Revisiting gray box model learning and Kalman filtering with subspace based model learning

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Outline

1. Motivations
2. Subspace based model learning (SML): a reminder
3. Kalman filter tuning with SML
4. Gray box model learning with SML
5. Discussion

MOTIVATIONS

Engineers need accurate models

- In many practical situations, engineers have to determine dynamical models of real systems from available data sets and prior knowledge.

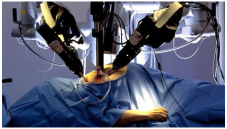
AIRBUS-ONERA-LIAS project

Fluttering detection
DDM from short duration data

GE-LAMIH-RU-LIAS project

Fouling detection in heat exchangers

SINTERS-ICUBE-LIAS project



Flexibility estimation and control for remote surgery

MICHELIN-LIAS project

Tire-road interaction estimation for autonomous cars

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Common denominator of these projects

- During these projects, we were asked to generate (new) estimators of physical parameters and signals.
  - "The best" parameter estimates were produced with nonlinear optimization based solutions.
  - "The best" signal estimates were produced with Kalman filters.
- Precise results were achieved solely through meticulous tuning of these techniques.
  - Nonlinear optimization works well when initial guesses are in the vicinity of the global optimum.
  - Kalman filters are efficient when the noise covariance matrices are well selected *a priori*.
- We primarily developed new tuning solutions based on initial estimates generated by subspace based model learning.

SUBSPACE BASED MODEL LEARNING: A REMINDER

Problem formulation

- Subspace based model learning (SML) methods mainly focus on state space models of the form

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{w}_k, \tag{1a}$$
$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{v}_k. \tag{1b}$$

where the noise sources are assumed to be realizations of zero mean white noises statically independent of the input sequence such that

$$\mathbb{E}\left[\begin{bmatrix}\mathbf{v}_i \\ \mathbf{w}_i\end{bmatrix}\begin{bmatrix}\mathbf{v}_j^\top & \mathbf{w}_j^\top\end{bmatrix}\right] = \begin{bmatrix}\boldsymbol{V} & \boldsymbol{S} \\ \boldsymbol{S}^\top & \boldsymbol{W}\end{bmatrix}\delta_{ij}. \tag{2}$$

Problem formulation (cont'd)

- By assuming that

  A1 the input vector sequence is quasi stationary and exciting of sufficient order,

  A2 the pair $(\boldsymbol{A}, \boldsymbol{C})$ is observable and the pair $(\boldsymbol{A}, \begin{bmatrix} \boldsymbol{B} & \boldsymbol{V}^{1/2} \end{bmatrix})$ is reachable,

  standard SML solutions aim at estimating

  - the order n_x of the system,
  - an approximated minimum variance estimate of $\boldsymbol{x}_k$, $k \in \mathbb{T}$,
  - $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ up to a similarity transformation.

Main steps

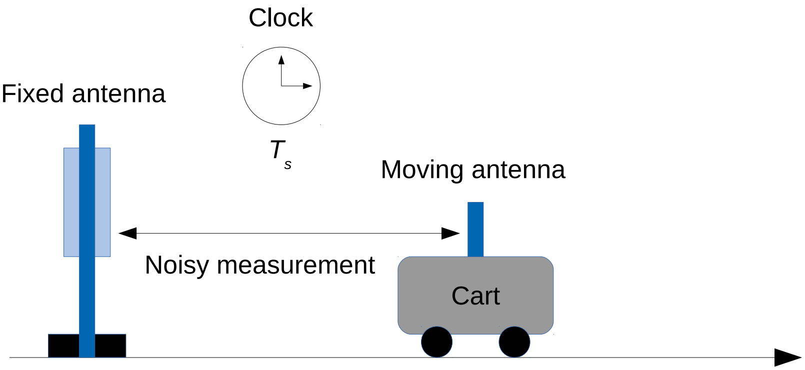
- Under Assumption A1 and A2, standard SML solutions consists in
  1. selecting future and past indexes f and p with the constraint that $f > n_x$ and p large "enough",
  2. building "past" and "future" Hankel matrices $\boldsymbol{Z}_p$, $\boldsymbol{U}_f$ and $\boldsymbol{Y}_f$,
  3. computing the RQ factorization of $\begin{bmatrix}\boldsymbol{U}_f^\top & \boldsymbol{Z}_p^\top & \boldsymbol{Y}_f\end{bmatrix}^\top$,
  4. extracting $\boldsymbol{R}_{32}$ and $\boldsymbol{R}_{22}$ from this RQ factorization, then computing the SVD of $\boldsymbol{R}_{32}\boldsymbol{R}_{22}^{-1}\boldsymbol{Z}_p$ to determine $\hat{n}_x$ as well as an estimate $\hat{\boldsymbol{X}}_{[f,M]}$ of the state on a user defined horizon,
  5. resorting to a LLS solution for determining $\hat{\boldsymbol{A}}$ $\hat{\boldsymbol{B}}$ and $\hat{\boldsymbol{C}}$.
- Keep in mind that all these estimates are valid up to a similarity transformation!!!!

KALMAN FILTER TUNING WITH SUBSPACE BASED MODEL LEARNING

KALMAN FILTERING: A REMINDER

Toy example

- Let us assume we want[1] to determine from remote noisy measurements the position and speed (state) of a cart moving straightforward.



[1]See *Understanding the basis of the Kalman filter via a simple and intuitive derivation*, R. Faragher, IEEE Signal Processing Magazine, 2012.

Main steps

- In order to reach this goal, we need[2]
  - a model of the cart dynamics, *i.e.*,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k,$$

  - a model of the measuring process, *i.e.*,

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k,$$

[2] We focus on LTI systems only.

