Social Balance, Gossiping and Riccati Equations

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joint work with Vincent Traag (KNAW)
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Joos-fest, September 2013

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Joos's interests included Riccati differential equations

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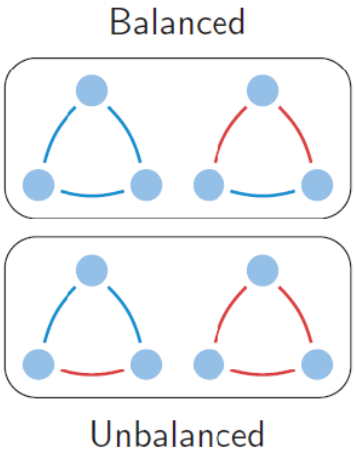

Heider's theory on social balance (1946)

Structure Thm (Cartwright & Harary, 1956): Social balance iff, up to permutation, X takes one of two block forms:

$$X = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Thus, at most 2 factions in the network.
Individuals in same faction are friends.
Individuals in different factions are enemies.

Examples: World Peace (?), WW II (allied powers/axis powers), Cold War (NATO/Warsaw pact), democracy (majority/opposition),...

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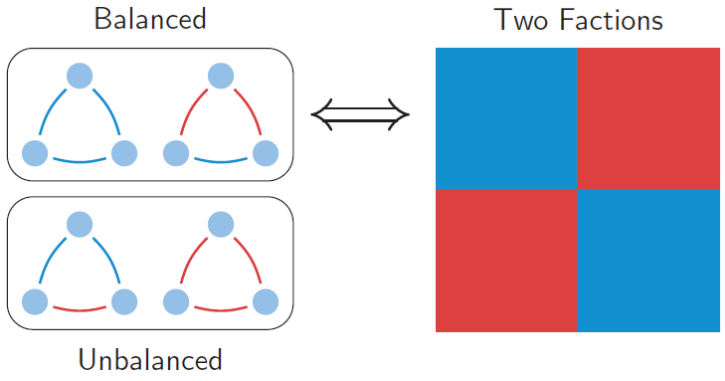

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- Kulakowsky etal in 2005 numerically studied:

$$\dot{X} = X^2, \text{ i.e. } \dot{X}_{ij} = \sum_k X_{ik} X_{kj} \qquad (2)$$

and observed emergence of social balance.
Model interpretation: i updates opinion about j, based on gossip from k: the product of
(i) i's opinion about gossiping partner k, and
(ii) what k thinks about j.
Note: Self-esteem X_{ii} has a positive effect, X_{ii}^2.

Dynamic models of social balance

Technical issue: Finite-time blow up, even if $n = 1$:

$$\dot{x} = x^2, \;\; x(0) = x_0,$$

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To remedy this, one considers social balance in a normalized sense:

$$\lim_{t \to \bar{t}} \frac{X(t)}{|X(t)|_F}, \;\; |X|_F := \Big(\text{tr}(XX^T)\Big)^{1/2}$$

where $\bar{t}$ is finite when $X(t)$ blows up in finite time. What matters for social balance are the signs of the entries $X_{ij}(t)$.

Dynamic models of social balance

Strogatz et al. in 2011 proved that, if $X(0) = X(0)^T$:

$$\lim_{t \to 1/\lambda_1} \frac{X(t)}{|X(t)|_F} = U_1 U_1^T$$

provided eigenvalues of $X(0)$: $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1 > 0$, and $X(0)U_1 = \lambda_1 U_1$.

Thus, if, up to permutation,:

$$U_1 = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + \\ - \end{pmatrix} \implies U_1 U_1^T = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

then social balance is achieved in finite time.

Can we relax symmetry of $X(0)$?

