flexible but so far, hard to use: "work in progress"...

- **Applications:** Banks, Transports (Air, Railways,...), Public Services, Municipalities, Post, School, Education, Research, University, Insurance, Hospitals, Finance, Mutual funds, Industry, Electric plants, Food industry, Agronomy, Macroeconomic, Economy of development, Regional economy,... (*Journal of Productivity Analysis*)

III. Nonparametric Approaches

Nonparametric Estimators: FDH -1-

- **Envelopment Estimators**: estimate $\boldsymbol{\Psi}$ by $\widehat{\boldsymbol{\Psi}}$ which **"envelops"** at best the cloud of n data points $\mathcal{X}$.

- **Free Disposal Hull: FDH** Deprins, Simar, Tulkens (1984)

$$\widehat{\Psi}_{FDH}(\mathcal{X}) = \Big\{(x,y) \in \mathbb{R}_+^{p+q} | y \le y_i,\ x \ge x_i, \quad (x_i, y_i) \in \mathcal{X}\Big\}$$

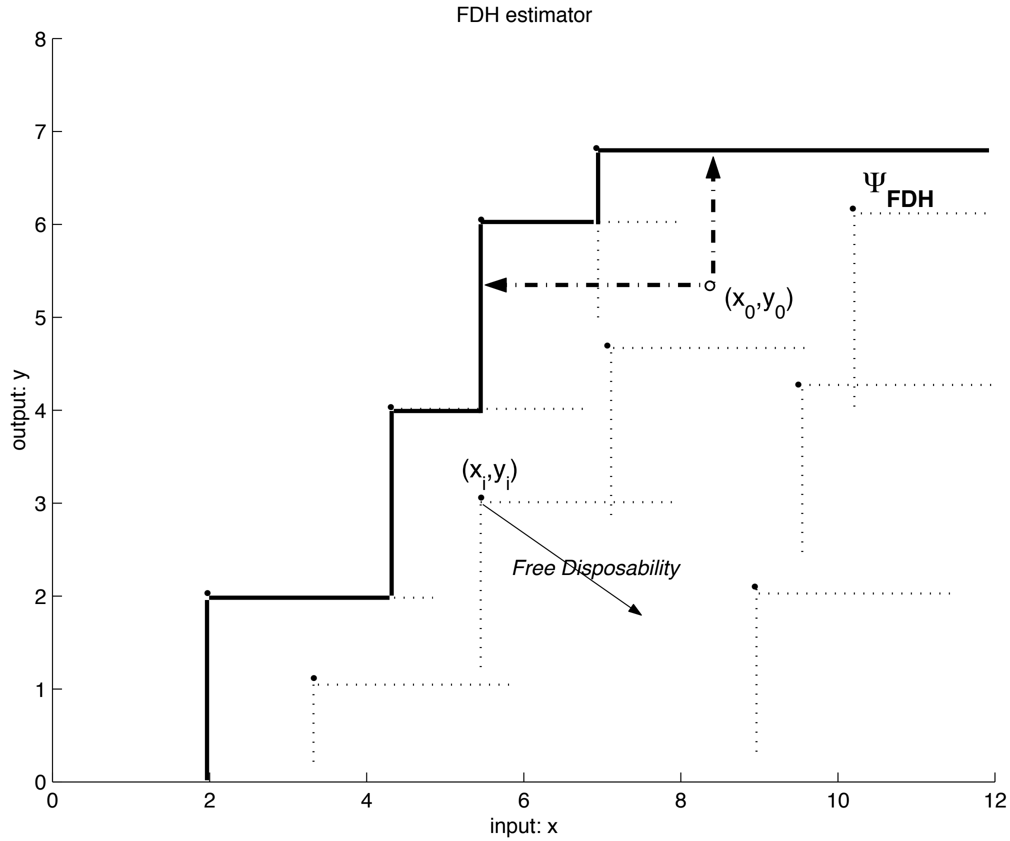
- **FDH efficiency scores**

$$\begin{aligned}\hat{\theta}(x_0, y_0) &= \inf\{\theta \mid (\theta x_0, y_0) \in \widehat{\Psi}_{FDH}(\mathcal{X})\} \\ \hat{\lambda}(x_0, y_0) &= \sup\{\lambda \mid (x_0, \lambda y_0) \in \widehat{\Psi}_{FDH}(\mathcal{X})\}.\end{aligned}$$

- **Practical computations**: fast and easy (sorting algorithms)
  - The set **dominating** points: $D_0 = \{i \mid (x_i, y_i) \in \mathcal{X},\ x_i \le x_0,\ y_i \ge y_0\}$

$$\boldsymbol{\hat{\theta}(x_0, y_0) = \min_{i \in D_0} \ \max_{j=1,\dots,p} \left(\frac{x_i^j}{x_0^j}\right); \qquad \hat{\lambda}(x_0, y_0) = \max_{i \in D_0} \ \min_{j=1,\dots,q} \left(\frac{y_i^j}{y_0^j}\right)}$$

Nonparametric Estimators: FDH -2-



FDH estimator $\widehat{\Psi}_{FDH}$ of the production set Ψ: the $\bullet$ are the observations.

Nonparametric Estimators: DEA -1-

- **Data Envelopment Analysis: DEA** If Ψ is convex:
  - Take the **convex hull** of $\widehat{\Psi}_{FDH}$ (Farrell, 1957, Charnes, Cooper and Rhodes, 1978)

$$\widehat{\Psi}_{DEA} = \quad \{(x,y) \in I\!\!R^{p+q} | y \leq \sum_{i=1}^{n} \gamma_i y_i ; x \geq \sum_{i=1}^{n} \gamma_i x_i \text{ for } (\gamma_1, \ldots, \gamma_n)$$
$$\text{such that } \sum_{i=1}^{n} \gamma_i = 1; \gamma_i \geq 0, i = 1, \ldots, n\}.$$

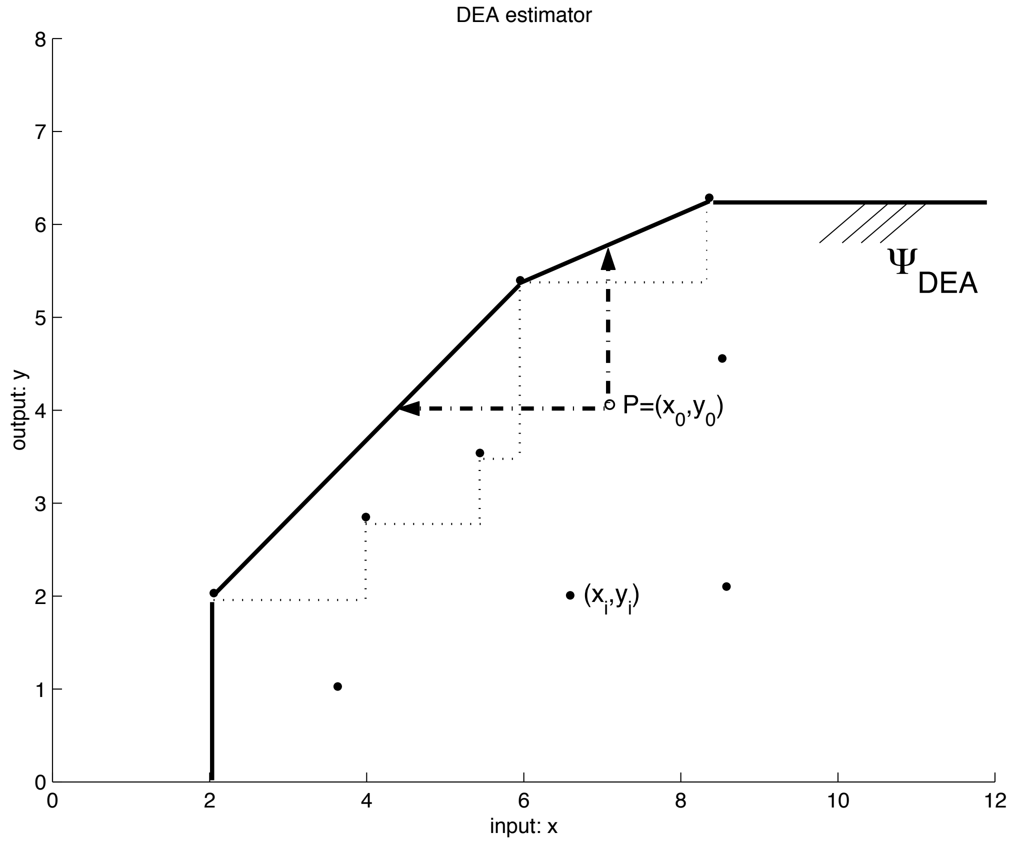
- **Estimation of efficiency score**

$$\begin{aligned} \hat{\theta}(x,y) &= \inf\{\theta \mid (\theta x, y) \in \widehat{\Psi}_{DEA}(\mathcal{X})\} \\ \hat{\lambda}(x,y) &= \sup\{\lambda \mid (x, \lambda y) \in \widehat{\Psi}_{DEA}(\mathcal{X})\} \end{aligned}$$

- **Computation** through **linear programs**.

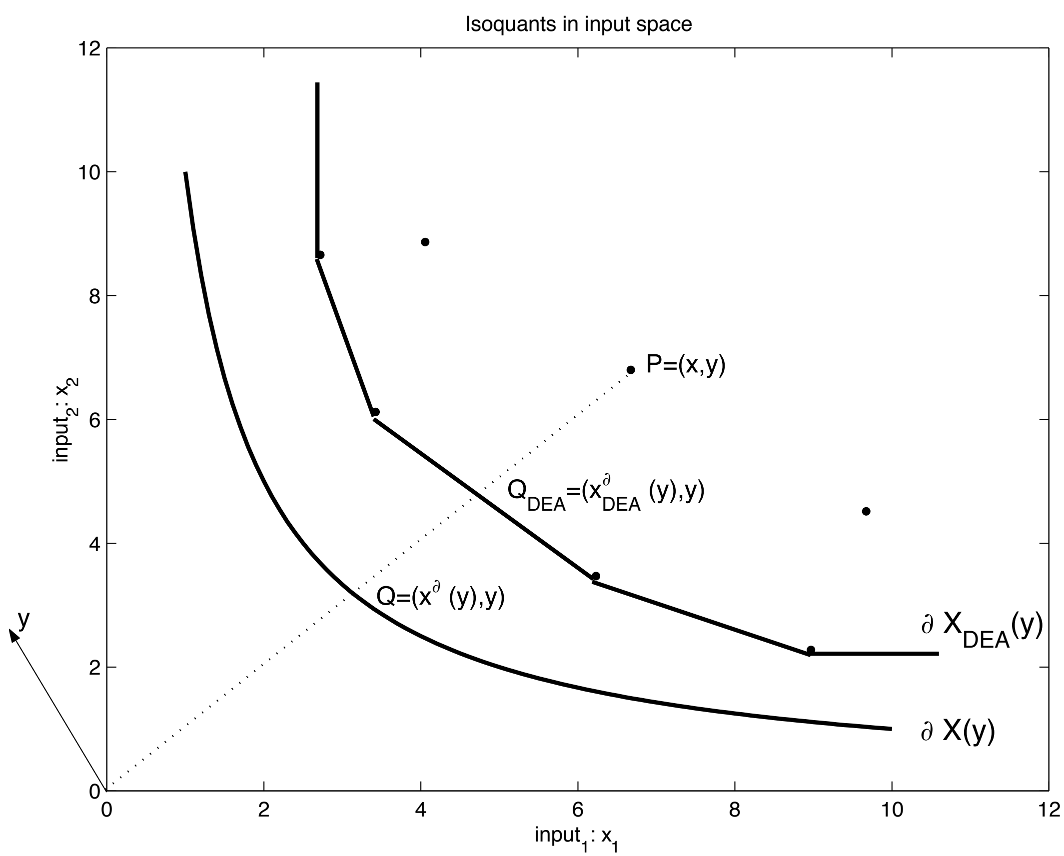
  Available free software: **FEAR** (Wilson, 2008)

Nonparametric Estimators: DEA -2-



DEA estimator $\widehat{\Psi}_{DEA}$ of the production set Ψ: the $\bullet$ are the observations.

Nonparametric Estimators: DEA -3-



Properties of DEA estimators: Relations between $\hat{\theta}_{DEA}(x,y)$ and $\theta(x,y)$?

Statistical Inference: State of the Art -1-

Properties: recent survey, Simar and Wilson (2008)

- **Consistency and rate of convergence**:

$$\left(\hat{\theta}(x,y) - \theta(x,y)\right) = O_p(n^{-\tau}), \text{ as } n \to \infty?$$

  - **FDH**: Korostelev, Simar and Tsybakov (1995a) and Park, Simar and Weiner (2000). Rate is $n^{-1/(p+q)}$.