Main steps

- In order to reach this goal, we need
  - a model of the cart dynamics, *i.e.*,

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k + \boldsymbol{w}_k,$$

  - a model of the measuring process, *i.e.*,

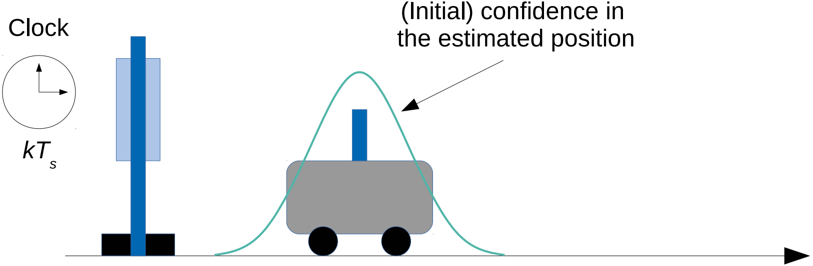
$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{v}_k,$$

  - a description of the noise and uncertainties acting on the system, *i.e.*,

$$\mathbb{E}\left\{\mathbf{w}_k\right\} = \mathbf{0},$$
$$\mathbb{E}\left\{\mathbf{w}_k\mathbf{w}_j^\top\right\} = \boldsymbol{W}_k\delta_{kj},\ \ \boldsymbol{W}_k \succ 0,$$

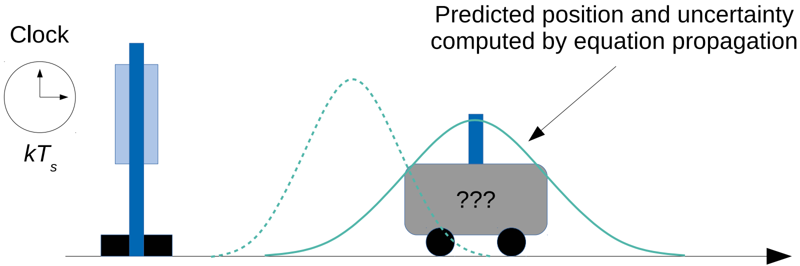
$$\mathbb{E}\left\{\mathbf{v}_k\right\} = \mathbf{0},$$
$$\mathbb{E}\left\{\mathbf{v}_k\mathbf{v}_j^\top\right\} = \boldsymbol{V}_k\delta_{kj},\ \ \boldsymbol{V}_k \succ 0.$$

Main steps (cont'd)



- Prior:
  - the model matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$,
  - the estimated state $\boldsymbol{x}_{k-1}^{+}$ and its covariance matrix $\boldsymbol{X}_{k-1}^{+}$.

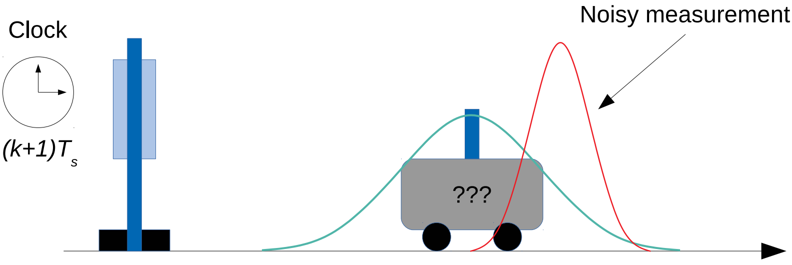
Main steps (cont'd)



- "Prediction" Kalman filter equations are

$$\hat{\boldsymbol{x}}_k^- = \boldsymbol{A}\hat{\boldsymbol{x}}_{k-1}^+ + \boldsymbol{B}\boldsymbol{u}_{k-1},$$
$$\boldsymbol{X}_k^- = \boldsymbol{A}\boldsymbol{X}_{k-1}^+\boldsymbol{A}^\top + \boldsymbol{W}_k.$$

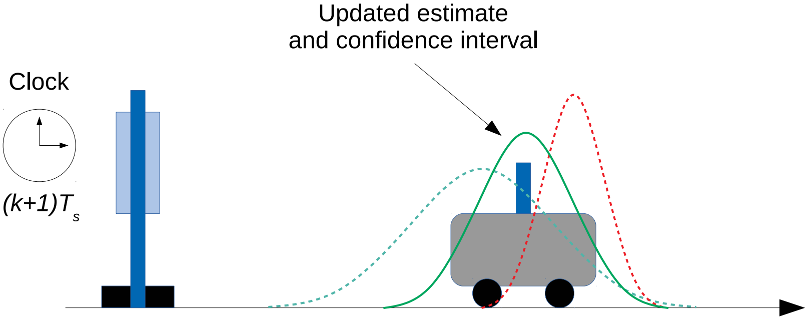
Main steps (cont'd)



- We get noisy measurements, *i.e.*,

$$\boldsymbol{y}_k = \boldsymbol{C}\boldsymbol{x}_k + \boldsymbol{v}_k.$$

Main steps (cont'd)



- "Update" Kalman filter equations are

$$\boldsymbol{K}_k = \boldsymbol{X}_k^- \boldsymbol{C}^\top \left(\boldsymbol{C} \boldsymbol{X}_k^- \boldsymbol{C}^\top + \boldsymbol{V}_k \right)^{-1},$$

$$\hat{\boldsymbol{x}}_k^+ = \hat{\boldsymbol{x}}_k^- + \boldsymbol{K}_k (\boldsymbol{y}_k - \boldsymbol{C} \hat{\boldsymbol{x}}_k^-),$$

$$\boldsymbol{X}_k^+ = \left(\boldsymbol{I}_{n_x \times n_x} - \boldsymbol{K}_k \boldsymbol{C} \right) \boldsymbol{X}_k^-.$$

Basic idea of the solution

- The sequences $(\boldsymbol{v}_i)_{i \in \mathbb{T}}$ and $(\boldsymbol{w}_i)_{i \in \mathbb{T}}$ are used to describe
  - the noise acting on the real system,
  - the (in)accuracy of the model representation,

  thanks to their covariance matrices $\boldsymbol{V}_k$ and $\boldsymbol{W}_k$, $k \in \mathbb{T}$.
- Because $\boldsymbol{V}$ and $\boldsymbol{W}$ are used to describe the confidence we have in the model and the measurements, we aim at determining them by comparing
  - the model used in the Kalman filter,
  - a model estimated from the available data sets.
- Herein, the data driven model learning solution is a SML method.