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Put $X(0)$ in Jordan canonical form:

$$X(0) = U \begin{pmatrix} A_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & A_k & 0 & \dots & 0 \\ 0 & \dots & 0 & B_1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & B_l \end{pmatrix} U^{-1},\ UU^{-1} = I_n$$

$$A_i = \begin{pmatrix} a_i & 1 & \dots & 0 \\ 0 & a_i & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_i \end{pmatrix},\ B_j = \begin{pmatrix} C_i & I_2 & \dots & 0 \\ 0 & C_i & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_i \end{pmatrix},$$

$$C_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix},\ \beta_j \neq 0$$

$$\Longrightarrow \text{ Solution: } X(t) = U \begin{pmatrix} A_1(t) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_l(t) \end{pmatrix} U^{-1}$$

Thm Let $a_1 > a_2 \geq \cdots \geq a_k$ with $a_1 > 0$ simple, and $X(0)U_1 = a_1 U_1$, and $V_1^T X(0) = a_1 V_1^T$. Then

$$\lim_{t \to 1/a_1} \frac{X(t)}{|X(t)|_F} = \frac{U_1 V_1^T}{|U_1 V_1^T|_F} = \frac{U_1 V_1^T}{||U_1||.||V_1||}$$

Remark: Although convergence to rank 1 matrix, there is generically no social balance! Up to permutation:

$$U_1 V_1^T = \begin{pmatrix} + \\ \hline - \end{pmatrix} \left(+ \;\; - \;\; | \;\; + \;\; - \right) = \begin{pmatrix} + & - & + & - \\ - & + & - & + \end{pmatrix}$$

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Def: Network is socially balanced if:

$$X_{ij}X_{ik}X_{jk} > 0, \ \forall i, j, k$$

Equivalent to previous definition, since $(i, j, k) \to (i, k, j)$, so Structure Thm remains valid: Social balance iff

$$X = \begin{pmatrix} + \end{pmatrix} \text{ or } \begin{pmatrix} + & - \\ - & + \end{pmatrix}$$

Relaxing symmetry of $X(0)$ for $\dot{X} = XX^T$.

Symm. and anti-symm. parts satisfy:

$$\begin{aligned} \dot{S} &= (S+A)(S-A) \\ \dot{A} &= 0 \Longrightarrow A(t) = A_0 \end{aligned}$$

S-eqn. is a matrix RDE. Eliminate linear terms:

$$\widehat{S}(t) = \mathrm{e}^{-tA_0}\, S(t)\, \mathrm{e}^{tA_0}$$

$$\Longrightarrow \quad \dot{\widehat{S}} = \widehat{S}^2 - A_0^2,\ \widehat{S}(0) = S_0$$

Block-diagonalize A_0 with $VV^T = I_n$:

$$A_0 = V \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & B_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{pmatrix} V^T,\ B_j = \begin{pmatrix} 0 & \omega_j \\ -\omega_j & 0 \end{pmatrix},\ \omega_j \neq 0$$

Then setting $\tilde{S} = V\hat{S}V^T$ and $\tilde{S}_0 = VS_0V^T$:

$$\dot{\tilde{S}} = \tilde{S}^2 + D^2,\ \tilde{S}(0) = \tilde{S}_0,\ D^2 := \begin{pmatrix} 0 & 0 \\ 0 & \tilde{D}^2 \end{pmatrix},\ \tilde{D} := \begin{pmatrix} \omega_1 I_2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_k I_2 \end{pmatrix}$$

This is a matrix RDE, whose solution follows from:

$$\begin{pmatrix} \dot{P} \\ \dot{Q} \end{pmatrix} = \begin{pmatrix} 0 & -I_n \\ D^2 & 0 \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix},\ \begin{pmatrix} P(0) \\ Q(0) \end{pmatrix} = \begin{pmatrix} I_n \\ \tilde{S}_0 \end{pmatrix}$$