    Recent Extensions: Daouia, Florens and Simar (2010)
  - **DEA**: Korostelev, Simar and Tsybakov (1995b) and Kneip, Park and Simar (1998). Rate is $n^{-2/(p+q+1)}$. Park, Jeong and Simar (2010) (CRS case), rate is $n^{-2/(p+q)}$.

- **Nice!** but not very useful for the practitionners.

- **Curse of dimensionality**: bad rates if $p+q\uparrow$.

Statistical Inference: State of the Art -2-

Is Inference possible ?

- **Asymptotic sampling distribution**:

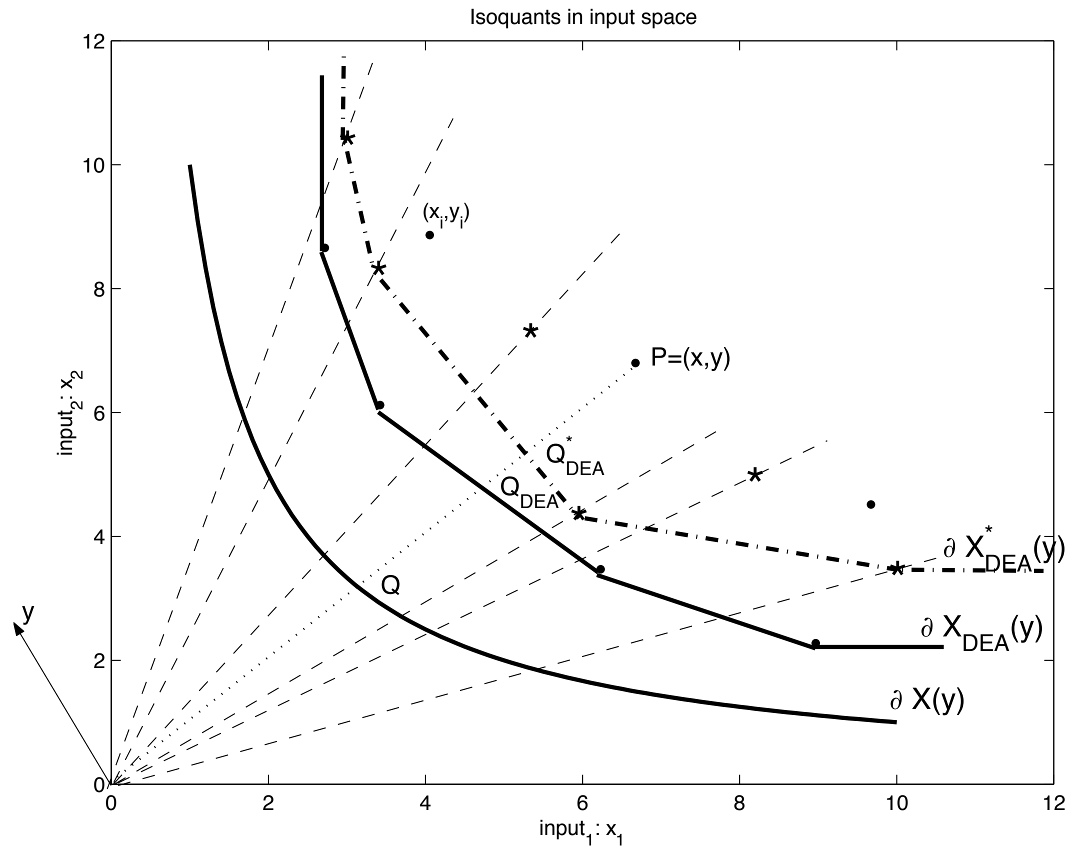
$$n^{\tau}\left(\hat{\theta}(x,y)-\theta(x,y)\right)\sim Q(\eta),\ \text{ as } n\to\infty?$$

  - **FDH**: Park, Simar and Weiner (2000), Badin, Simar (2009), Daouia, Florens and Simar (2010); $Q(\eta)$ is a **Weibull distribution** with unknown parameters to be estimated: not easy to handle and need large sample sizes if $p+q$ increases.
  - **DEA**: Gijbels, Mammen, Park and Simar (1999), Kneip, Simar and Wilson (2008), Park, Jeong, Simar (2010); $Q(\eta)$ is a **Regular distribution** depending on unknown parameters but no closed forms available (untractable for practical purposes) when p or $q>1$.
- **No hope ?** Yes: the bootstrap.

The Bootstrap -1-

Basic Idea

- **The "Real World"**: The Data Generating Process $\mathcal{P}$
  (x_i, y_i) in $\mathcal{X}$ are realizations of iid random variables (X, Y) with probability density function $f(x, y)$ with support Ψ, and $\text{Prob}\big((X, Y) \in \Psi\big) = 1$.
  - $\widehat{\Psi}(\mathcal{X})$ is an estimator of Ψ (FDH or DEA)
  - $\hat{\theta}(x, y) = \inf\{\theta \mid (\theta x, y) \in \widehat{\Psi}(\mathcal{X})\}$ is an estimator of $\theta(x, y)$
- **The "Bootstrap World"**: Consider a DGP $\widehat{\mathcal{P}}$, a consistent estimator of $\mathcal{P}$.
  We can use $\widehat{\Psi}(\mathcal{X})$ (FDH or DEA) and **some appropriate** $\hat{f}(x, y)$ with support $\widehat{\Psi}(\mathcal{X})$, and $\text{Prob}\big((X, Y) \in \widehat{\Psi}(\mathcal{X})\big) = 1$.
- **Bootstrap Analogy**:
  Define a new data set $\mathcal{X}^* = \{(x_i^*, y_i^*),\ i = 1, \ldots, n\}$ drawn from $\widehat{\mathcal{P}}$.
  - $\widehat{\Psi}(\mathcal{X}^*)$ is an estimator of $\widehat{\Psi}(\mathcal{X})$: here, $\widehat{\Psi}(\mathcal{X}^*)$ is the FDH or DEA set computed with $\mathcal{X}^*$ as reference data set.
  - $\hat{\theta}^*(x, y) = \inf\{\theta \mid (\theta x, y) \in \widehat{\Psi}(\mathcal{X}^*)\}$ is an estimator of $\hat{\theta}(x, y)$



The Bootstrap idea:

the • are the original observations (x_i, y_i) generated by the **unknown** $\mathcal{P}$, and the * are the pseudo-observations (x_i^*, y_i^*) generated by the **known** $\widehat{\mathcal{P}}$.

The Bootstrap -2-

- **The Key Relation** : If the Bootstrap is **consistent**, for large n,

$$(\hat{\theta}^*(x,y) - \hat{\theta}(x,y)) \mid \widehat{\mathcal{P}} \quad \approx \quad (\hat{\theta}(x,y) - \theta(x,y)) \mid \mathcal{P}.$$

  – The right part is **unknown** and/or difficult to handle

  – The left part can be approximated by **Monte-Carlo** simulation methods

- **Inference is now available**

  – Bias correction and Standard errors of $\hat{\theta}(x,y)$ are available

  – Confidence intervals for $\theta(x,y)$ can be builded

- **How to generate $\mathcal{X}^*$?** Naive bootstrap **looks easy**: n random drawns of (x_i^*, y_i^*) from $\mathcal{X}$.

- **But naive bootstrap is inconsistent** Simar and Wilson (1998, 1999a, 1999b)

  – The efficient facet, which determines in the original sample $\mathcal{X}$ the value of $\hat{\theta}$, appears **too often**, and with a **fixed** probability, in $\mathcal{X}^*$ and this fixed probability **does not vanish** even when $n \to \infty$.

The Bootstrap -3-

Two Solutions: see Simar and Wilson (1998, 2000, 2011a), Jeong and Simar (2006), Kneip, Simar and Wilson (2008)

- **Subsampling**: draw from $\widehat{\mathcal{P}}$ pseudo-samples of size $m = n^{\kappa}$ where $\kappa < 1$.
  - How to chose m in practice: Simar and Wilson (2011a).
- **Smoothing**: Use smoothed density estimate $\hat{f}(x, y)$ and smooth the boundary of $\widehat{\Psi}$ when defining $\widehat{\mathcal{P}}$: not easy to implement due to the double smoothing.
  - Simplification: homogeneous bootstrap, Simar and Wilson (1998), similar to homoskedastic assumption in regression. But restrictive...
  - Consistent efficient algorithm in the heterogeneous case: Kneip, Simar and Wilson (2011).

Testing issues: Returns to scale, Simar and Wilson (2002), Comparison of groups of firms, Simar and Zelenyuk (2006, 2007), Testing significancy of variables and/or aggregation of variables, Simar and Wilson (2001), and work in progress (convexity,...).

Extensions available: **Hyperbolic distances**, Wilson (2011), **Directional distances**, Simar and Vanhems (2012), Simar, Vanhems and Wilson (2012).

An Example: Program Follow Through (PFT)

- Charnes, Cooper, Rhodes (1981): analysis of an experimental education program administered in US schools: data for 49 schools that implemented PFT, and 21 schools that did not, for a total of 70 observations. 5 inputs and 3 outputs
  - x_1: Education level of the mother (percentage of high school graduates among the mothers),
  - x_2: Highest occupation of a family member (according a pre-arranged rating scale),
  - x_3: Parental visit to school index (number of visits to the school)
  - x_4: Parent counseling index (time spent with child on school related topics)
  - x_5: Number of teachers of the school.

  There are three outputs (results to standard tests):
  - y_1: Total Reading Score (MAT: Metropolitan Achievement Test),
  - y_2: Total Mathematics Score (MAT) and
  - y_3: Coopersmith Self-Esteem Inventory (measure of self-esteem).
- We look for **output efficiency** of the Schools $\lambda(x, y)$ using DEA estimators.

Units	$\hat{\lambda}(x,y)$	Units	$\hat{\lambda}(x,y)$
1	1.0323	50	1.0436
2	1.1093	51	1.0871
3	1.0684	52	1.0000
4	1.1074	53	1.1465
5	1.0000	54	1.0000
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- **Questions**:
  - What is the real value of $\lambda(x,y)$ (bias correction, confidence intervals)?
  - Comparaison of the 2 groups of school:
    * Mean of Group A (49 PFT schools): $\overline{\hat{\lambda}}_A = 1.0589$
    * Mean of Group B (21 Non-PFT schools): $\overline{\hat{\lambda}}_B = 1.0384$ (more efficient?)
  - Is it **significant**?

• The Bootstrap

Units	Eff. Scores	Eff. Bias-Corrected	Bias	Std	Lower Bound	Upper Bound
1	1.0323	1.0671	-0.0348	0.0246	1.0343	1.1268
2	1.1093	1.1387	-0.0294	0.0162	1.1111	1.1702
3	1.0684	1.0979	-0.0295	0.0186	1.0703	1.1396
4	1.1074	1.1264	-0.0190	0.0098	1.1094	1.1463
5	1.0000	1.0530	-0.0530	0.0444	1.0020	1.1651
50	1.0436	1.0725	-0.0289	0.0221	1.0450	1.1239
51	1.0871	1.1102	-0.0231	0.0125	1.0895	1.1373
52	1.0000	1.0558	-0.0558	0.0435	1.0021	1.1542
53	1.1465	1.1718	-0.0253	0.0121	1.1485	1.1954
54	1.0000	1.0520	-0.0520	0.0418	1.0019	1.1484

- After **bias correction** the mean are:
  - Group A (PFT): 1.0940
  - Group B (Non-PFT): 1.0740
- **Formal Test:** $H_0 : E[\lambda(X,Y)|A] = E[\lambda(X,Y)|B]$ vs $H_0 : E[\lambda(X,Y)|A] > E[\lambda(X,Y)|B]$
  - p-value of $H_0 = 0.5590$: $\Rightarrow$ **We do not reject** H_0.

IV. Challenges

Challenges: Drawbacks of DEA/FDH and Solutions

- **Sensitivity to extreme/outliers**: **robust** methods and/or detection of outliers
  - **Order-m frontiers**: Cazals, Florens and Simar (2002), Simar (2003), Daraio and Simar (2006), Daouia, Florens and Simar (2012).
  - **Order-α quantile frontiers**: Aragon, Daouia and Thomas (2005), Daouia and Simar (2005, 2007), Daouia, Florens and Simar (2008, 2010).
- **Lack of Economic interpretation**: **Semiparametric Model**, parametric approximations of nonparametric frontiers, Simar (1992), Florens and Simar (2005), Daouia, Florens and Simar (2008)
- **Heterogeneity:** How to explain inefficiency by environmental/external factors ?
  - **Two-stage methods**, Simar and Wilson (2007, 2011b).
  - **Conditional measures of efficiency**, Cazals, Florens and Simar (2002), Daraio and Simar (2005, 2006, 2007a, 2007,b), Jeong, Park and Simar (2010), Badin, Daraio and Simar (2010, 2012a, 2012b).
- **No noise is allowed**: deterministic frontiers $\text{Prob}\big((X,Y) \in \Psi\big) = 1$: **Nonparametric Stochastic Frontiers?**: Simar (2007), Kumbhakar, Park, Simar and Tsionas (2008), Simar and Zelenyuk, (2011), Kneip, Simar and Van Keilegom (2012), flexible semiparametric models.