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NOISE COVARIANCE MATRIX ESTIMATION

Main ingredients

- When a Kalman filter is designed, we have access to
  - I/O data samples,
  - a "reliable" DT state space representation of the system dynamics, *i.e.*, $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$.
- When the I/O data is rich enough, it can be used with a SML algorithm to generate
  - an approximated minimum variance estimate of $\boldsymbol{x}_k$, $k \in \mathbb{T}$,
  - estimates of $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$,

  up to a similarity transformation.

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Notation interlude

For any vector $\boldsymbol{r}_k \in \mathbb{R}^{n_r \times 1}$ and parameters M, i and $\ell \in \mathbb{N}_*^+$, we define

$$\boldsymbol{R}_{[i,M]} = \begin{bmatrix} \boldsymbol{r}_i & \boldsymbol{r}_{i+1} \cdots & \boldsymbol{r}_{i+M-1} \end{bmatrix} \in \mathbb{R}^{n_r \times M}.$$

Similarity transformation

- The SML algorithm gives access to $\hat{\boldsymbol{X}}_{[i,M]}$ in an unknown basis.
- Knowing $(\boldsymbol{A}, \boldsymbol{C})$ and $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{C}})$, the similarity transformation $\boldsymbol{T}$ between $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ and $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{C}})$ satisfies

$$\boldsymbol{\Gamma}_i(\boldsymbol{A}, \boldsymbol{C})\boldsymbol{T} = \boldsymbol{\Gamma}_i(\hat{\boldsymbol{A}}, \hat{\boldsymbol{C}}),$$

where

$$\boldsymbol{\Gamma}_i(\boldsymbol{A}, \boldsymbol{C}) = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{i-1} \end{bmatrix}, \; i \geq n_x.$$

- Thus, we can get $\breve{\boldsymbol{X}}_{[i,M]}$ is the "correct basis" as follows

$$\breve{\boldsymbol{X}}_{[i,M]} = \boldsymbol{\Gamma}_i^{\dagger}(\boldsymbol{A}, \boldsymbol{C})\boldsymbol{\Gamma}_i(\hat{\boldsymbol{A}}, \hat{\boldsymbol{C}})\hat{\boldsymbol{X}}_{[i,M]}.$$

Covariance matrix estimation

- We can finally estimate

$$\begin{bmatrix} \hat{\boldsymbol{W}}_{[i,M-1]} \\ \hat{\boldsymbol{V}}_{[i,M-1]} \end{bmatrix} = \begin{bmatrix} \breve{\boldsymbol{X}}_{[i+1,M]} \\ \boldsymbol{Y}_{[i,M-1]} \end{bmatrix} - \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \breve{\boldsymbol{X}}_{[i,M-1]} \\ \boldsymbol{U}_{[i,M-1]} \end{bmatrix},$$

and

$$\begin{bmatrix} \hat{\boldsymbol{V}} & \hat{\boldsymbol{S}} \\ \hat{\boldsymbol{S}}^\top & \hat{\boldsymbol{W}} \end{bmatrix} = \lim_{M\to\infty} \frac{1}{M} \begin{bmatrix} \hat{\boldsymbol{W}}_{[i,M-1]} \\ \hat{\boldsymbol{V}}_{[i,M-1]} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{W}}^\top_{[i,M-1]} & \hat{\boldsymbol{V}}^\top_{[i,M-1]} \end{bmatrix}.$$

NUMERICAL ILLUSTATIONS

Toy example

- We consider

$$\boldsymbol{A} = \begin{bmatrix} 0.603 & 0.603 & 0 & 0 \\ -0.603 & 0.603,0 & 0 & \\ 0 & 0 & -0.603 & -0.603 \\ 0 & 0 & 0.603 & -0.603 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{bmatrix} 1.1650 & -0.6965 \\ 0.6268 & 1.6961 \\ 0.0751 & 0.0591 \\ 0.3516 & 1.7971 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0.2641 & -1.4462 & 1.2460 & 0.5774 \\ 0.8717 & -0.7012 & -0.6390 & -0.3600 \end{bmatrix},$$

$$\boldsymbol{K} = 4 \times \begin{bmatrix} 0.1242 & -0.0895 \\ -0.0828 & -0.0128 \\ 0.0390 & -0.0968 \\ -0.0225 & 0.1459 \end{bmatrix},$$

$$\boldsymbol{R}_e = \begin{bmatrix} 0.0176 & -0.0267 \\ -0.0267 & 0.0497 \end{bmatrix}.$$

- We generate 10^3 realizations of the noise sequence and we select a data length $N = 1000$.

Estimates of the elements of $\boldsymbol{V}$

		$\hat{v}_{11}$	$\hat{v}_{12}$	$\hat{v}_{22}$
Theo. value		0.0176	-0.0267	0.0497
Sample cov.	avg.	0.0176	-0.0267	0.0497
	std.	0.0008	0.0012	0.0022
ICM	avg.	0.0588	-0.0750	0.0888
	std.	0.0412	0.0149	0.0057
DCM	avg.	0.031	-0.041	0.028
	std.	0.0041	0.0013	0.006
CMM	avg.	0.023	-0.073	0.031
	std.	0.0087	0.0066	0.0092
New meth.	avg.	0.0198	-0.0272	0.0516
	std.	0.0011	0.0013	0.0024

Estimates of the elements of $\boldsymbol{W}$

		$\hat{w}_{11}$	$\hat{w}_{12}$	$\hat{w}_{22}$	$\hat{w}_{23}$	$\hat{w}_{34}$	$\hat{w}_{44}$
Theo. value		0.0202	-0.0045	0.0149	-0.0198	0.0012	-0.0031
Sample cov.	avg.	0.0202	-0.0045	0.0149	-0.0198	0.0012	-0.0031
	std.	0.8886e-03	0.2054e-03	0.6563e-03	0.8750e-03	0.0509e-03	0.1487e-03
ICM	avg.	0.0526	-0.0150	0.0355	-0.0454	0.0041	-0.0103
	std.	0.0146	0.0079	0.0066	0.0067	0.0042	0.0036
DCM	avg.	0.0113	-0.0058	0.0186	-0.0285	0.003	-0.0103
	std.	0.004	0.0033	0.0068	0.0067	0.0039	0.0033
CMM	avg.	0.0170	-0.0041	0.0124	-0.0234	0.0041	-0.0043
	std.	0.0097	0.0082	0.0064	0.0062	0.0052	0.0028
New meth.	avg.	0.0196	-0.0041	0.0145	-0.0190	0.0015	-0.0026
	std.	0.0017	0.0006	0.0011	0.0011	0.0004	0.0005

