Direct Numerical Computation of Polynomial Multiplication Maps

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Macaulay framework

- Consider a system of polynomials

$$f(x, y) = 1 + x + y = 0$$
$$g(x, y) = 2x^2 - 2y - 6 = 0$$

- The null space of a 'Macaulay matrix' $\boldsymbol{M}_2$ is then spanned by generalized Vandermonde vectors associated with the common roots $(1, -2), (-2, 1)$ [2]

$$\begin{array}{rc} & \begin{array}{cccccc} 1 & x & y & x^2 & xy & y^2 \end{array} \\ \begin{array}{r} f(x,y) \\ xf(x,y) \\ yf(x,y) \\ g(x,y) \end{array} & \underbrace{\begin{bmatrix} 1 & 1 & 1 & & & \\ & 1 & & 1 & 1 & \\ & & 1 & & 1 & 1 \\ -6 & & -2 & 2 & & \end{bmatrix}}_{\boldsymbol{M}_2} \end{array} \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ -2 & 1 \\ 1 & 4 \\ -2 & -2 \\ 4 & 1 \end{bmatrix} \begin{array}{l} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{array} = \mathbf{0}$$

Macaulay framework: Multiplication maps

- From the null space, the multiplication maps $\boldsymbol{D}_x, \boldsymbol{D}_y$ can be computed:

$$\boldsymbol{N} = \begin{array}{c} \left[\begin{array}{rr} 1 & 1 \\ 1 & -2 \\ -2 & 1 \\ 1 & 4 \\ -2 & -2 \\ 4 & 1 \end{array}\right] \end{array} \begin{array}{l} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{array} \implies \begin{array}{c} \begin{array}{l} 1 \\ x \\ y \end{array} \left[\begin{array}{rr} 1 & 1 \\ 1 & -2 \\ -2 & 1 \end{array}\right] \underbrace{\left[\begin{array}{rr} 1 & 0 \\ 0 & -2 \end{array}\right]}_{\boldsymbol{D}_x} = \left[\begin{array}{rr} 1 & -2 \\ 1 & 4 \\ -2 & -2 \end{array}\right] \begin{array}{l} x \\ x^2 \\ xy \end{array} \\ \\ \begin{array}{l} 1 \\ x \\ y \end{array} \left[\begin{array}{rr} 1 & 1 \\ 1 & -2 \\ -2 & 1 \end{array}\right] \underbrace{\left[\begin{array}{rr} -2 & 0 \\ 0 & 1 \end{array}\right]}_{\boldsymbol{D}_y} = \left[\begin{array}{rr} -2 & 1 \\ -2 & -2 \\ 4 & 1 \end{array}\right] \begin{array}{l} y \\ xy \\ y^2 \end{array} \end{array}$$

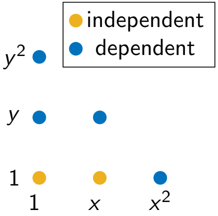
- In practice: **eigenvalue problem** to obtain the roots from a numerical null space basis $\boldsymbol{Z} = \boldsymbol{N}\boldsymbol{K}$:

$$\boldsymbol{S}_1 \boldsymbol{N} \underbrace{\boldsymbol{K}\boldsymbol{D}_{x_i}\boldsymbol{K}^{-1}}_{\boldsymbol{A}_{x_i}} = \boldsymbol{S}_{x_i}\boldsymbol{N}.$$

Macaulay framework: general case

Macaulay matrix $\boldsymbol{M}_d \Longrightarrow$ Null space basis $\boldsymbol{N}_d \Longrightarrow$ Multiplication maps $\boldsymbol{A}_{x_i} \Longrightarrow$ Eigenvalues

- Details:
  - Gap: linearly dependent degree block (d large enough)
  - Rank checks and system solving using SVD
  - Ways to avoid explicit construction of $\boldsymbol{M}_d$



- Drawbacks:
  - Large sizes of matrices involved: # monomials $= \binom{n+d-1}{n-1}$
  - Computation of 'unnecessary objects' $\boldsymbol{M}_d, \boldsymbol{N}_d$
  - Zero dimensional solution set required

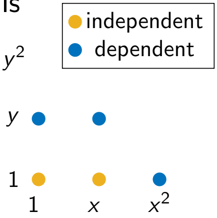
Symbolic methods

- **Not all monomials are required** to set up multiplication maps [1]
- We can substitute monomials: e.g. reconsider $f(x,y), g(x,y)$

$$\begin{aligned} y &= -1 - x \\ x^2 &= \phantom{-1 - x +{}} y + 3 \end{aligned} \implies \begin{aligned} y &= -1 - x \\ x^2 &= -1 - x + 3 \\ xy &= -x - x^2 \end{aligned} \implies \begin{aligned} y &= -1 - x \\ x^2 &= 2 - x \\ xy &= -2 \end{aligned}$$

- In vector representation: $\mathcal{B} = \{1, x\}$ as basis polynomials

$$N = \begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 2 & -1 \\ -2 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ 1 \\ x \\ y \\ x^2 \\ xy \end{array}$$