IV.1 Sensitivity to Outliers

Robust Frontier -1

Probabilitic Formulation of DGP

- **The DGP**: $H(x,y) = \text{Prob}(X \leq x, Y \geq y)$, **$\Psi$ is the support of $H(x,y)$**
- **Farrell-Debreu Efficiency score** (case of input orientation)

$$H(x,y) = \text{Prob}(X \leq x \,|\, Y \geq y)\,\text{Prob}(Y \geq y) = F_{X|Y}(x|y)\,S_Y(y)$$

$$\theta(x_0,y_0) = \inf\{\theta|(\theta x_0,y_0) \in \Psi\} = \inf\{\theta|F_{X|Y}(\theta x_0|y_0) > 0\}$$

- **Nonparametric Estimator**: Plug-in the empirical version of $H(x,y)$

$$\widehat{H}_n(x,y) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i \leq x, Y_i \geq y), \text{ then } \widehat{F}_{X|Y,n}(x|y) = \frac{\widehat{H}_n(x,y)}{\widehat{H}_n(\infty,y)}$$

- **The FDH estimators**: Cazals, Florens and Simar (2002)
  - $\widehat{\Psi}_{FDH}$ is the support of $\widehat{H}_n(x,y)$
  - Estimation (input) efficiency score: $\widehat{\theta}(x_0,y_0) = \inf\{\theta \,|\, \widehat{F}_{X|Y,n}(\theta x_0|y_0) > 0\}$

Robust Frontier -2-

Partial order frontiers. Economic interpretation (case of univariate output)

Another benchmark frontier less extreme than the "full frontier".

- **Order-m:** Cazals, Florens, Simar (2002)
  - a unit (x, y) is benchmarked against the average maximal output reached by m peers randomly drawn from the population of units using less input than x.
  - As $m \to \infty$, order-m frontier converges to the **full-frontier**.
- **Order-α quantile:** Aragon, Daouia, Thomas (2005), Daouia and Simar (2007)
  - a unit (x, y) is benchmarked against the output level not exceeded by $100(1 - \alpha)\%$ of firms in the population of units using less input than x.
  - As $\alpha \to 1$, order-α frontier converges to the **full-frontier**.

Robust Frontier -2-

Partial order frontiers: Mathematical definition for univariate output

- **Full Frontier Benchmark:** $\varphi(x) = \inf\{y|F_{Y|X}(y|x) \geq 1\}$ and
- **Less Extreme Benchmarks:**
  - **Order-m frontier:**

$$\begin{aligned} \varphi_m(x) &= E\left[\max(Y^1,\ldots,Y^m)|X \leq x\right] \\ &= \int_0^\infty (1-[F_{Y|X}(y|x)]^m)\,dy \end{aligned}$$

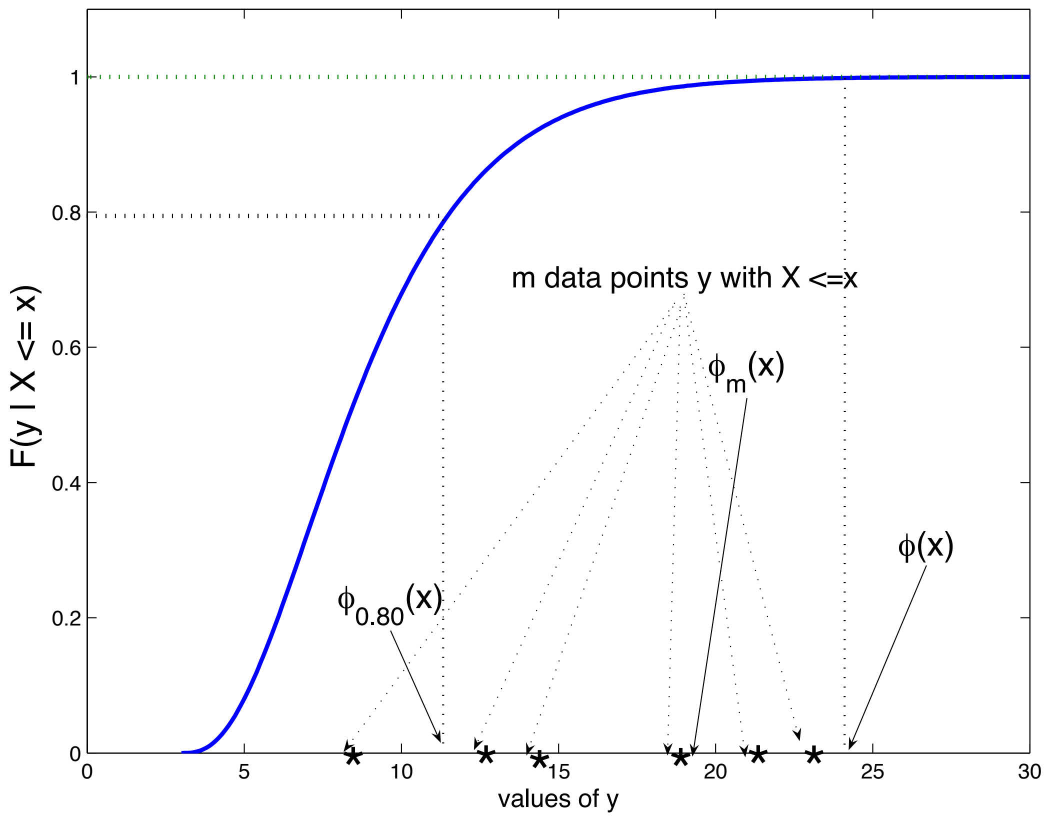
  - **Order-α quantile frontier:**

$$\begin{aligned} \varphi_\alpha(x) &= F^{-1}_{Y|X}(\alpha|x) \\ &= \inf\{y \in \mathbb{R}_+|F_{Y|X}(y|x) \geq \alpha\} \end{aligned}$$

Properties

$$\text{as } m \to \infty,\ \varphi_m(x) \to \varphi(x) \quad \text{and as} \quad \alpha \to 1,\ \varphi_\alpha(x) \to \varphi(x)$$

Robust Frontier -3-



Picture of $F_{Y|X}(y|x) = \text{Prob}(Y \leq y|X \leq x)$

Illustration of **full and partial frontiers**: one output with $m = 6$ and $\alpha = 0.80$

Robust Frontier -4-

Nonparametric estimators of partial order frontier

- **Plug-in principle**

$$\hat{\varphi}_{m,n}(x) = \int_0^\infty (1 - [\widehat{F}_{n,Y|X}(y|x)]^m)\, dy$$
$$\hat{\varphi}_{\alpha,n}(x) = \inf\{y \in \mathbb{R}_+ | \widehat{F}_{n,Y|X}(y|x) \geq \alpha\}$$

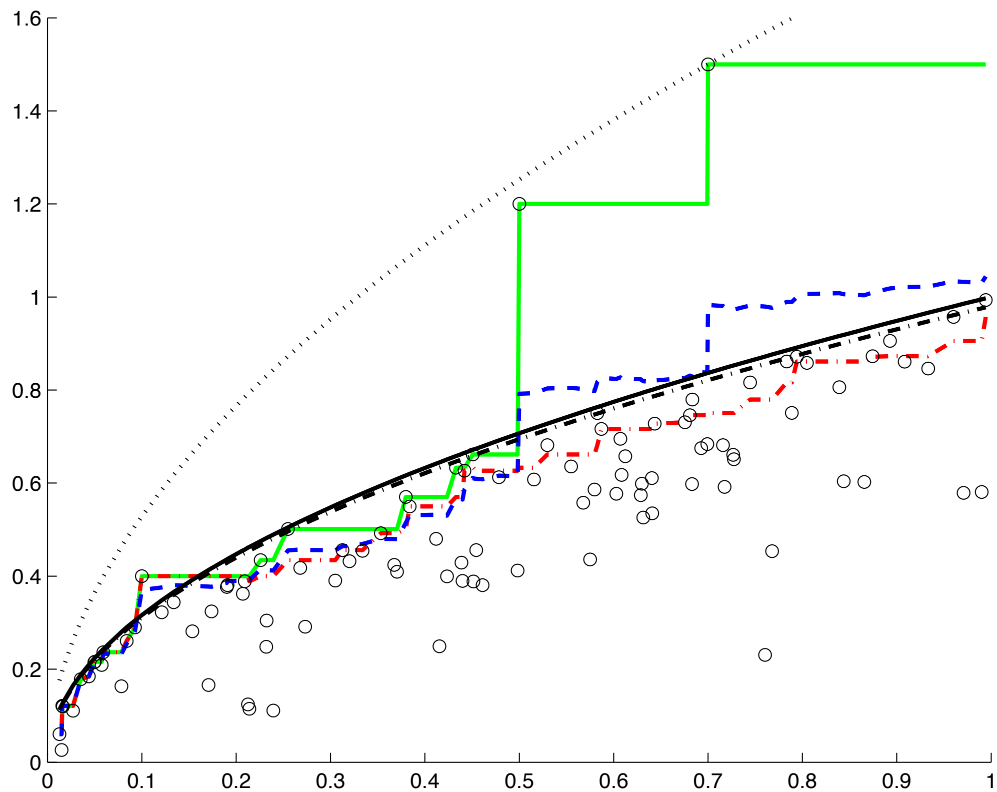
- **Properties**
  - **$\sqrt{n}$-consistency and asymptotic normality:**

$$\sqrt{n}(\hat{\varphi}_{m,n}(x) - \varphi_m(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_m^2(x)) \text{ and } \sqrt{n}(\hat{\varphi}_{\alpha,n}(x) - \varphi_\alpha(x)) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma_\alpha^2(x))$$

  - **Convergence to FDH estimator:**

$$\text{as } m \to \infty,\ \hat{\varphi}_{m,n}(x) \to \hat{\varphi}_{FDH,n}(x) \quad \text{and as} \quad \alpha \to 1,\ \hat{\varphi}_{\alpha,n}(x) \to \hat{\varphi}_{FDH,n}(x)$$

- **Choice of m and α:** tune the **percentage of points left out** estimated partial frontier, see Simar (2003), Daraio, Simar (2005, 2007a).



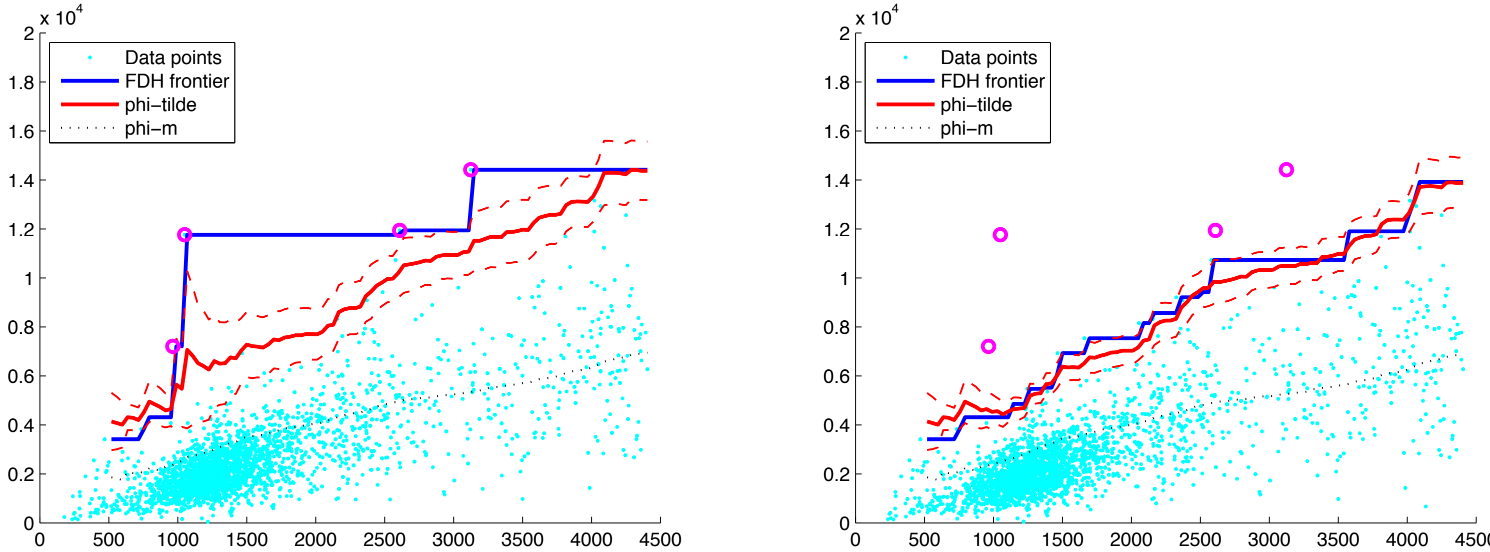
In solid black line, the **true** frontier $y = x^{0.5}$. In green solid, the **FDH** frontier estimate, in blue dashed the estimated **order-m** frontier and in dash-dot red the estimate of the **order-α** frontier. In black dotted, the shifted OLS estimate and in dash-dot black, the parametric stochastic fit, $m = 20$ and $\alpha = 0.95$.