Estimates of the elements of $\boldsymbol{S}$

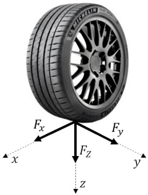
		$\hat{s}_{11}$	$\hat{s}_{13}$	$\hat{s}_{21}$	$\hat{s}_{24}$
Theo. value		0.0183	-0.0045	0.0131	-0.0172
Sample cov.	avg.	0.0183	-0.0045	0.0131	-0.0172
	std.	0.0008	0.0002	0.0006	0.0008
New meth.	avg.	0.0181	-0.0039	0.0137	-0.0169
	std.	0.0011	0.0007	0.0010	0.0010

MICHELIN PROJECT

PhD thesis with Michelin

- Future autonomous vehicles will require well-developed Advanced Driver Assistance Systems (ADAS) to assist human beings in driving.
- One path chosen by Michelin for ADAS improvement consists in providing ADAS with information related to the state of the road.
- Such information is included in the grip potential quantity.
- Benefits for passenger security (to name a few) are
  - detection of roads with low grip area,
  - evaluation of the driving conditions,
  - reduction of the impact of rear end collisions.

Problem formulation



- The grip potential is

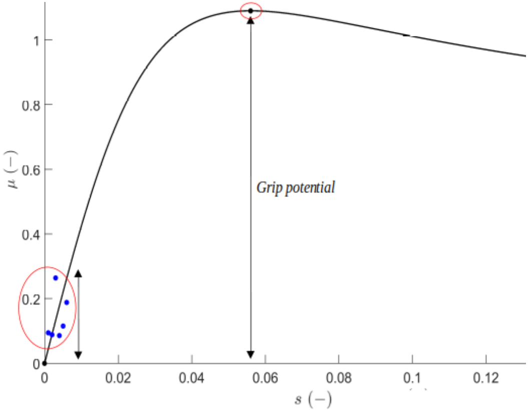
$$\mu_{\max} = \max\left(\frac{\sqrt{F_x^2 + F_y^2}}{F_z}\right),$$

i.e., the maximum effort a tire can generate before sliding on the road.

Problem formulation (cont'd)

Problem

Estimate the grip potential under standard driving conditions from sensors fitted on production vehicles.



Problem formulation (cont'd)

- Getting (noisy) data requires to measure the friction μ and the slip ratio s.
- No dedicated sensors exist on production vehicles.
- These signals must be estimated knowing that, for the longitudinal dynamics,

$$\mu = \frac{F_x}{F_z},$$
$$s = \frac{\omega R_{\mathsf{rol}} - v_x}{\max\left(\omega R_{\mathsf{rol}} - v_x\right)}.$$

- A Kalman filter was suggested to reconstruct the components of μ and s accurately.
- Efficient Kalman filter tuning solutions must be introduced to guarantee accurate estimates.

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Michelin results

- Data is generated with VI-CRT (realistic simulator).
- A nonlinear state space model is used, *i.e.*,

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), t, \boldsymbol{\theta}),$$
$$\boldsymbol{y}(t) = \boldsymbol{g}(\boldsymbol{x}(t), t, \boldsymbol{\theta}),$$

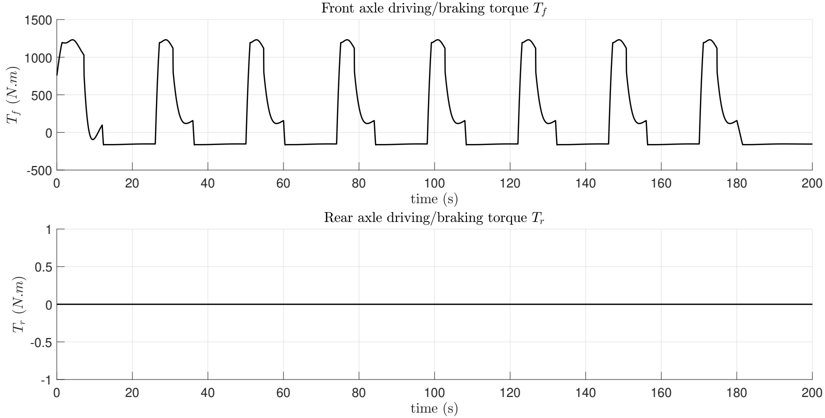
with

$$\boldsymbol{x} = \begin{bmatrix} v_x & \omega_f & \omega_r & F_{x_f} & F_{x_r} & \dot{F}_{x_f} & \dot{F}_{x_r} & \kappa & \dot{\kappa} \end{bmatrix}^\top,$$
$$\boldsymbol{u} = \begin{bmatrix} T_f & T_r \end{bmatrix}^\top,$$
$$\boldsymbol{y} = \begin{bmatrix} v_x & \omega_f & \omega_r & \dot{\kappa} \end{bmatrix}^\top,$$

involving

- a single track model for the vehicle dynamics,
- an effective tire radius model,
- a suspension model and a load transfer model.

