# Updating tensor decompositions

**Michiel Vandecappelle**
**michiel.vandecappelle@kuleuven.be**
Nico Vervliet
Martijn Boussé
Lieven De Lathauwer

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Tensor-Based Signal Processing
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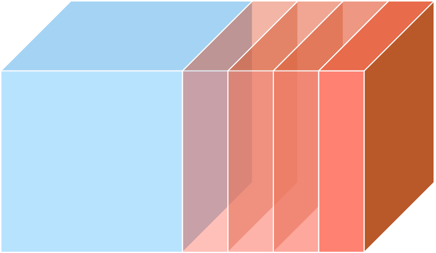
# Tensors can change over time



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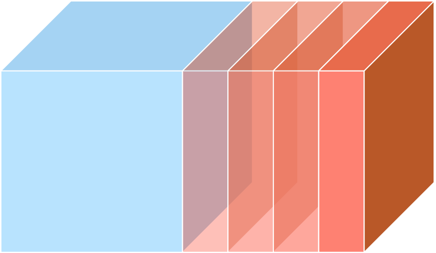
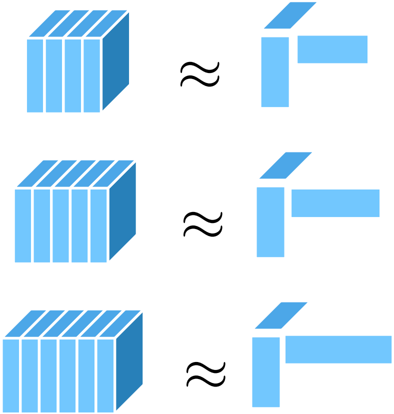


- Data only becomes available gradually
- The data is non-stationary
- The full tensor does not fit into memory

# Tensors can change over time

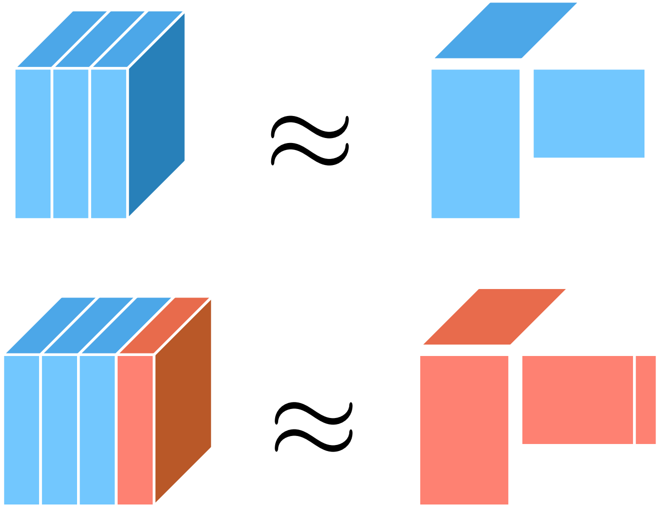
- Example: EEG data
  - New EEG samples are obtained
  - Old data becomes outdated
  - Test subjects are added/removed
  - Mobile EEG

# Recomputing the full decomposition can be infeasible or unnecessary



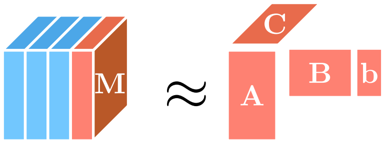
- The new data arrives too fast
- The full tensor takes up too much memory
- We want to locally improve the accuracy of the decomposition

# Can tensor decompositions adapt to changes in the tensor?



# CPD updating

- A new slice $\mathbf{M}$ gets added to the tensor



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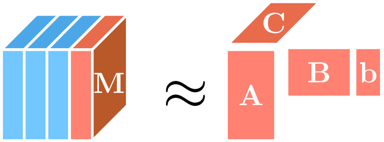

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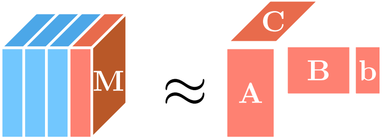

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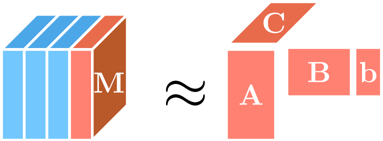
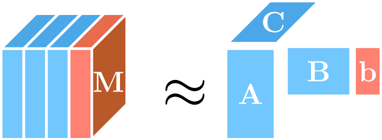
- The algorithm should be fast, accurate and memory-efficient