Robust Frontier -5-

Robust Nonparametric Estimator of Full-Frontier $\varphi(x)$, Daouia, Florens, Simar (2010, 2012)

- If $m = m(n)$ (and $\alpha = \alpha(n)$) converges to ∞ (to 1) when $n \to \infty$, but at a slow rate, we obtain an estimator (after bias correction) that converges to the full frontier with a Normal limiting distribution
  - Easy to build confidence intervals for $\varphi(x)$ using Normal Tables.
- **For finite n**, $\hat{\varphi}_{m(n),n}(x)$ and $\hat{\varphi}_{\alpha(n),n}(x)$ provide estimators of $\varphi(x)$ that will not envelop all the data points and so, are **more robust to extreme and outliers**.

Robust Frontier -5-



Post Offices in France (from Daouia, Florens, Simar, 2012).

Left panel: estimation with the 4 extreme points.

Right panel: estimation without these 4 points

IV.2 Lack of Economic Interpretation

Parametric Approximation of Deterministic Frontiers -1-

- **Parametric models:** easy economic interpretation of the model (returns to scale, elasticities, elasticities of substitution, ...)
- **Standard parametric approches:** **some drawbacks**
  - strong restrictive assumptions on the stochastic part of the models
  - sensitive to extreme/outliers
  - most are "regression-based" models and capture the shape of the cloud of points near its center (not at the efficient boundary)
- **Two stage semiparametric approaches:** Simar (1992), Florens, Simar (2005), Daouia, Florens, Simar (2008)
  - First estimate the efficient frontier using **nonparametric** or **robust nonparametric methods**;
  - Then fit, by standard OLS, the approriate **parametric model** on the obtained nonparametric frontier

Parametric Approximation of Deterministic Frontiers -2-

- **More sensible estimator** of the parametric frontier model and **allows for some noise** by tunning the robustness parameter.

- **Asymptotic theory** of the resulting estimators (for fix m and fix α):

If FDH is used as 1st step: $\hat{\theta}_n \xrightarrow{p} \theta_0$

If order-m is used is used as 1st step: $\sqrt{n}(\hat{\theta}_n^m - \theta_0^m) \xrightarrow{\mathcal{L}} \mathcal{N}_k(0, V_m)$

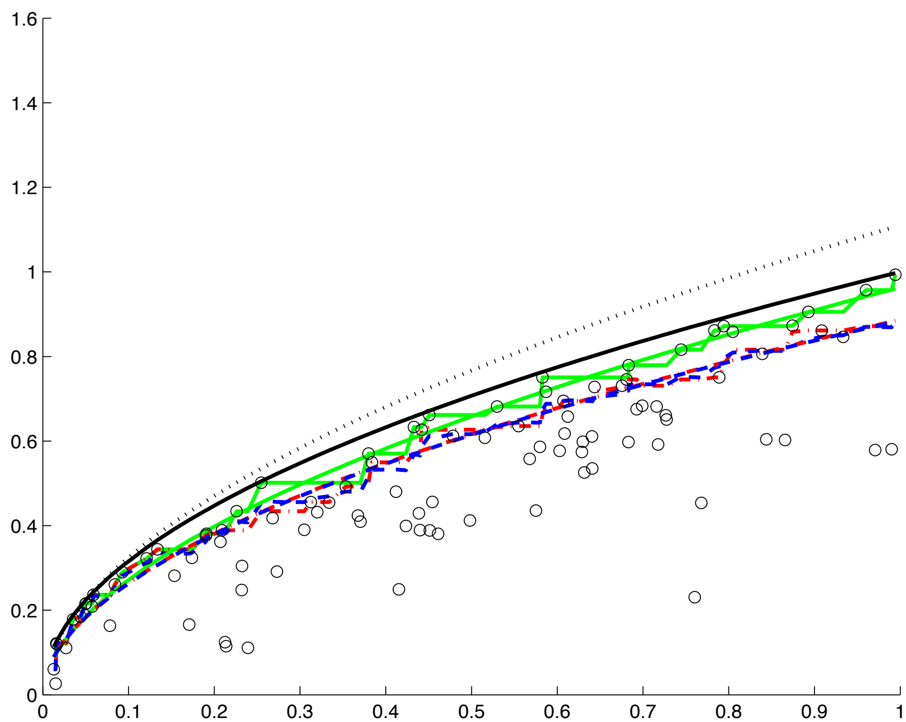
If order-α is used is used as 1st step: $\sqrt{n}(\hat{\theta}_n^\alpha - \theta_0^\alpha) \xrightarrow{\mathcal{L}} \mathcal{N}_k(0, V_\alpha)$

where θ_0, $(\theta_0^m,\, \theta_0^\alpha)$, are the **pseudo-true values** of the parameters of the best approximation of the corresponding frontier $\varphi(x)$, $(\varphi_m(x),\, \varphi_\alpha(x))$.

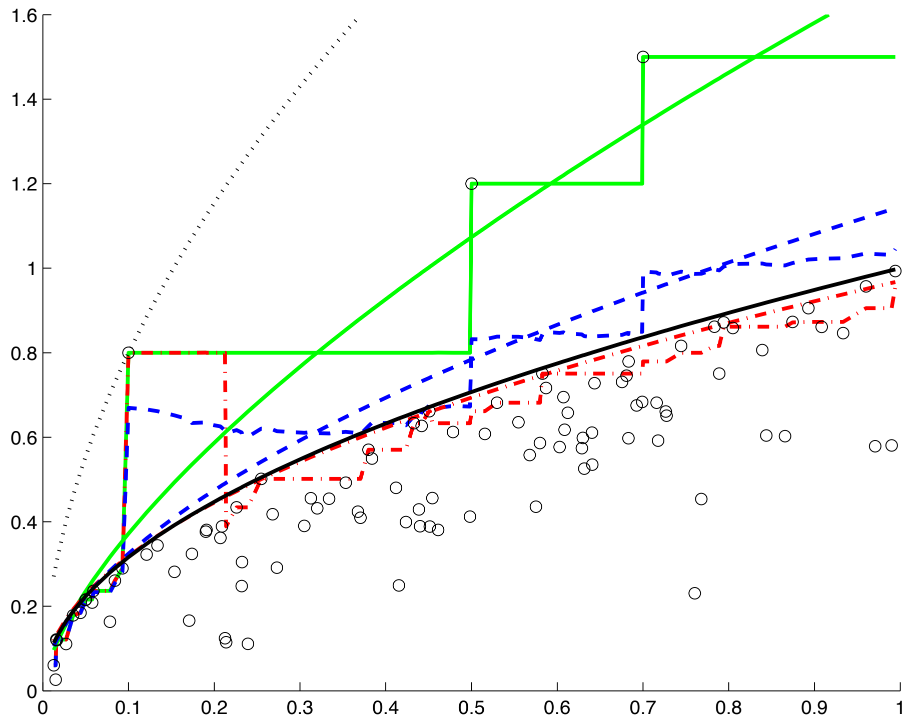
- If $m(n) \to \infty$ and $\alpha(n) \to 1$ as $n \to \infty$ at appropriate rates:

$$\hat{\theta}_n \xrightarrow{a.s.} \theta_0\,; \qquad \hat{\theta}_n^{m(n)} \xrightarrow{a.s.} \theta_0\,; \qquad \hat{\theta}_n^{\alpha(n)} \xrightarrow{a.s.} \theta_0$$

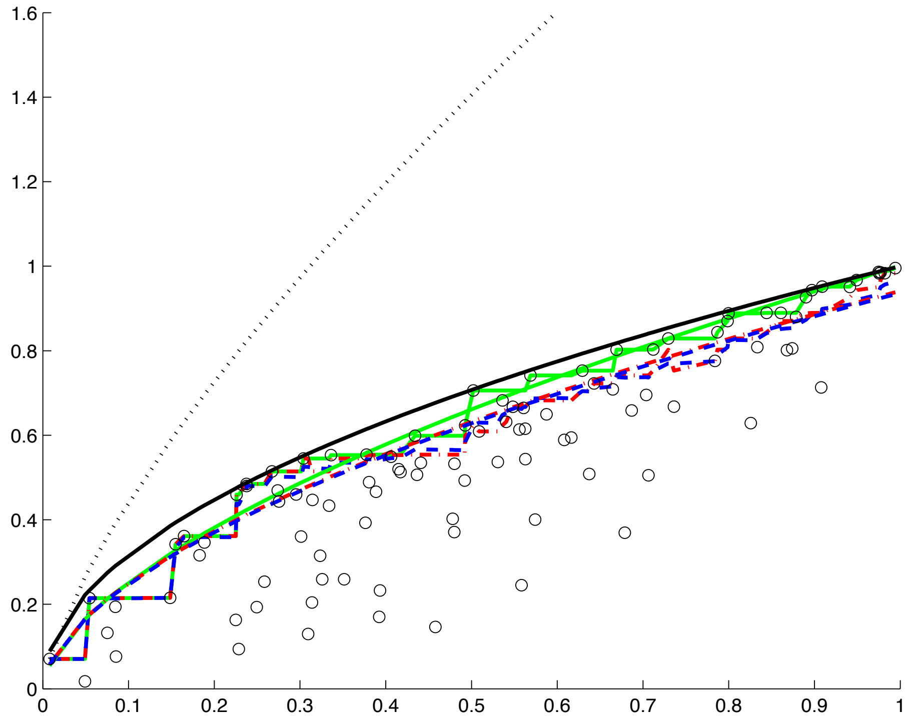
- **Multivariate case:** multi-input/muti-output, see Daraio and Simar (2007a)



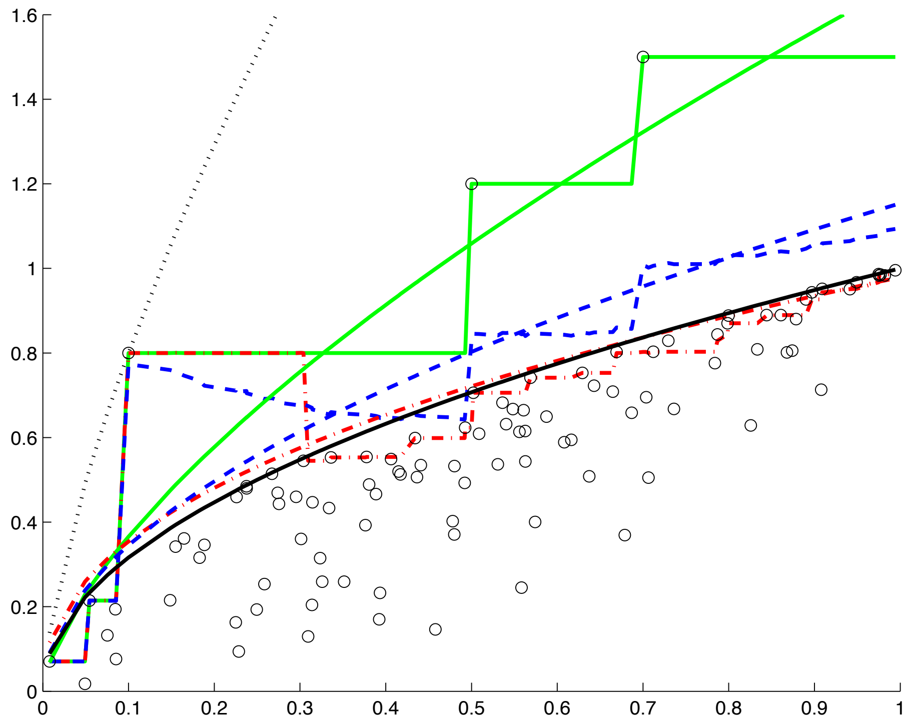
In solid black line, the true frontier $y = x^{0.5}$ **homoscedastic inefficiency**. In cyan solid, the FDH frontier, in blue dashed the order-m frontier and in dash-dot red the order-α frontier. Here, $m = 20$ and $\alpha = .9622$. In black dotted, the shifted OLS estimate.



Same with 3 outliers included.



Same with **heteroscedastic inefficiency**. In cyan solid, the FDH frontier estimate, in blue dashed the order-m frontier and in dash-dot red the order-α frontier. Here, $m = 20$ and $\alpha = .9622$. In black dotted, the shifted OLS estimate.



Same with 3 outliers included.