Input data



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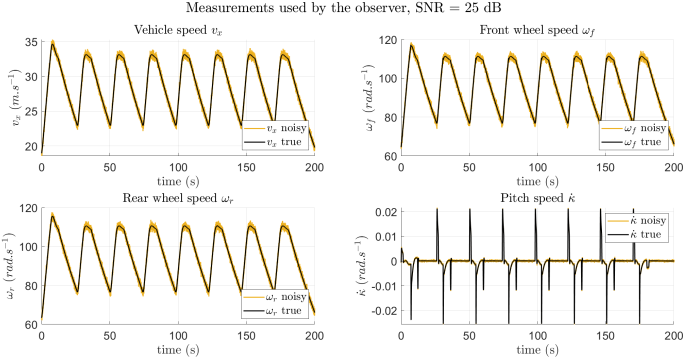
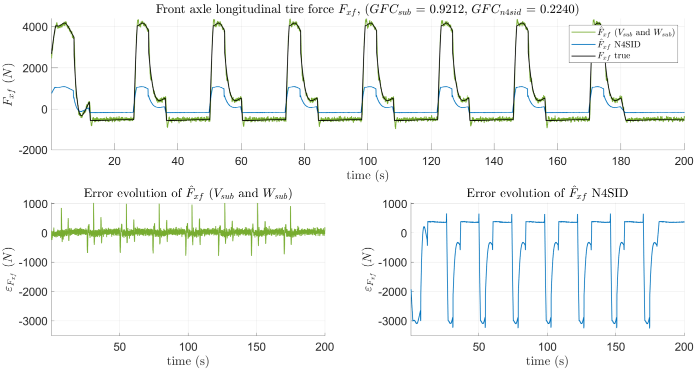
Output data

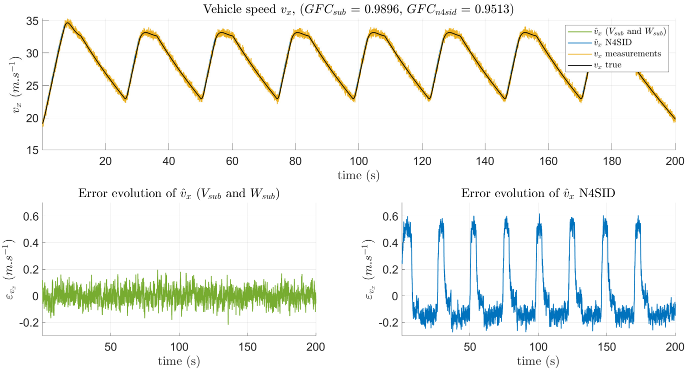


Measurements used by the observer, SNR = 25 dB
Vehicle speed v_x
v_x $(m.s^{-1})$
v_x noisy
v_x true
time (s)
Front wheel speed ω_f
ω_f $(rad.s^{-1})$
ω_f noisy
ω_f true
time (s)
Rear wheel speed ω_r
ω_r $(rad.s^{-1})$
ω_r noisy
ω_r true
time (s)
Pitch speed $\dot{\kappa}$
$\dot{\kappa}$ $(rad.s^{-1})$
$\dot{\kappa}$ noisy
$\dot{\kappa}$ true
time (s)

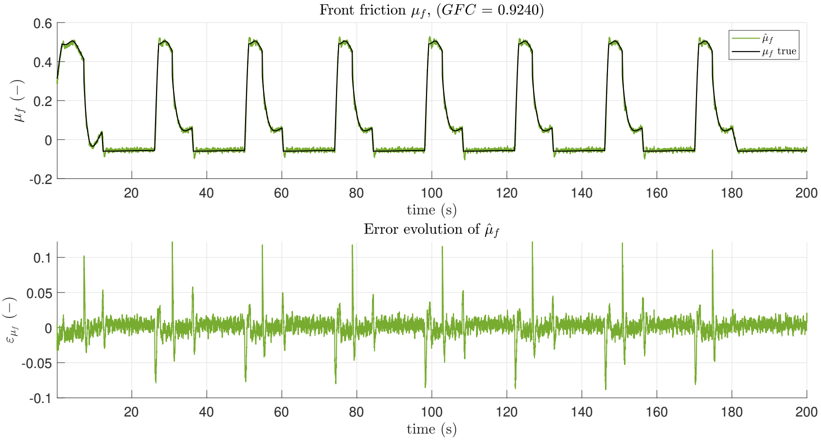
Long. tire force estimation



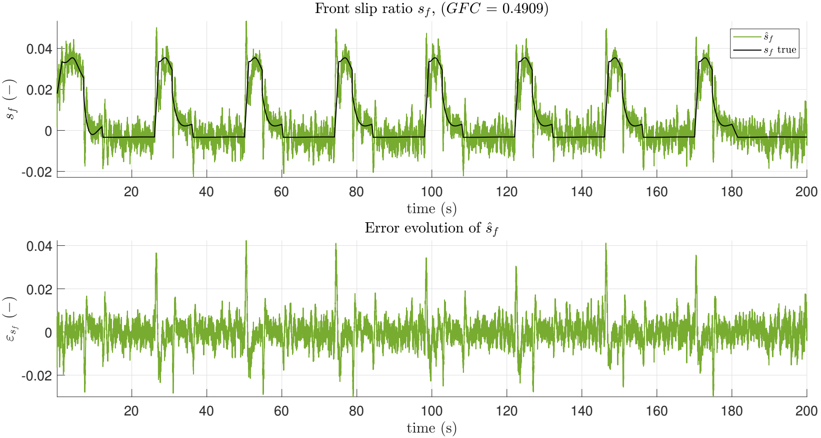
Vehicle speed estimation



Friction reconstruction

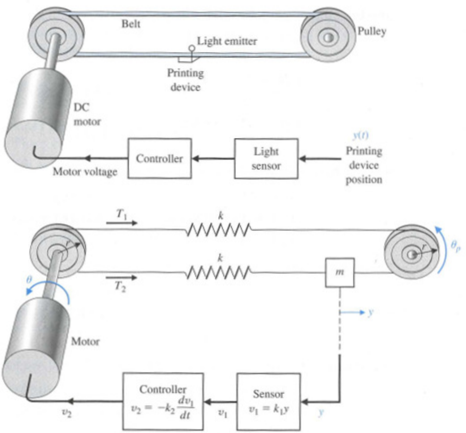


Slip ratio reconstruction



GRAY BOX MODEL LEARNING WITH SUBSPACE BASED MODEL LEARNING

Running example: a printer belt



Picture from "Modern control systems", 2010, pp. 222–228.