Then:

$$\tilde{S}(t) = Q(t)P^{-1}(t),\ \begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = \frac{\left(\begin{pmatrix} I_{n-2k} & 0 \\ 0 & c \end{pmatrix} - \begin{pmatrix} tI_{n-2k} & 0 \\ 0 & \tilde{D}^{-1}s \end{pmatrix} \tilde{S}_0 \right)}{\begin{pmatrix} 0 & 0 \\ 0 & \tilde{D}s \end{pmatrix} + \begin{pmatrix} I_{n-2k} & 0 \\ 0 & c \end{pmatrix} \tilde{S}_0},$$

$$s := \begin{pmatrix} \sin(\omega_1 t) I_2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sin(\omega_k t) I_2 \end{pmatrix} \text{ (and } c \to \cos)$$

Generic case

- In the generic case $\tilde{S}(t)$ has a unique eigenvalue which blows up in finite time $\bar{t}$.

- Therefore social balance at $t \to \bar{t}$ for $\dot{X} = XX^T$:

$$\begin{aligned}
\lim_{t\to\bar{t}} \frac{X(t)}{|X(t)|_F} &= \lim_{t\to\bar{t}} \frac{S(t)+A_0}{|S(t)+A_0|_F} = \lim_{t\to\bar{t}} \frac{S(t)}{|S(t)|_F} \\
&= \mathrm{e}^{\bar{t}A_0}\, V^T \left(\lim_{t\to\bar{t}} \frac{\tilde{S}(t)}{|\tilde{S}(t)|_F} \right) V\, \mathrm{e}^{-\bar{t}A_0} \\
&= \mathrm{e}^{\bar{t}A_0}\, V^T U_{i^*} U_{i^*}^T V\, \mathrm{e}^{-\bar{t}A_0}
\end{aligned}$$

Conclusions

For $\dot{X} = X^2$, social balance is generically not achieved if $X(0)$ is not symmetric.

For $\dot{X} = XX^T$, social balance is generically achieved for arbitrary $X(0)$.

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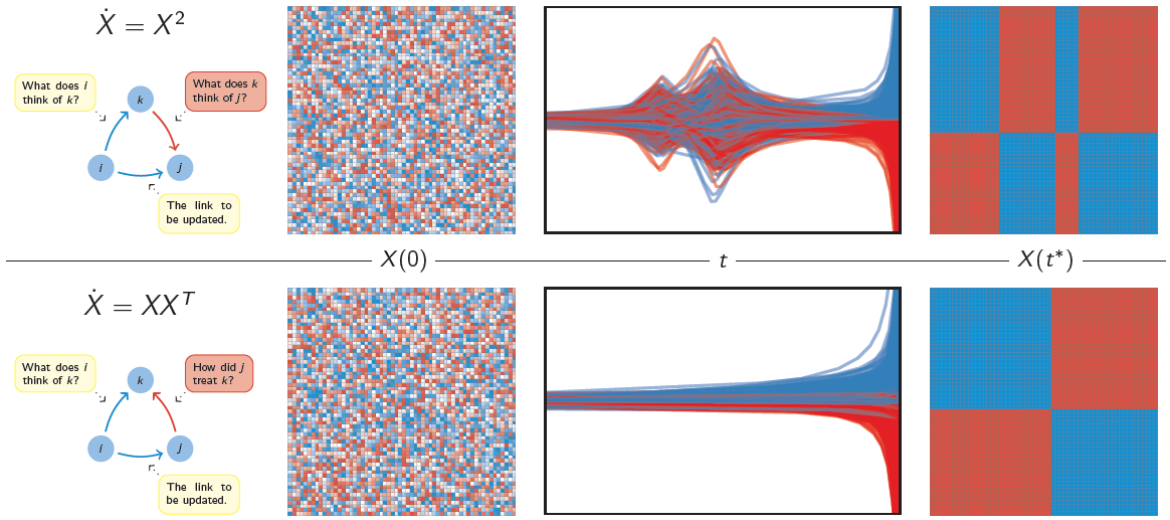