IV.3 Heterogeneity

Introducing Environmental Factors -1-

- **Motivation**
  - The analysis of productive efficiency should have two components:
    1. Estimation of a production frontier (best-practice) which serve as a benchmark against which **efficiency** of a producer can be measured;
    2. Incorporation into the analysis of **exogenous variables** (Z) which are neither inputs, nor outputs, and so are **not under the control of the producer**, but which may influence the process.
  - How to explain inefficiencies of firms by these factors?
  - How to introduce heterogeneity in the production process?

Introducing Environmental Factors -2-

- **One-stage approaches** Banker and Morey (1986)
  - Z is like an input(favorable) or like an output (defavorable) $\Rightarrow$ Adapt FDH/DEA
  - Free disposability ? Convexity ? RTS assumption ?
  - Which direction for Z?
  - What if the effect of Z changes?
    (say, favorable if $Z \leq z_0$ and then defavorable or neutral for $Z > z_0$)
- **Two-stage approaches** Simar and Wilson (2007, 2011b)
  - DEA efficiency scores are regressed on Z (in an **appropriate** way)
  - **Implicit Separability Condition**:
    - Z does not influence Ψ
    - Z only affects the probability of being more or less efficient
    - The second stage regression is nonstandard (correlation among efficiency scores, bias,...): **inference by bootstrap.**

Traditional 2-stage approaches

- **First stage** get efficiency estimates $\widehat{\lambda}(X_i, Y_i)$ (or $\widehat{\theta}(X_i, Y_i)$, $\widehat{\gamma}(X_i, Y_i)$,...) with respect to $\widehat{\Psi}$ (by DEA or FDH, ...)
- **Second stage** regression of $\widehat{\lambda}(X_i, Y_i)$ on Z.
  - Parametric models (truncated regression, logistic, etc,...)
  - Nonparametric models (truncated, etc,...)
- **Problems:** $\boldsymbol{\Psi^z = \{(x, y) | Z = z,}$ **x can produce y}** $\boldsymbol{\}}$ Simar and Wilson (2007, 2011b):
  - If $\Psi^z \neq \Psi$, what is the **Economic meaning** of $\lambda(x, y)$ (and so, of $\widehat{\lambda}(X_i, Y_i)$), for a unit facing environmental conditions z?
  - Separability issue: condition for giving economic meaning to $\widehat{\Psi}$ and $\widehat{\lambda}(x, y)$.

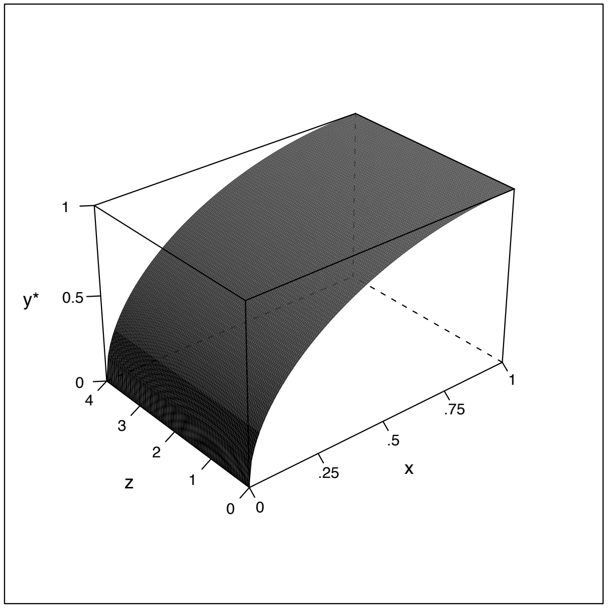
    "Separability" condition: $\boldsymbol{\Psi^z = \Psi,}$ for all $\boldsymbol{z \in \mathcal{Z}}$**.**

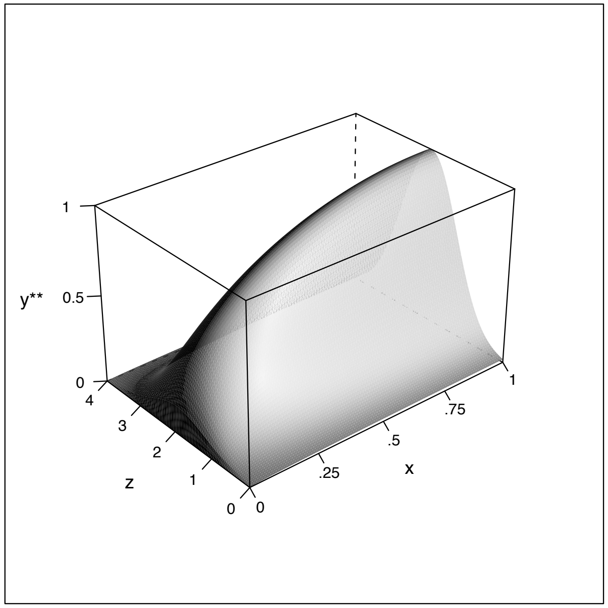
  - Even if separability holds, **Inference** in second stage is nonstandard (bootstrap).

"Separability" Condition

$$g(X) = [1 - (X-1)^2]^{1/2}$$

$$Y^* = g(X)e^{-(Z-2)^2 U} \qquad Y^{**} = g(X)e^{-(Z-2)^2}e^{-U}$$





Left Panel: **Separable**, Right Panel: **Not Separable**

Conditional Efficiency -1-

- **Conditional Measures** Cazals, Florens, Simar (2002), Daraio Simar (2005, 2007a, 2007b), Jeong, Park, Simar (2010)
  - **The DGP** (A Model for the Production process) is now characterized by
    - $F(x, y|z) = \text{Prob}(X \le x, Y \le y|Z = z)$ or
    - $H(x, y|z) = \text{Prob}(X \le x, Y \ge y|Z = z)$
    - The attainable set is Ψ^z: the support of $F(x, y|z)$
  - **Natural and very easy:** A firm combines inputs $X \in \mathbb{R}^p_+$ and outputs $Y \in \mathbb{R}^q_+$ facing the environmental conditions $Z \in \mathbb{R}^r$
    - No **separability** conditions
    - No **prior** information of the role of Z (favorable or not to the process)
  - Note that the **separability condition** of 2-stages methods relies on:

$$\Psi \equiv \Psi^z \quad \text{for all} \quad z.$$

Conditional Efficiency -2-

- **Conditional efficiency score**
  - Same idea as the unconditional measure:

$$\lambda(x, y|z) = \sup\{\lambda \mid H_{XY|Z}(x, \lambda y|z) > 0\} = \sup\{\lambda \mid S_{Y|X,Z}(\lambda y|x, z) > 0\},$$

  where
  $S_{Y|X,Z}(y|x, z) = H_{XY|Z}(x, y|z)/H_{XY|Z}(x, 0|z) = \text{Prob}(Y \geq y \mid X \leq x, Z = z)$.

- **Nonparametric estimator**: kernel smoothing on Z (here continuous)

$$\widehat{H}_{XY,n|Z}(x, y|Z = z) = \frac{\sum_{i=1}^{n} \mathbb{I}(X_i \leq x, Y_i \geq y) K((Z_i - z)/h)}{\sum_{i=1}^{n} K((Z_i - z)/h)}$$

$$\widehat{S}_{Y|X,Z}(y|x, z) = \frac{\sum_{i=1}^{n} \mathbb{I}(Y_i \geq y, X_i \leq x) K_h(Z_i, z)}{\sum_{i=1}^{n} \mathbb{I}(X_i \leq x) K_h(Z_i, z)}$$

Conditional Efficiency -3-

- **Conditional FDH efficiency estimator**: Kernels with compact support,

$$\widehat{\lambda}_{FDH}(x,y|z) = \sup\{\lambda|\widehat{S}_{Y|X,Z}(\lambda y|x,z) > 0\} = \max_{\{i|X_i \le x, ||Z_i - z|| \le h\}} \left\{ \min_{j=1,\ldots,q} \frac{Y_i^j}{y^j} \right\}.$$

- **Conditional FDH attainable set**:

$$\widehat{\Psi}^Z_{FDH} = \{(x,y) \in \mathbb{R}^{p+q}_+ \mid x \ge x_i, y \le y_i \text{ for } i \text{ s.t. } ||Z_i - z|| \le h\}$$

- **DEA versions**: Convexify the FDH attainable set, see Daraio, Simar (2007b)

$$\widehat{\Psi}^Z_{DEA} = \{(x,y) \in \mathbb{R}^{p+q}_+ \mid x \ge \sum_{\{i|||Z_i - z|| \le h\}} \gamma_i x_i, \quad y \le \sum_{\{i|||Z_i - z|| \le h\}} \gamma_i y_i$$

$$\text{for } \gamma_i \text{ s.t. } \sum_{\{i|||Z_i - z|| \le h\}} \gamma_i = 1\},$$

$$\widehat{\lambda}_{DEA}(x,y|z) = \sup\{\lambda \mid (x,\lambda y) \in \widehat{\Psi}^Z_{DEA}\}.$$

Conditional Efficiency -4-

- **Properties**
  - Optimal bandwidth selection by data-driven methods, Badin, Daraio, Simar (2010)
  - Asymptotic properties: similar to FDH/DEA with n replaced by nh^r, Jeong, Park, Simar (2010)
  - Allow to detect the direction of the “influence” of Z on efficiency, see Daraio, Simar (2005, 2007a), Badin, Daraio, Simar (2012a, 2012b)
  - Inference (confidence intervals) by bootstrap
  - Robust versions (using order-m and order-α) are also available
  - Z can be continuous, categorical or discrete

Conditional Efficiency -5-

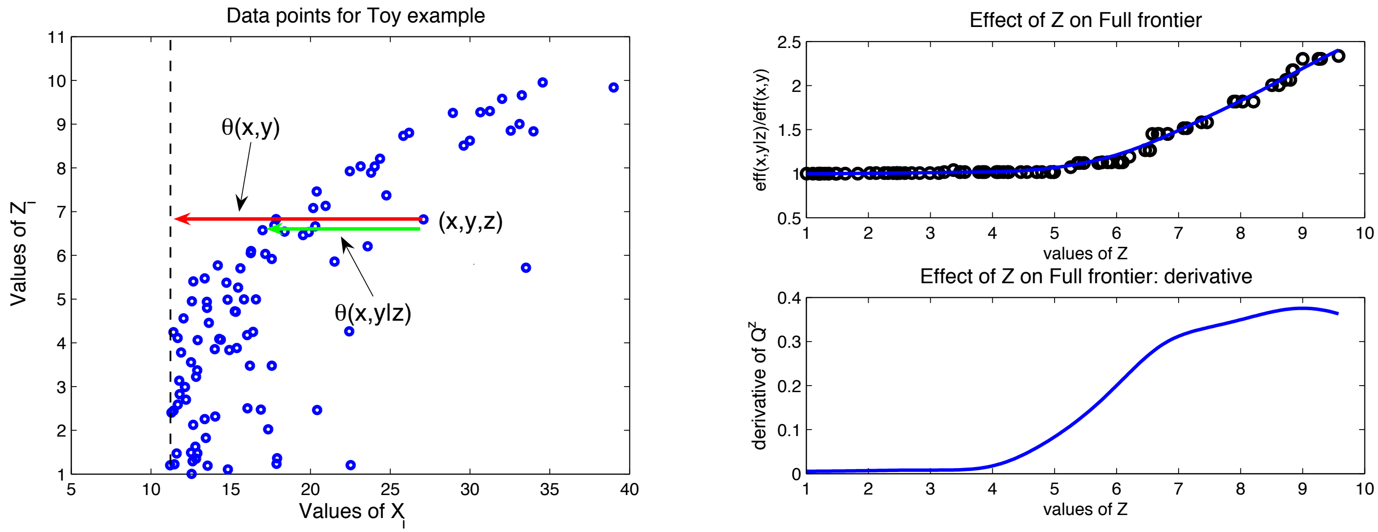
- **Usefulness**
  - Define a **"pure measure of technical efficiency"**, Badin, Daraio, Simar (2012a, 2012b)
    - Eliminate most of the influence of Z on $\widehat{\lambda}(x, y|z)$ by using a flexible location-scale nonparametric model: $\widehat{\lambda}(x, y|z) = \mu(z) + \sigma(z)\varepsilon$, where $\mu(z)$ and $\sigma(z)$ are unspecified functions
    - $\hat{\epsilon}_i$ allows to rank firms facing different operating conditions.
- **N.B.: An other approach**: Florens, Simar, Van Keilegom (2011).
  - First eliminate influence of Z on inputs X and outputs Y by using two flexible location-scale nonparametric models
  - The residuals are **"pure inputs and outputs"** $\tilde{X}_i$ and $\tilde{Y}_i$
  - Search for the frontier in these new units, to define "pure measure of technical efficiency"
  - Full frontier and order-m frontiers

Conditional Efficiency, Example -1-

- **A Toy example**:
  - No output ($Y_i \equiv 1$) and one input (input orientation)
  - Z has no effect on X when $Z \leq 5$ and then a defavorable effect on X when $Z > 5$.
  - The input are generated according

$$X_i = 5^{1.5}\, \mathbb{1}(Z_i <= 5) + Z_i^{1.5}\, \mathbb{1}(Z_i > 5) + U_i,$$

where $Z_i \sim \mathrm{U}(1, 10)$, $U_i \sim \mathrm{Expo}(\mu = 3)$ and $n = 100$.