Running example (cont'd)

- This system satisfies

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t),$$

with

$$x_1(t) = r\phi(t) - z(t), \quad x_2(t) = \dot{z}(t), \quad x_3(t) = \dot{\phi}(t),$$

and

- $y(t)$ the speed of the printing device,
- $u(t)$ the torque applied to drive the belt.

Running example (cont'd)

- This system satisfies

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t),$$

with

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & -1 & \theta_1 \\ \theta_2 & 0 & 0 \\ \theta_3 & 0 & \theta_4 \end{bmatrix}, \qquad \boldsymbol{B}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ 0 \\ \theta_5 \end{bmatrix},$$
$$\boldsymbol{C}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & \theta_6 & 0 \end{bmatrix}.$$

Running example (cont'd)

- This system satisfies

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t),$$

with

$$\theta_1 = r, \qquad \theta_2 = \frac{2k}{m}, \qquad \theta_3 = -\frac{2kr}{J},$$
$$\theta_4 = -\frac{b}{J}, \qquad \theta_5 = \frac{k_m}{RJ}, \qquad \theta_6 = k_1.$$

Problem

From the available I/O data, estimate consistently (and uniquely) a gray box (GB) model of this system.

A STANDARD SOLUTION

Output error method

- Fix a model parameterization

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & -1 & \theta_1 \\ \theta_2 & 0 & 0 \\ \theta_3 & 0 & \theta_4 \end{bmatrix}, \qquad \boldsymbol{B}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ 0 \\ \theta_5 \end{bmatrix},$$

$$\boldsymbol{C}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & \theta_6 & 0 \end{bmatrix}, \qquad \boldsymbol{D}(\boldsymbol{\theta}) = 0.$$

Output error method

- Fix a model parameterization
- Check the identifiability

Structural identifiability

We must assume that k_1, m, r, and R are known *a priori* to guarantee that the model is structurally identifiable.

Output error method

- Fix a model parameterization
- Check the identifiability
- Compare with a signal norm the outputs of the system and the model, *e.g.*,

$$V_N(\boldsymbol{\theta}) = \sum_{k=0}^{N-1} \|\boldsymbol{y}(t_k) - \boldsymbol{\gamma}(t_k, \boldsymbol{\theta})\|_{\boldsymbol{Q}}^2,$$

with

$$\dot{\boldsymbol{x}}(t, \boldsymbol{\theta}) = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{x}(t, \boldsymbol{\theta}) + \boldsymbol{B}(\boldsymbol{\theta})\boldsymbol{u}(t),$$
$$\boldsymbol{\gamma}(t, \boldsymbol{\theta}) = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{x}(t, \boldsymbol{\theta}).$$

Output error method

- Fix a model parameterization
- Check the identifiability
- Compare with a signal norm the outputs of the system and the model, *e.g.*,

$$V_N(\boldsymbol{\theta}) = \sum_{k=0}^{N-1} \|\boldsymbol{y}(t_k) - \boldsymbol{\gamma}(t_k, \boldsymbol{\theta})\|_{\boldsymbol{Q}}^2 .$$

Remark

Minimizing $V_N(\boldsymbol{\theta})$ can be efficiently performed by using, *e.g.*, the Levenberg-Marquardt or a BFGS like algorithm.

Simulation example

- Simulation procedure:
  - one noise free data set for model learning (PRBS)
  - A double Monte Carlo simulation is run

Simulation example

- Simulation procedure:
  - one noise free data set for model learning (PRBS)
  - A double Monte Carlo simulation is run
    - for the noise, 100 zero mean white Gaussian noises are generated such that the SNR $= 10\ dB$,

Simulation example

- Simulation procedure:
  - one noise free data set for model learning (PRBS)
  - A double Monte Carlo simulation is run
    - for the noise, 100 zero mean white Gaussian noises are generated such that the SNR $= 10\ dB$,
    - for the initialization, 100 initial guesses are generated as follows

$$\theta_i^{init} = \theta_i^{real}(0.5 + \lambda),$$

where λ is a random value drawn from the standard uniform distribution on the open interval $(0, 1)$.

Parameter estimation results

	NOMINAL	BEST	STD	MEAN	MEDIAN
k_1	1.4	1.4	0	1.4	1.4
m	0.2	0.2	0	0.2	0.2
r	0.15	0.15	0	0.15	0.15
R	2	2	0	2	2
b	0.25	0.24399	0.0030617	0.25072	0.25106
J	0.01	0.0091905	0.00034345	0.0099452	0.0099247
k	20	19.9461	0.051731	20.0234	20.0317
k_m	2	1.9554	0.024181	2.006	2.0088

Quick analysis

- Getting such good results require
  - reliable initial guesses,
  - tuning the nonlinear optimization efficiently,

  in order to help the algorithm converge towards the global minimum of V_N.

Some important questions

- Can we suggest alternatives to this output error approach?
- Can we generate reliable initial guesses for nonlinear optimization?

FROM BLACK BOX TO GRAY BOX WITH LINEAR ALGEBRA

Problem formulation

- Let use assume that
  - A reliable fully parameterized continuous[2] time linear time invariant state space model $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is available.
  - The parameterization $(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{B}(\boldsymbol{\theta}), \boldsymbol{C}(\boldsymbol{\theta}))$ is identifiable.
- We know that matrices $\boldsymbol{T}$ or $\boldsymbol{S}$ exist such that

$$\mathfrak{A}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{A}(\boldsymbol{\theta}), \quad \mathfrak{B} = \boldsymbol{T}\boldsymbol{B}(\boldsymbol{\theta}), \quad \mathfrak{C}\boldsymbol{T} = \boldsymbol{C}(\boldsymbol{\theta}),$$
$$\boldsymbol{S}\mathfrak{A} = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{S}, \quad \boldsymbol{S}\mathfrak{B} = \boldsymbol{B}(\boldsymbol{\theta}), \quad \mathfrak{C} = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{S}.$$

Problem to solve

Determine the matrices $\boldsymbol{T}$ or $\boldsymbol{S}$ (and by extension $\boldsymbol{\theta}$) by solving the aforementioned set of equations.