# Existing algorithms are sequential

- Estimate $\mathbf{b}$ with its least squares solution:

$$\left[(\mathbf{C}_{\mathsf{old}}^{\mathsf{T}}\mathbf{C}_{\mathsf{old}}) * (\mathbf{A}_{\mathsf{old}}^{\mathsf{T}}\mathbf{A}_{\mathsf{old}})\right]^{-1}(\mathbf{C}_{\mathsf{old}} \odot \mathbf{A}_{\mathsf{old}})^{\mathsf{T}}\mathsf{vec}(\mathbf{M})$$



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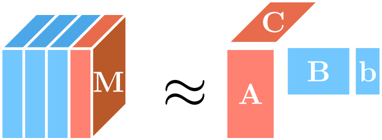

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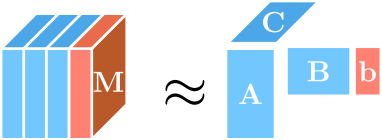
- Update $\mathbf{A}$ and $\mathbf{C}$
- Refine $\mathbf{b}$:

$$\left[(\mathbf{C}_{\text{new}}^{\mathsf{T}}\mathbf{C}_{\text{new}}) * (\mathbf{A}_{\text{new}}^{\mathsf{T}}\mathbf{A}_{\text{new}})\right]^{-1}(\mathbf{C}_{\text{new}} \odot \mathbf{A}_{\text{new}})^{\mathsf{T}}\text{vec}(\mathbf{M})$$



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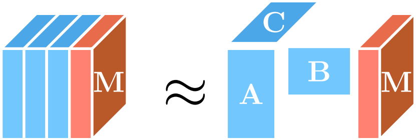

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- Replace old slices by an approximation (CPD, MLSVD . . . )
- Derive efficient expressions for objective function, gradient and Hessian approximation $\mathbf{J}^\mathsf{T}\mathbf{J}$ by exploiting structure of old slices
  - Split the computation in parts depending on the old data and parts depending on new slices
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- Apply Conjugate Gradients to solve the system $\mathbf{J}^\mathsf{T}\mathbf{J}\mathbf{p} = -\mathbf{g}$
  - Forming and inverting $\mathbf{J}^\mathsf{T}\mathbf{J}$ is not needed
  - Products of the form $\mathbf{J}^\mathsf{T}\mathbf{J}\mathbf{x}$ can be computed efficiently

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- Apply Conjugate Gradients to solve the system $\mathbf{J}^\mathsf{T}\mathbf{J}\mathbf{p} = -\mathbf{g}$
- Use block-Jacobi preconditioner $\mathbf{M}$
  - Left-multiply both sides of $\mathbf{J}^\mathsf{T}\mathbf{J}\mathbf{p} = -\mathbf{g}$ by $\mathbf{M}^{-1}$
  - Good choice of $\mathbf{M}$ improves convergence of CG algorithm

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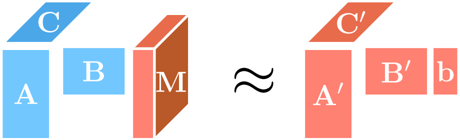
- Limit number of NLS and CG iterations



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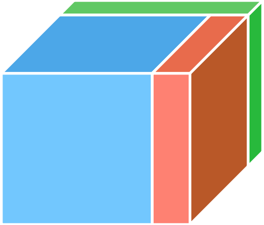
- Applicable to $(N-1)$-th order updates of $N$th-order tensors in any number of modes simultaneously



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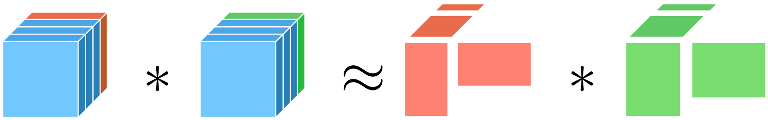
- Extendable to structured or coupled CPDs
- Weighted least squares (WLS) CPD updating
  - CPD of weight tensor is updated
  - This CPD is used in WLS update of the data tensor CPD



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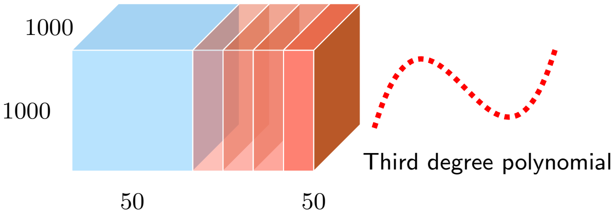

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- Model with slowly evolving second mode and $20$dB SNR
- Comparison of batch and updating methods
  - Batch CPD-NLS and CPD-ALS (Sorber et al. 2013)
  - PARAFAC-SDT and PARAFAC-RLST (Nion et al. 2009)
  - CPD updating