Effect of Z on the ratios $\hat{\theta}_n(x, y \mid z)/\hat{\theta}_n(x, y)$.

Conditional Efficiency, Examples -2a-

- **2 inputs/ 2 outputs : output orientation**
  - The efficient frontier is described by: $y^{(2)} = 1.0845(x^{(1)})^{0.3}(x^{(2)})^{0.4} - y^{(1)}$.
  - $X_i^{(j)} \sim U(1,2)$ and $\tilde{Y}_i^{(j)} \sim U(0.2,5)$ for $j = 1,2$.
  - The output efficient random points on the frontier are

$$Y_{i,eff}^{(1)} = \frac{1.0845(X_i^{(1)})^{0.3}(X_i^{(2)})^{0.4}}{S_i + 1}$$
$$Y_{i,eff}^{(2)} = 1.0845(X_i^{(1)})^{0.3}(X_i^{(2)})^{0.4} - Y_{i,eff}^{(1)}.$$

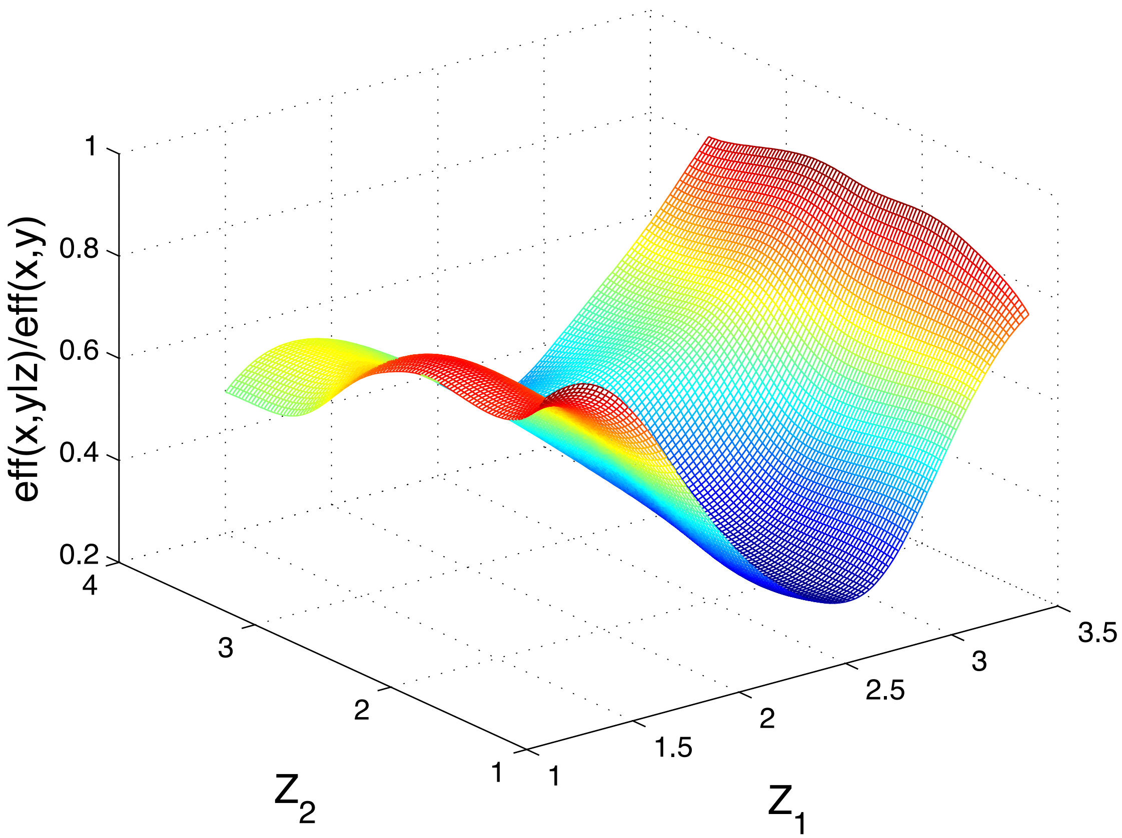
    where $S_i = \tilde{Y}_i^{(2)}/\tilde{Y}_i^{(1)}$ represent the generated random rays in the output space.
  - The efficiencies are simulated according to $\exp(-U_i)$
  - The observed output are defined by $Y_i = Y_{i,eff} * \exp(-U_i)$ where $U_i \sim Exp(\mu_U = 1/2)$.
  - $n = 100$.

Conditional Efficiency, Examples -2b-

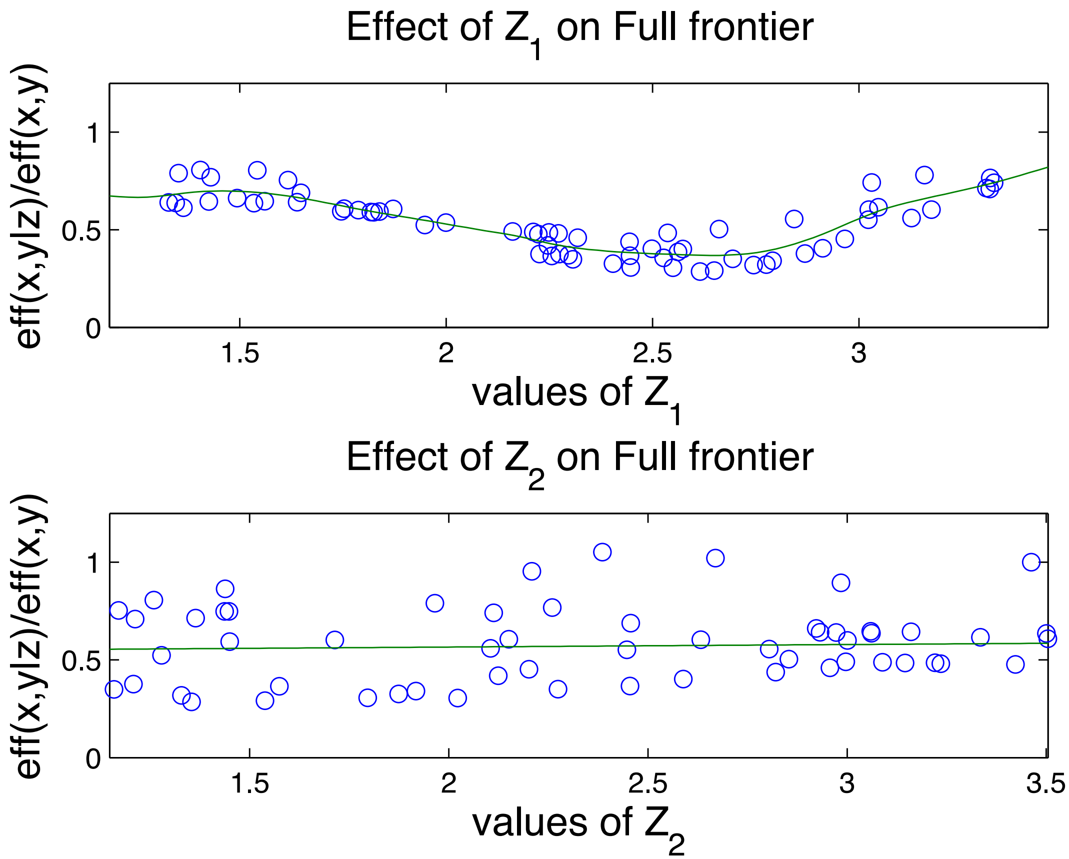
- **Environmental factors Z bivariate**
  - We generate two independent uniform variables $Z_j \sim U(1,4)$ to build the bivariate variable $Z = (Z_1, Z_2)$.
  - The influence of Z on the production process is described by:

$$Y_i^{(1)} = (1 + 2 * |Z_1 - 2.5|^3) * Y_{i,eff}^{(1)} * \exp(-U_i)$$
$$Y_i^{(2)} = (1 + 2 * |Z_1 - 2.5|^3) * Y_{i,eff}^{(2)} * \exp(-U_i).$$

    - Z_1 pushes the efficient frontier above when far from 2.5, in both directions, with a **cubic effect**,
    - Z_2 has **no effect** on the frontier or on the distribution of inefficiencies: Z_2 is irrelevant.
    - Note that there is no interaction between Z_1 and Z_2 (independent) and no interaction between X and Z.
  - Remember: only $n = 100$ observations, with $p = q = r = 2$!



Smoothed nonparametric surface regression of $\widehat{\lambda}_n(x, y|z)/\widehat{\lambda}_n(x, y)$ on Z_1 and Z_2.



Simulated example with multivariate Z. Marginal views of the surface regression of $\widehat{\lambda}_n(x,y|z)/\widehat{\lambda}_n(x,y)$ on z at the observed points (X_i, Y_i, Z_i), viewed as a function of Z_1 (top panel) and as a function of Z_2 (bottom panel).

IV.4 Introducing Noise

Nonparametric Stochastic Frontiers -1-

• **Basic Idea:** localize (using kernels) an anchorage parametric model, Kumbhakar, Park, Simar, Tsionas (2007)

$$Y_i = r(X_i) + v_i - u_i$$

– $u|X = x \sim |\mathcal{N}(0, \sigma_u^2(x))|$ and $v|X = x \sim \mathcal{N}(0, \sigma_v^2(x))$ and u and v being independent conditionally on X.

– $r(x), \sigma_u^2(x)$ and $\sigma_v^2(x)$ are **unknown functional parameters**

– Estimation by **Local Maximum Likelihhood** method: $r(x), \sigma_u^2(x)$ and $\sigma_v^2(x)$ are approximated by local polynomials (linear or quadratic).

– Asymptotic properties are available

– Bandwidths selection by **LOO-LS cross-validation**: **numerical burden!**

Nonparametric Stochastic Frontiers -2-

- **Multivariate extension**: Simar (2007), Simar, Zelenyuk (2011)
  - Use (partial-)polar coordinates: $(x, y) \Leftrightarrow (\omega, \eta, x)$, where $\omega \in \mathbb{R}_+$ is the modulus and $\eta \in [0, \pi/2]^{q-1}$ is the amplitude (angle) of the vector y.
  - The joint density $f_{X,Y}(x, y)$ induces a density on (ω, η, x):

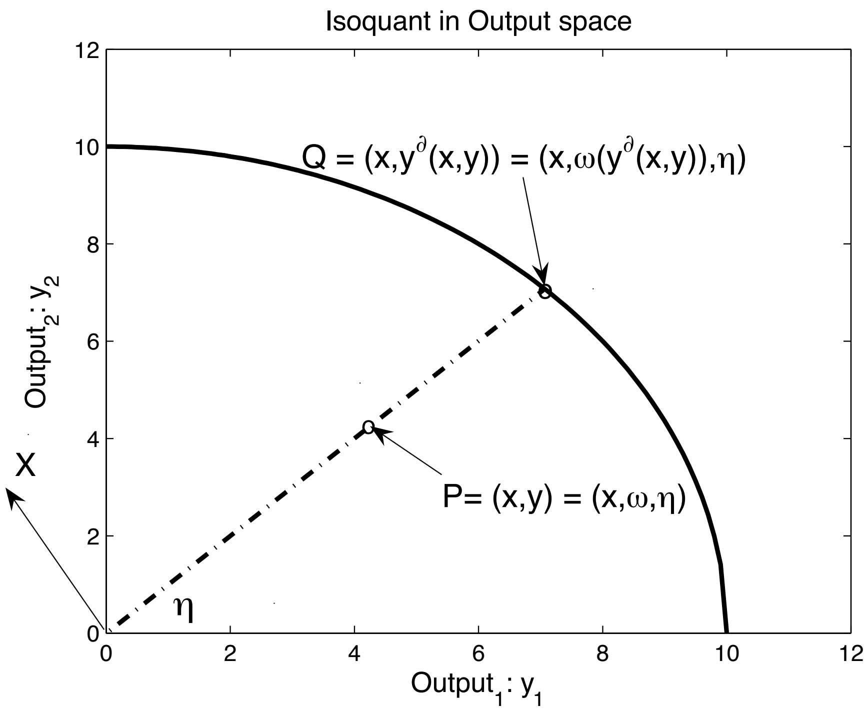
$$f_{\omega,\eta,X}(\omega, \eta, x) = f_\omega(\omega \mid \eta, x)\, f_{\eta,X}(\eta, x)$$

  - For a given (x, y) the frontier point $y^\partial(x, y) = \lambda(x, y)\, y$ has a modulus:

$$\omega(y^\partial(x, y)) = \sup\{\omega \in \mathbb{R}^+ \mid f_\omega(\omega \mid \eta, x) > 0\}$$

- **Back to a univariate frontier problem!**
  - Given (η, x) find $\omega(y^\partial(x, y))$.