[2] Several transformations exist to convert a DT LTI SS model into a CT LTI SS model efficiently.

Solution

- In [Prot and Mercère, 2020], we have shown that the similarity transformation $\boldsymbol{T}$ or $\boldsymbol{S}$ can be determined by
  - extracting a large system of equations linear in terms of the similarity transformation,
  - solving this system of linear equations by using standard linear algebra tools,
  - completing this two step procedure by a numerical optimization solution (dedicated now to a very small number of unknowns) if necessary,

  when
  - the dependence on $\boldsymbol{\theta}$ is affine,
  - the system to identify is in the model class.

A toy example

- Let us consider the gray box model ($\theta_1 = -1$ and $\theta_2 = -2$ when necessary)

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & \theta_1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} \theta_2 \\ 3 \end{bmatrix} \boldsymbol{u}(t),$$
$$\boldsymbol{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t).$$

- Let us assume that

$$\mathfrak{A} = \begin{bmatrix} 2.1961e{+}01 & -5.9101e{+}01 \\ 7.4341 & -1.9961e{+}01 \end{bmatrix}, \quad \mathfrak{B} = \begin{bmatrix} 3.6753 \\ 1.3522 \end{bmatrix},$$
$$\boldsymbol{T} = \begin{bmatrix} 9.5974e{-}01 & 5.8527e{-}01 \\ 3.4039e{-}01 & 2.2381e{-}01 \end{bmatrix}, \quad \mathfrak{C}^\top = \begin{bmatrix} 1.4360e{+}01 \\ -3.7552e{+}01 \end{bmatrix}.$$

A toy example (cont'd)

- Instead of focusing on the unknowns θ_1 and θ_2, we concentrate on the known parameters, *i.e.*,

$$\boldsymbol{S}\mathfrak{A} = \begin{bmatrix} 1 & \theta_1 \\ 0 & 1 \end{bmatrix} \boldsymbol{S}, \qquad \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{S}\mathfrak{A} = \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{S},$$

$$\boldsymbol{S}\mathfrak{B} = \begin{bmatrix} \theta_2 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \end{bmatrix} \boldsymbol{S}\mathfrak{B} = 3$$

$$\mathfrak{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{S} \qquad \mathfrak{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{S}.$$

- Written differently, we get

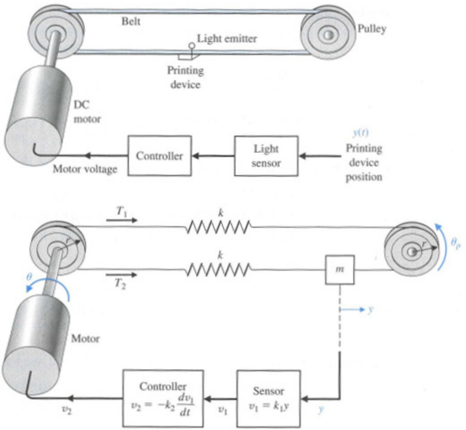
$$\begin{bmatrix} \mathfrak{A}^\top \otimes \boldsymbol{I}_{2\times 2} - \boldsymbol{I}_{2\times 2} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \boldsymbol{I}_{2\times 2} \otimes \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \mathfrak{B}^\top \otimes \boldsymbol{I}_{2\times 2} \end{bmatrix} \text{vec}(\boldsymbol{S}) = \begin{bmatrix} \boldsymbol{0}_{1\times 2} \\ \text{vec}(\mathfrak{C}) \\ 3 \end{bmatrix}.$$

A toy example (cont'd)

- $\boldsymbol{S}$ is obtained by solving this system of linear equations, *i.e.*,

$$\boldsymbol{S} = \begin{bmatrix} 1.4360e{+}01 & -3.7552e{+}01 \\ -2.1840e{+}01 & 6.1580e{+}01 \end{bmatrix} = \boldsymbol{T}^{-1}.$$

Printer belt (cont'd)



Picture from "Modern control systems", 2010, pp. 222–228.

Printer belt (cont'd)

- We consider the model parameterization

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & -1 & \theta_1 \\ \theta_2 & 0 & 0 \\ \theta_3 & 0 & \theta_4 \end{bmatrix}, \qquad \boldsymbol{B}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ 0 \\ \theta_5 \end{bmatrix},$$
$$\boldsymbol{C}(\boldsymbol{\theta}) = \begin{bmatrix} 0 & \theta_6 & 0 \end{bmatrix}.$$

- We know (in red)

$$\boldsymbol{A}(\boldsymbol{\theta}) = \begin{bmatrix} \color{red}{0} & \color{red}{-1} & \color{red}{\theta_1} \\ \theta_2 & \color{red}{0} & \color{red}{0} \\ \theta_3 & \color{red}{0} & \theta_4 \end{bmatrix}, \qquad \boldsymbol{B}(\boldsymbol{\theta}) = \begin{bmatrix} \color{red}{0} \\ \color{red}{0} \\ \theta_5 \end{bmatrix},$$
$$\boldsymbol{C}(\boldsymbol{\theta}) = \begin{bmatrix} \color{red}{0} & \color{red}{\theta_6} & \color{red}{0} \end{bmatrix}.$$

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Printer belt (cont'd)

- Thanks to the insight on four rows in $(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{B}(\boldsymbol{\theta}), \boldsymbol{C}(\boldsymbol{\theta}))$ as well as the availability of $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, we try to estimate $\boldsymbol{S}$ via the set of linear equations

$$\begin{bmatrix}1 & 0 & 0\end{bmatrix} \boldsymbol{S}\mathfrak{A} = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{S},$$
$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix} \boldsymbol{S}\mathfrak{B} = \begin{bmatrix}0\\0\end{bmatrix},$$
$$\mathfrak{C} = \boldsymbol{C}(\boldsymbol{\theta})\boldsymbol{S}.$$

Printer belt (cont'd)