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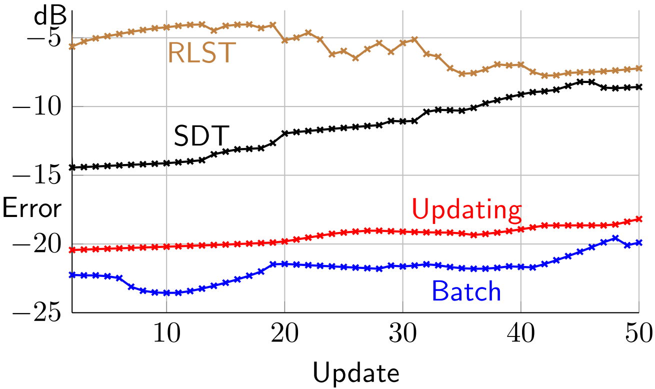

# Updating is both accurate and fast

- Model with slowly evolving second mode and $20$dB SNR
- Comparison of batch and updating methods
- Mean errors of the CPD approximation
- CPU-time of the CPD approximation (ms)

| $R$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|---|---|---|---|---|---|
| Updating | 60 | 81 | 104 | 140 | 169 |
| NLS | 2375 | 4464 | 2557 | 3563 | 5522 |
| ALS | 910 | 1222 | 1400 | 1401 | 2352 |
| SDT | 48 | 71 | 98 | 136 | 172 |
| RLST | 570 | 607 | 623 | 775 | 822 |

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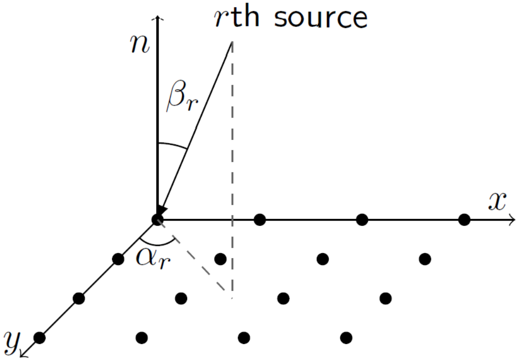
- Direction-of-arrival estimation using a uniform rectangular array (URA)



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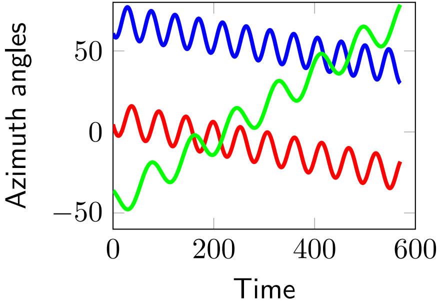


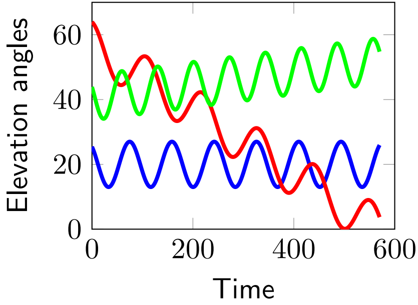

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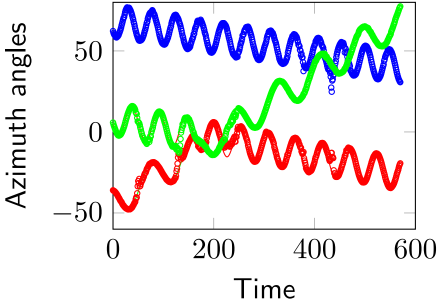