Nonparametric Stochastic Frontiers -3-



Polar coordinates in the output space for a particular section $Y(x)$. Output efficiency of $P = (x, y)$ is $\lambda(x, y) = |OQ|/|OP| = \omega(y^{\partial}(x, y))/\omega(x, y) \geq 1$.

Nonparametric Stochastic Frontiers -4-

- **The Model**:
  - The observations are made on noisy data in the output radial-direction
  - The data $\{(X_i, Y_i),\ i = 1, \ldots, n\}$ have polar coordinates (ω_i, η_i, X_i)

$$\omega_i = \omega(y^\partial(X_i, Y_i))\, e^{-u_i}\, e^{v_i},$$

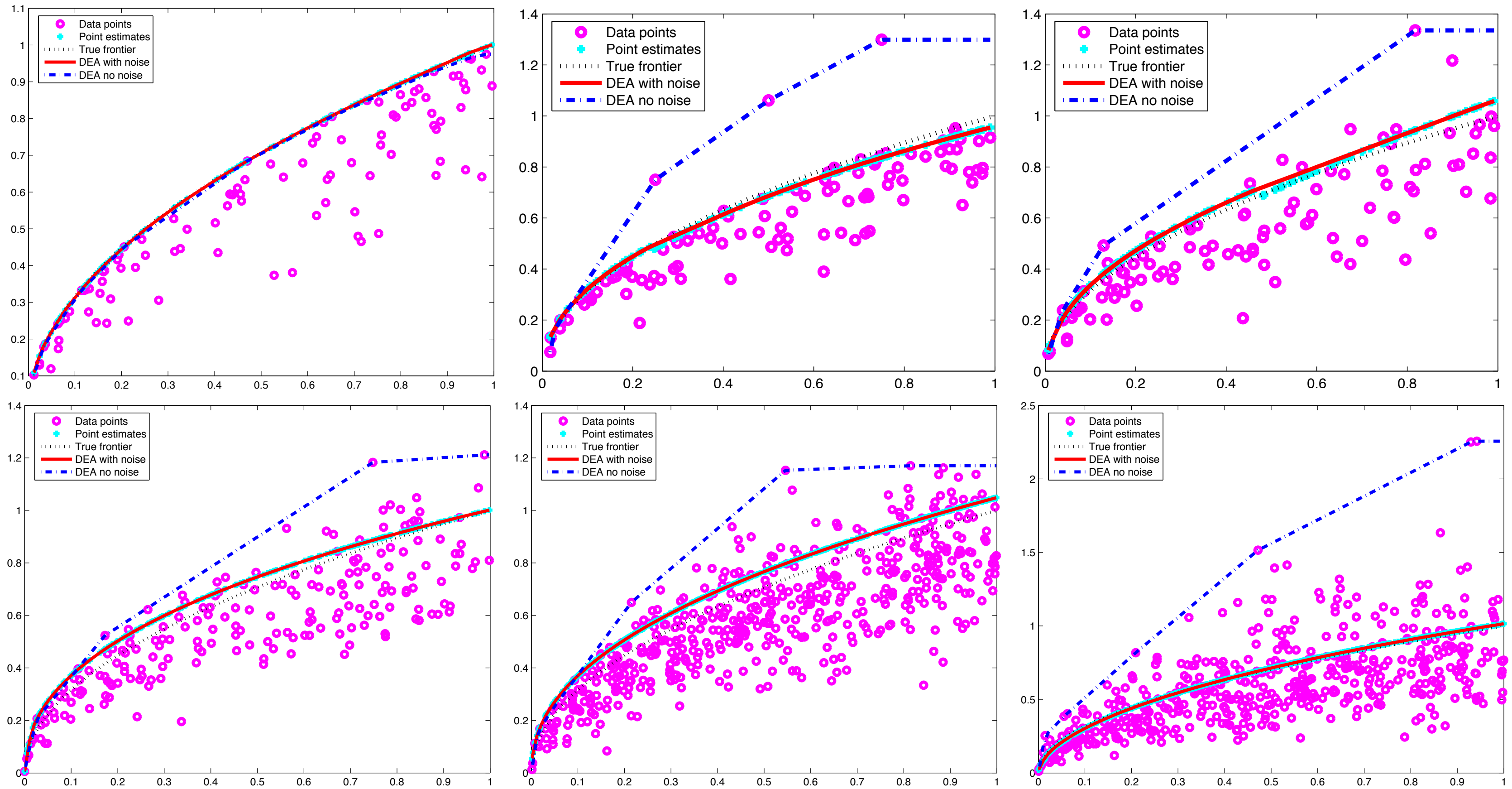
    where $u_i > 0$ is inefficiency and v_i is noise $(E(v_i|X_i, Y_i) = 0)$.
  - $\omega(y^\partial(X_i, Y_i))$ is only a function of (η_i, X_i).
- **In the log-scale**, the model could be written as

$$\log \omega_i = r(\eta_i, X_i) - u_i + v_i,$$

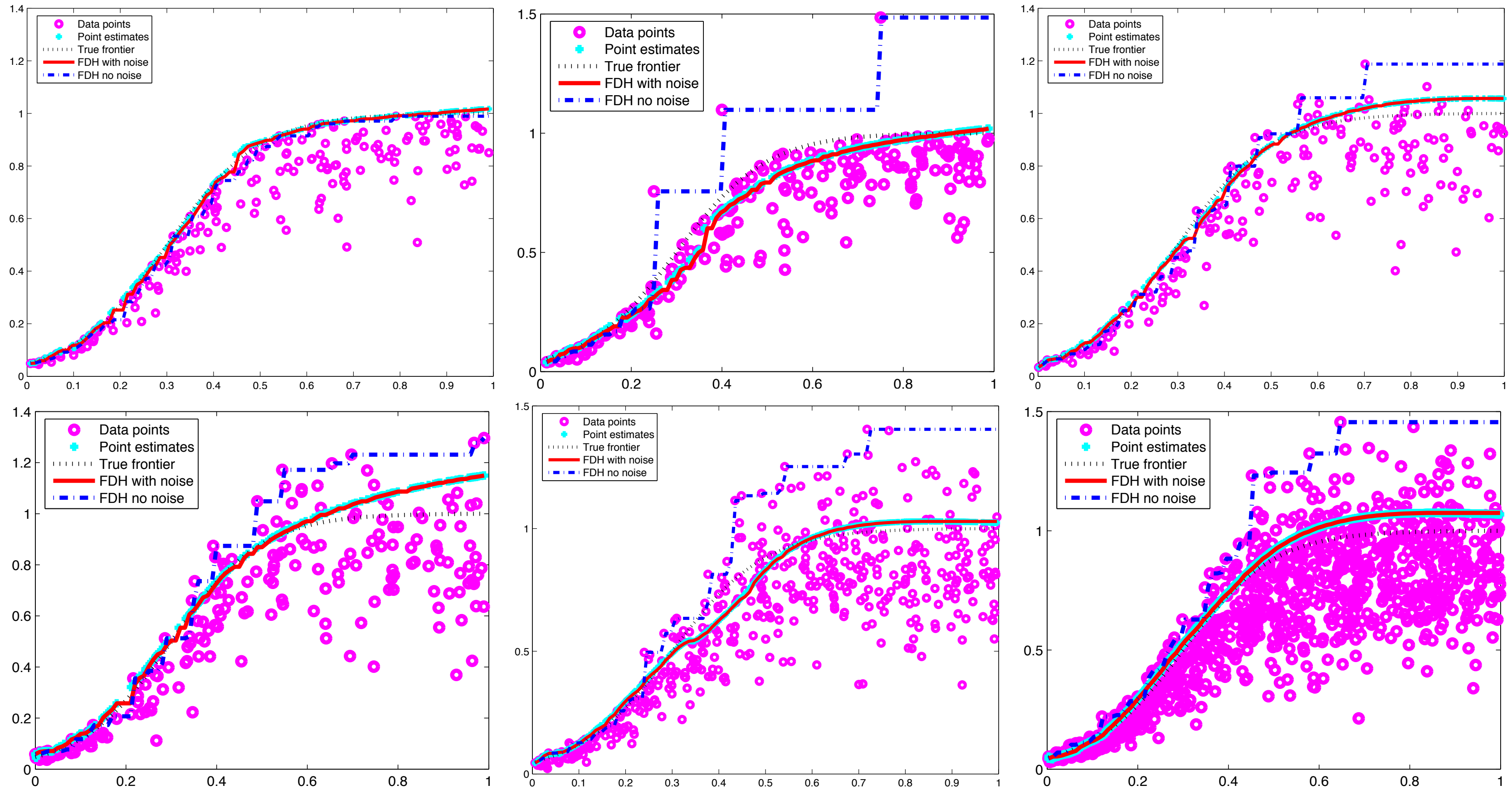
with $u_i > 0$ and $E(v_i|\eta_i, X_i) = 0$.

Nonparametric Stochastic Frontiers -5-

- **Stochastic Versions of DEA/FDH :** Two-stage procedure
  - [1] "Whitening the noise": Compute the consistent estimator of the frontier levels $\hat{r}(\eta_i, X_i)$ for each data points
    - This gives points (X_i, Y_i^*) where $Y_i^* = \exp(\hat{r}(\eta_i, X_i))Y_i/\omega_i$
  - [2] Run a DEA (or FDH) program with reference set (X_i, Y_i^*).
- **Summary:**
  - Very encouraging results
  - Computationally demanding (cross-validation for bandwidth selection)
  - Below, some bivariate examples (see multivariate examples in Simar and Zelenyuk, 2011)



(a) $n = 100, \rho_{\text{nts}} = 0$, (b) $n = 103, \rho_{\text{nts}} = 0 + 3$ outliers, (c) $n = 100, \rho_{\text{nts}} = 1$, (d) $n = 200, \rho_{\text{nts}} = 1$, (e) $n = 500, \rho_{\text{nts}} = 1$, (f) $n = 500, \rho_{\text{nts}} = 2$.



(a) $n = 200, \rho_{\text{nts}} = 0$, (b) $n = 203, \rho_{\text{nts}} = 0 + 3$ outliers, (c) $n = 200, \rho_{\text{nts}} = 0.5$, (d) $n = 200, \rho_{\text{nts}} = 1$, (e) $n = 500, \rho_{\text{nts}} = 1$, (f) $n = 1000, \rho_{\text{nts}} = 1$.

Conclusions -1-

- **Nonparametric models $\mathcal{NP}$ are Econometric/Statistical Models**
  - Flexible and can be "robustified",
  - Inference is available (bootstrap)
  - Noise can be introduced, but not easy.
  - Environmental factors (heterogeneity) can be introduced
  - Any directional distance can be used
- **$\mathcal{P}$ and $\mathcal{NP}$ are complimentary models**
  - $\mathcal{NP}$ models can be used to check (test) $\mathcal{P}$ models (not the contrary).
  - Parametric approximations of $\mathcal{NP}$ models can be useful for economic analysis.
  - Semiparametric models should be developed.

Conclusions -2-

- **Other challenges**
  - Panel Data: introduce dynamic behavior of units
  - Theory for functions of DEA/FDH scores: Kneip, Simar and Wilson (2012)
    - Useful for justifying and deriving testing procedures: **Work in progress!!**
    - RTS, Convexity, using subsampling, Simar and Wilson (2011a),
    - Testing Separabilty, Daraio, Simar and Wilson (2010), still problems...
    - Testing by avoiding bootstrap? Kneip, Simar, Wilson (?)
  - Nonparametric Stochastic Frontiers
    - Kneip, Simar, Van Keilegom (2012): Gaussian noise and using penalized nonparametric techniques (sieve estimation)
    - Florens, Simar (?): Gaussian noise and deconvolution with Tikhonov regularization.
  - ...

References

- **Basic References**

  - Fried, H., Lovell, K. and S. Schmidt (eds) (2008),*The Measurement of Productive Efficiency*, 2nd Edition, Oxford University Press.
  - Kumbhakar, S.C. and C.A.K. Lovell (2000), *Stochastic Frontier Analysis*, Cambridge University Press.
  - Daraio, C. and L. Simar (2007a), *Advanced Robust and Nonparametric Methods in Efficiency Analysis. Methodology and Applications*, Springer, New-York.
  - Simar, L. and P.W. Wilson (2008), Statistical Inference in Nonparametric Frontier Models: recent Developments and Perspectives, in *The Measurement of Productive Efficiency*, 2nd Edition, Harold Fried, C.A.Knox Lovell and Shelton Schmidt, editors, Oxford University Press, 2008.