- By using the vectorization tool

$$\underbrace{\begin{bmatrix} \mathfrak{A}^\top \otimes \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} - \boldsymbol{I}_{3\times 3} \otimes \begin{bmatrix} 0 & -1 & \theta_1 \end{bmatrix} \\ \mathfrak{B}^\top \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ \boldsymbol{I}_{3\times 3} \otimes \boldsymbol{C}(\boldsymbol{\theta}) \end{bmatrix}}_{\boldsymbol{N}} \text{vec}(\boldsymbol{S}) = \underbrace{\begin{bmatrix} \mathbf{0}_{3\times 1} \\ \mathbf{0}_{2\times 1} \\ \text{vec}(\mathfrak{C}) \end{bmatrix}}_{\boldsymbol{\alpha}}$$

- With $\boldsymbol{N} \in \mathbb{R}^{8\times 9}$, we have 8 equations for 9 unknown parameters in $\boldsymbol{S}$.
- Because $\ker(\boldsymbol{N}) \neq \{\mathbf{0}\}$, the solution of this set of equations is not unique!
- We can however compute a (non unique) solution of it.

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Printer belt (cont'd)

- We can compute $\ker(\boldsymbol{N})$ and show that $\ker(\boldsymbol{N}) \in \mathbb{R}^{9\times 2}$.
- Furthermore,

$$\mathsf{vec}\left(\hat{\boldsymbol{S}}\right) = \boldsymbol{N}^{\dagger}\left(\boldsymbol{\alpha} + \boldsymbol{\alpha}_{\mathsf{null}}\right),$$

  where $\boldsymbol{\alpha}_{\mathsf{null}}$ stands for any vector belonging to $\ker(\boldsymbol{N})$.
- In order to bypass the ambiguities relevant to $\ker(\boldsymbol{N})$, we can use the prior on $\boldsymbol{A}(\boldsymbol{\theta})$ unused until now.

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Printer belt (cont'd)

- We introduce the cost function

$$\arg \min_{\boldsymbol{z} \in \mathbb{R}^{2\times 1}} \left\| \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\mathcal{A}}(\boldsymbol{z}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\|_2^2$$

$$\text{s.t. } \begin{cases} \boldsymbol{\mathcal{A}}(\boldsymbol{z}) = \boldsymbol{\mathcal{S}}(\boldsymbol{z}) \mathfrak{A} \boldsymbol{\mathcal{S}}^{-1}(\boldsymbol{z}) \\ \boldsymbol{\mathcal{S}}(\boldsymbol{z}) = \text{reshape}\big(\boldsymbol{N}^\dagger \boldsymbol{\alpha} + \ker(\boldsymbol{N}) \boldsymbol{z}, n_x, n_x\big) \end{cases}.$$

- Thanks to linear algebra, we have to determine 2 unknown parameters instead of 4.
- This cost function can be minimized with a L-BFGS algorithm without strong prior.

Parameter estimation results

- We use (again) the 100 noisy data sets to estimate 100 black box models by using N4SID + d2c.
- We select the best black box model $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ giving the best fit on a validation data set.
- The initialization of the BFGS algorithm is performed with 100 different initial vectors $\boldsymbol{z}$ randomly generated from a standard zero-mean normal distribution.
- The best $\boldsymbol{z}$ is chosen by selecting the $\hat{S}$ having the best condition number.

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Parameter estimation results (cont'd)

- We can compute $(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{B}(\boldsymbol{\theta}), \boldsymbol{C}(\boldsymbol{\theta}))$ from $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and $\hat{\boldsymbol{S}}$.

	NOMINAL	ESTIMATED
k_1	1.4	1.4
m	0.2	0.2
r	0.15	0.15
R	2	2
b	0.25	0.24401
J	0.01	0.0091916
k	20	19.9462
k_m	2	1.9555

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Explanations with the printer belt (cont'd)

- Let us assume that k is known as well.
- Because θ_2 is now known *a priori*, we know the first two rows of $\boldsymbol{A}(\boldsymbol{\theta})$.
- By combining this prior with the knowledge of $\boldsymbol{C}(\boldsymbol{\theta})$, we get

$$\underbrace{\begin{bmatrix} \mathfrak{A}^\top \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \boldsymbol{I}_{3\times3} \otimes \begin{bmatrix} 0 & -1 & \theta_1 \\ \theta_2 & 0 & 0 \end{bmatrix} \\ \boldsymbol{I}_{3\times3} \otimes \boldsymbol{C}(\boldsymbol{\theta}) \end{bmatrix}}_{\boldsymbol{N}}$$

$$\times \text{vec}(\boldsymbol{S}) = \underbrace{\begin{bmatrix} \mathbf{0}_{6\times1} \\ \text{vec}(\mathfrak{C}) \end{bmatrix}}_{\boldsymbol{\alpha}}.$$

- Now $\boldsymbol{N} \in \boldsymbol{R}^{9\times9}$ and $\ker(\boldsymbol{N}) = \{\mathbf{0}\}$!

Parameter estimation results (cont'd)

- We can compute $\hat{\boldsymbol{S}}$ without any nonlinear optimization!!!
- Again, $(\boldsymbol{A}(\boldsymbol{\theta}), \boldsymbol{B}(\boldsymbol{\theta}), \boldsymbol{C}(\boldsymbol{\theta}))$ from $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and $\hat{\boldsymbol{S}}$.

	NOMINAL	ESTIMATED
k_1	1.4	1.4
m	0.2	0.2
r	0.15	0.15
R	2	2
b	0.25	0.25483
J	0.01	0.0099987
k	20	20
k_m	2	2.0367

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DISCUSSION

References I

Mussot, V., Mercère, G., Dairay, T., Arvis, V., and Vayssettes, J. (2021).
Noise covariance matrix estimation with subspace model identification for Kalman filtering.
International Journal of Adaptive Control and Signal Processing, 35:591–611.

Prot, O. and Mercère, G. (2020).
Combining linear algebra and numerical optimization for gray-box affine state-space model identification.
IEEE Transactions on Automatic Control, 65:3272–3285.