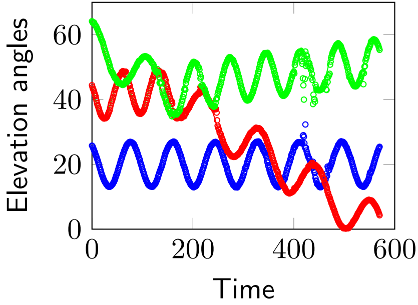

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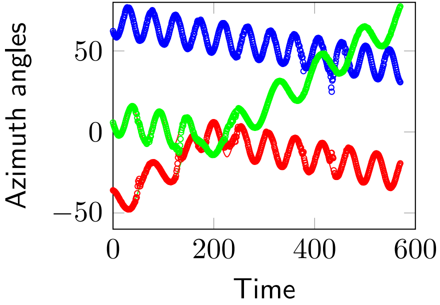
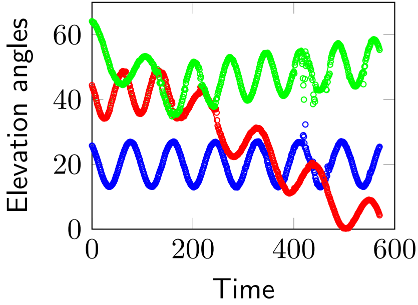
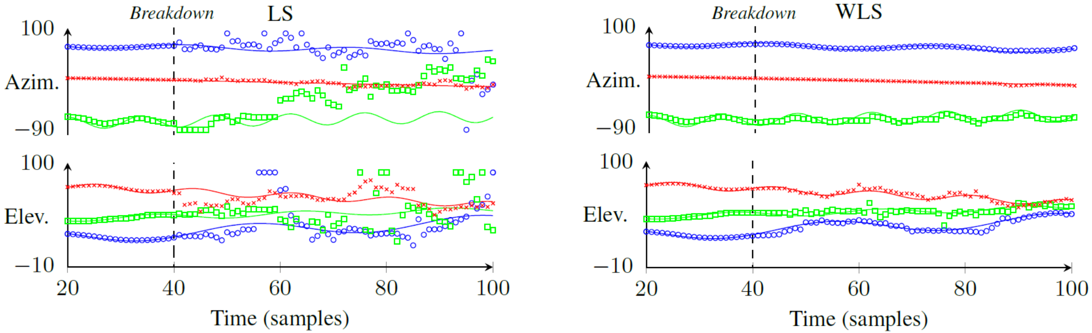
- NLS updating: 6-8ms per update



# NLS updating enables real-time CPD applications

- WLS NLS CPD updating
  - Direction-of-arrival estimation with three moving sources
  - Some sensors break down, leading to bad readings
  - Less weight is given to readings of these sensors



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- Low memory requirements $\mathcal{O}(\text{New slice})$

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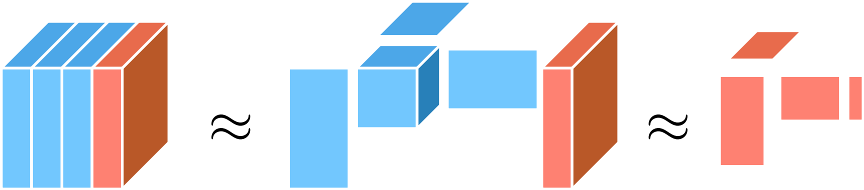

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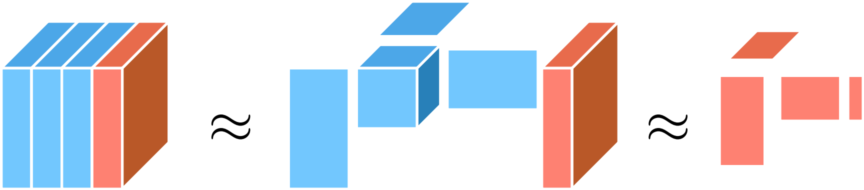

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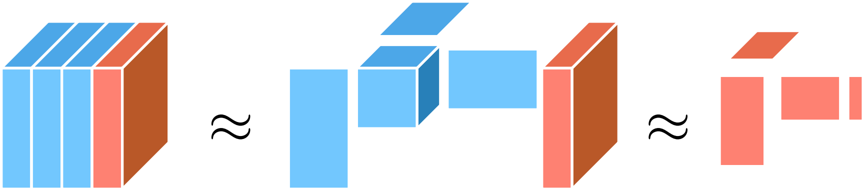

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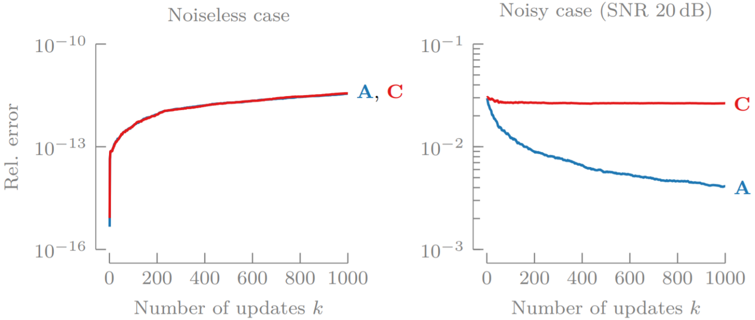
  - Compromise between storing full tensor and only CPD
  - Track enough data for rank-increases
  - Countering numerical error accumulation
    - Numerical error increases for all modes
    - Statistical error decreases for non-updated modes



# Further possibilities for updating algorithms

- Other decompositions (MLSVD, BTD)
- Different cost functions
- Tensor data changing in other ways

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