- **References list**

- Aigner, D.J., Lovell, C.A.K. and P. Schmidt (1977), Formulation and estimation of stochastic frontier models, *Journal of Econometrics*, 6, 21-37.
- Aragon, Y. and Daouia, A. and Thomas-Agnan, C. (2005), Nonparametric Frontier Estimation: A Conditional Quantile-based Approach, *Econometric Theory*, 21, 358–389.
- Badin, M., Daraio, C. and L. Simar (2010), Optimal Bandwidth Selection for Conditional Efficiency Measures: a Data-driven Approach, *European Journal of Operational Research*, 201, 633–640
- Badin, L., Daraio, C. and L. Simar (2012a), How to measure the impact of environmental factors in a nonparametric production model? Discussion paper 2011/19, Institut de Statistique, UCL, in press *European Journal of Operational Research*.
- Badin, L., Daraio, C. and L. Simar (2012b), Explaining Inefficiency in Nonparametric Production Models: the State of the Art. Discussion paper 2011/33, Institut de Statistique, UCL, in press *Annals of Operations Research*.
- Badin, L. and L. Simar (2009), A bias corrected nonparametric envelopment estimator of frontiers, *Econometric Theory*, 25, 5, 1289–1318.
- Banker, R.D. and R.C. Morey (1986), Efficiency analysis for exogenously fixed inputs and outputs, *Operations Research*, 34(4), 513–521.

- Cazals, C. Florens, J.P. and L. Simar (2002), Nonparametric Frontier Estimation: a Robust Approach , in *Journal of Econometrics*, 106, 1–25.
- Chambers, R. G., Y. Chung, and R. Färe (1998), Profit, directional distance functions, and nerlovian efficiency, *Journal of Optimization Theory and Applications*, 98, 351–364.
- Charnes, A., Cooper W.W. and E. Rhodes (1978), Measuring the inefficiency of decision making units, *European Journal of Operational Research* 2 (6), 429-444.
- Daouia, A., J.P. Florens and L. Simar (2008), Functional Convergence of Quantile-type Frontiers with Application to Parametric Approximations, *Journal of Statistical Planning and Inference*, 138, 708–725.
- Daouia, A., Florens, J.P. and L. Simar (2010), Frontier estimation and extreme values theory. *Bernoulli*, 16(4), 1039–1063.
- Daouia, A., Florens, J.P. and L. Simar (2012), Regularization of Non-parametric Frontier Estimators, *Journal of Econometrics*, 168, 285–299.
- Daouia, A. and L. Simar (2005), Robust Nonparametric Estimators of Monotone Boundaries, *Journal of Multivariate Analysis*, 96, 311–331.
- Daouia, A. and L. Simar (2007), Nonparametric efficiency analysis: a multivariate conditional quantile approach, *Journal of Econometrics*, 140, 375–400.

- Daraio, C. and L. Simar (2005), Introducing environmental variables in nonparametric frontier models: a probabilistic approach, *Journal of Productivity Analysis*, vol 24, 1, 93–121.
- Daraio, C. and L. Simar (2006), A Robust Nonparametric Approach to Evaluate and Explain the Performance of Mutual Funds, *European Journal of Operational Research*, vol 175, 1, 516–542.
- Daraio, C. and L. Simar (2007a), *Advanced Robust and Nonparametric Methods in Efficiency Analysis. Methodology and Applications*, Springer, New-York.
- Daraio, C. and L. Simar (2007b), Conditional nonparametric frontier models for convex and non convex technologies: a unifying approach, *Journal of Productivity Analysis*, vol 28, 13–32.
- Daraio, C., Simar, L. and P.W. Wilson (2010), Testing whether Two-Stage Estimation is Meaningful in Non-Parametric Models of Production, Discussion paper 1031, Institut de Statistique, UCL.
- Debreu, G. (1951), The coefficient of ressource utilization, *Econometrica*, 19:3, 273-292.
- Deprins, D., Simar, L. and H. Tulkens (1984), Measuring labor inefficiency in post offices. In *The Performance of Public Enterprises: Concepts and measurements*. M. Marchand, P. Pestieau and H. Tulkens (eds.), Amsterdam, North-Holland, 243–267.
- Farrell, M.J. (1957), The measurement of productive efficiency. *Journal of the Royal Statistical Society, Series A*, 120, 253–281.

- Färe, R. and S. Grosskopf (2000), Theory and application of directional distance functions, *Journal of Productivity Analysis* 13, 93–103.
- Färe, R., and S. Grosskopf (2004), *New Directions: Efficiency and Productivity*, Boston: Kluwer Academic Publishers.
- Färe, R., S. Grosskopf, and C. A. K. Lovell (1985), *The Measurement of Efficiency of Production*, Boston: Kluwer-Nijhoff Publishing.
- Florens, J.P. and L. Simar, (2005), Parametric Approximations of Nonparametric Frontier, *Journal of Econometrics*, vol 124, 1, 91–116
- Kneip, A., Simar, L. and I. Van Keilegom (2012), Boundary estimation in the presence of measurement error with unknown variance. Discussion paper 2012/02, Institut de Statistique, UCL.
- Fried, H., Lovell, K. and S. Schmidt (eds) (2008),*The Measurement of Productive Efficiency*, 2nd Edition, Oxford University Press.
- Gijbels, I., E. Mammen, B.U. Park and L. Simar (1999), On Estimation of Monotone and Concave Frontier Functions, *Journal of the American Statistical Association*, vol 94, 445, 220-228.
- Hall, P. and L. Simar (2002), Estimating a Changepoint, Boundary or Frontier in the Presence of Observation Error, *Journal of the American Statistical Association*, 97, 523–534.

- Jeong, S.O. , B. U. Park and L. Simar (2010), Nonparametric conditional efficiency measures: asymptotic properties. *Annals of Operations Research*, 173, 105–122.
- Jeong, S.O. and L. Simar (2006), Linearly interpolated FDH efficiency score for nonconvex frontiers, *Journal of Multivariate Analysis*, 97, 2141–2161.
- Kneip, A., Park, B.U. and Simar, L. (1998). : A note on the convergence of nonparametric DEA estimators for production efficiency scores. *Econometric Theory*, 14, 783–793.
- Kneip, A., Simar, L. and I. Van Keilegom (2010), Boundary Estimation in the Presence of Measurement Errors. Discussion paper 1046, Institut de Statistique, UCL.
- Kneip, A, L. Simar and P.W. Wilson (2008), Asymptotics and consistent bootstraps for DEA estimators in non-parametric frontier models, *Econometric Theory*, 24, 1663–1697.
- Kneip, A., Simar, L. and P.W. Wilson (2011), A Computational Efficient, Consistent Bootstrap for Inference with Non-parametric DEA Estimators. *Computational Economics.* , 38,483–515.
- Kneip, A., Simar, L. and P.W. Wilson (2012), Central Limit Theorems for DEA efficiency scores: when bias can kill the variance. Discussion paper, Institut de Statistique, UCL.
- Koopmans, T.C.(1951), An analysis of production as an efficient combination of activities, in Koopmans, T.C. (ed) *Activity Analysis of Production and Allocation*, Cowles Commision for Research in Economics, Monograph 13, John-Wiley, New-York.

- Korostelev, A., Simar, L. and A. Tsybakov (1995a), Efficient Estimation of Monotone Boundaries, *Annals of Statistics*, 23(2), 476–489.
- Korostelev, A., Simar, L. and A. Tsybakov (1995b), On Estimation of Monotone and Convex Boundaries, *Publications des Instituts de Statistique des Universités de Paris*, 1, 3–18.
- Kumbhakar, S.C. and C.A.K. Lovell (2000), *Stochastic Frontier Analysis*, Cambridge University Press.
- Kumbhakar, S.C. , Park, B.U., Simar, L. and E.G. Tsionas (2007), Nonparametric stochastic frontiers: a local likelihood approach, *Journal of Econometrics*, 137, 1, 1–27.
- Mouchart, M. and L. Simar (2002), Efficiency analysis of Air Controlers: first insights, Consulting report 0202, Institut de Statistique, Université Catholique de Louvain, Belgium.
- Park, B.U., Jeong, S.-O. and L. Simar (2010), Asymptotic Distribution of Conical-Hull Estimators of Directional Edges. *Annals of Statistics*, Vol 38, 6, 1320–1340.
- Park, B. Simar, L. and Ch. Weiner (2000), The FDH Estimator for Productivity Efficiency Scores: Asymptotic Properties, *Econometric Theory*, Vol 16, 855–877.
- Park, B., L. Simar and V. Zelenyuk (2008), Local Likelihood Estimation of Truncated Regression and its Partial Derivatives: Theory and Application, *Journal of Econometrics*, 146, 185–198.
- Ritter, C. and L. Simar (1997), Pitfalls of normal-gamma stochastic frontier models, *Journal of Productivity Analysis*, 8, 167–182.

- Schubert, T. and L. Simar (2011), Innovation and export activities in the German mechanical engineering sector: an application of testing restrictions in production analysis. *Journal of Productivity Analysis*, 36, 55–69.

- Shephard, R.W. (1970). *Theory of Cost and Production Function*. Princeton University Press, Princeton, New-Jersey.

- Simar, L. (1992), Estimating efficiencies from frontier models with panel data: a comparison of parametric, non-parametric and semi-parametric methods with bootstrapping, *Journal of Productivity Analysis*, 3, 167-203.

- Simar, L. (2003), Detecting Outliers in Frontiers Models: a Simple Approach, *Journal of Productivity Analysis*, 20, 391–424.

- Simar, L. (2007), How to Improve the Performances of DEA/FDH Estimators in the Presence of Noise, *Journal of Productivity Analysis*, vol 28, 183–201.

- Simar, L. and A. Vanhems (2012), Probabilist Characterization of Directional Distances and their Robust versions. *Journal of Econometrics*, 166, 342–354.

- Simar, L., Vanhems, A. and P.W. Wilson (2012), Statistical inference with DEA estimators of directional distances, *European Journal of Operational Research*, 220, 853–864.

- Simar, L. and P. Wilson (1998), Sensitivity of efficiency scores : How to bootstrap in Nonparametric frontier models, *Management Sciences*, 44, 1, 49–61.

- Simar, L. and P. Wilson (1999a), Some problems with the Ferrier/Hirschberg Bootstrap Idea, *Journal of Productivity Analysis*, 11, 67–80.
- Simar, L. and P. Wilson (1999b), Of Course we can bootstrap DEA scores ! But does it mean anything ? Logic trumps wishful thinking, *Journal of Productivity Analysis*, 11, 93–97.
- Simar L. and P. Wilson (2000), A General Methodology for Bootstrapping in Nonparametric Frontier Models, *Journal of Applied Statistics*, Vol 27, 6, 779–802.
- Simar L. and P. Wilson (2001), Testing Restrictions in Nonparametric Efficiency Models, *Communications in Statistics, simulation and computation*, 30 (1), 159–184.
- Simar L. and P. Wilson (2002), Nonparametric Test of Return to Scale, *European Journal of Operational Research*, 139, 115–132.
- Simar, L and P. Wilson (2007), Estimation and Inference in Two-Stage, Semi-Parametric Models of Production Processes, *Journal of Econometrics*, vol 136, 1, 31–64.
- Simar, L. and P.W. Wilson (2008), Statistical Inference in Nonparametric Frontier Models: recent Developments and Perspectives, in *The Measurement of Productive Efficiency*, 2nd Edition, Harold Fried, C.A.Knox Lovell and Shelton Schmidt, editors, Oxford University Press, 2008.
- Simar, L. and P.W. Wilson (2010), Inference From Cross-Sectional Stochastic Frontier Models. *Econometric Review*, 29, 1, 62–98.

- Simar, L. and P.W. Wilson (2011a), Inference by the m out of n bootstrap in nonparametric frontier models. *Journal of Productivity Analysis*, 36, 33–53.
- Simar, L. and P.W. Wilson (2011), Two-Stage DEA: *Caveat Emptor*. *Journal of Productivity Analysis*, 36, 205–218.
- Simar, L. and V. Zelenyuk (2006), On Testing Equality of Two Distribution Functions of Efficiency Score Estimated via DEA, *Econometric Review*, 25(4), 497-522.
- Simar, L. and V. Zelenyuk (2007), Statistical Inference for Aggregates of Farrell-type Efficiencies, *Journal of Applied Econometrics*, vol 22, 7, 1367–1394.
- Simar, L. and V. Zelenyuk (2011), Stochastic FDH/DEA estimators for frontier analysis. *Journal of Productivity Analysis*, 36, 1–20..
- Wilson, P.W. (2008), FEAR 1.0: A Software Package for Frontier Efficiency Analysis with R, *Socio-Economic Planning Sciences* 42, 247–254.
- Wilson, P.W. (2011), Asymptotic properties of some non-parametric hyperbolic efficiency estimators, in I. van Keilegom and P. W. Wilson, eds., *Exploring Research Frontiers in Contemporary Statistics and Econometrics*, Berlin: Springer-Verlag.