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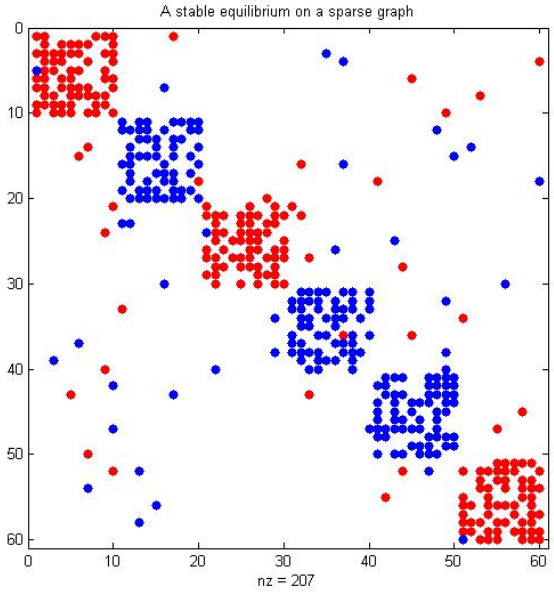

Different regions may behave almost independently and multi-factions may appear

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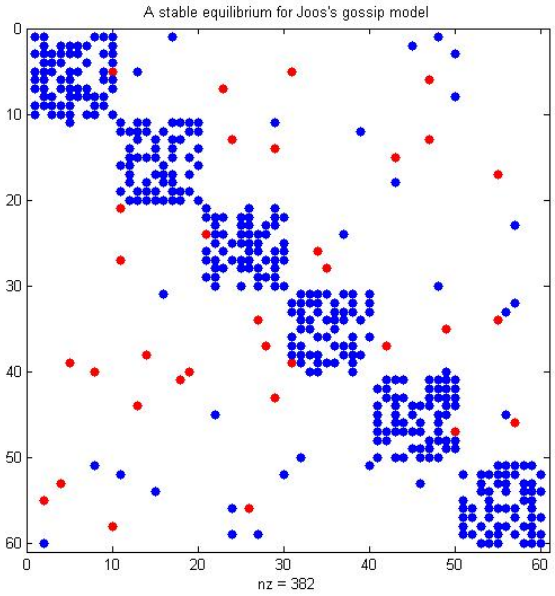

But the “Joos” effect is still not well understood ...

A few references in the area :

Heider (1946) Attitudes and cognitive organization. Journal of Psychology 21:107-112.

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What's next ?

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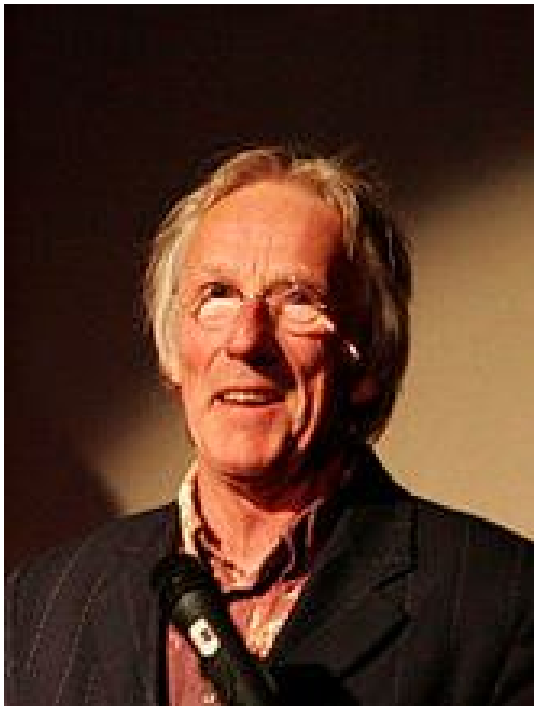
Freek de Jonge's reaction is to say

Na de dood, daar is leven, daar is leven, na de dood ...



What's next ?

but Joos knows what his afterday's life will be like ...



Good luck, Joos, with your "afterlife"