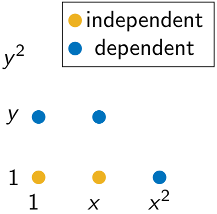


Symbolic methods

- Since $y = -1 - x$, y^2 is a linear combination of $y \cdot \mathcal{B} = \{y, xy\}$

$$\boldsymbol{N} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} 1 & x \end{array}$$

$$\boldsymbol{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 2 & -1 \\ -2 & 0 \end{bmatrix} \begin{array}{l} 1 \\ x \\ y \\ x^2 \\ xy \end{array}$$



- Multiplication maps carry out this substitution:

$$\begin{array}{c|cc} & 1 & x \\ \hline 1 & 1 & 0 \\ x & 0 & 1 \end{array} \boldsymbol{A}_x = \left.\begin{array}{cc|c} 1 & x & \\ \hline 0 & 1 & x \\ 2 & -1 & x^2 \end{array}\right\} x \cdot \mathcal{B}$$

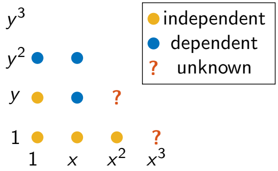
$$\begin{array}{c|cc} & 1 & x \\ \hline 1 & 1 & 0 \\ x & 0 & 1 \end{array} \boldsymbol{A}_y = \left.\begin{array}{cc|c} 1 & x & \\ \hline -1 & -1 & y \\ -2 & 0 & xy \end{array}\right\} y \cdot \mathcal{B}$$

Symbolic methods: some details

- Example:
  - Actively try to discover only border monomials [1]
  - **Reduction** of multiples of border elements onto basis elements using row echelon form $\Longrightarrow$ **fewer monomials**

$$
\begin{array}{c} \\ f(\mathbf{x}) \\ g(\mathbf{x}) \\ xf(\mathbf{x}) \\ yf(\mathbf{x}) \\ xg(\mathbf{x}) \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & x & y & x^2 & xy & y^2 & \mathbf{x^3} & \mathbf{x^2y} & xy^2 \end{array} \\
\begin{bmatrix}
5 & 4 & 3 & 2 & 1 & & & & \\
5 & 4 & 3 & 2 & & 1 & & & \\
 & 5 & & 4 & 3 & & 2 & 1 & \\
 & & 5 & & 4 & 3 & & 2 & 1 \\
 & 5 & & 4 & 3 & & 2 & & 1
\end{bmatrix}
\end{array}
\xrightarrow{rref}
\begin{array}{c} \\ f(\mathbf{x}) \\ g(\mathbf{x}) \\ h(\mathbf{x}) \\ b(\mathbf{x}) \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y \end{array} \\
\begin{bmatrix}
5 & 4 & 3 & 2 & 1 & & & \\
5 & 4 & 3 & 2 & & 1 & & \\
\frac{-10}{6} & \frac{7}{6} & \frac{-11}{6} & \frac{8}{6} & & & 1 & \\
\frac{-35}{3} & \frac{-28}{3} & \frac{-16}{3} & \frac{-14}{3} & & & & 1
\end{bmatrix}
\end{array}
$$

- Drawbacks:
  - Limited adaptability in choice of polynomial basis [4]
  - Row-echelon form to check linear (in)dependence



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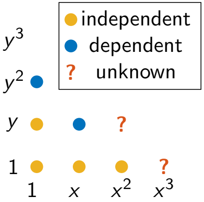
Rethinking the symbolic-style algorithm

- **Goal:** numerical-style approach with monomial substitution
  1. Rewrite the symbolic-style algorithm in the null space
  2. Add insights from the numerical methods



$$
N_2 = \begin{array}{c} \\ 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{array}
\begin{array}{c} \overbrace{\begin{array}{cccc} 1 & x & y & x^2 \end{array}}^{\mathcal{B}} \\
\left[\begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -5 & -4 & -3 & -2 \\ -5 & -4 & -3 & -2 \end{array}\right] \end{array}
$$

- **Example:** Adding dimensions $\rightarrow$ Adding monomials $\{x^3, x^2y\}$ to $\mathcal{B}$

$$
\mathbf{0} = \begin{array}{c} \\ f(\boldsymbol{x}) \\ g(\boldsymbol{x}) \end{array}
\begin{array}{c} \begin{array}{cccccccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y \end{array} \\
\left[\begin{array}{cccccccc} 5 & 4 & 3 & 2 & 1 & & & \\ 5 & 4 & 3 & 2 & & 1 & & \end{array}\right] \end{array}
\begin{array}{c} \begin{array}{ccc} & \mathbf{x^3} & \mathbf{x^2y} \end{array} \\
\left[\begin{array}{ccc} \boldsymbol{N}_2 & & \\ & 1 & \\ & & 1 \end{array}\right] \end{array}
\begin{array}{c} \\ \\ \mathbf{x^3} \\ \mathbf{x^2y} \end{array}
$$

Partial multiplication maps

- The following maps then express multiplication: $\mathcal{B} \mapsto x \cdot \mathcal{B}$ and $\mathcal{B} \mapsto y \cdot \mathcal{B}$
  - Images of $\mathcal{B}_1 = \{1, x, y\}$ are known: **perform substitution** where possible
  - $\mathcal{B}_2 = \{x^2\}$ maps to new dimensions $\mathcal{B}_{new} = \{x^3, x^2y\}$
  - **Numerical insight:** column basis can be unknown $\rightarrow$ linear system

$$
\begin{array}{rc}
 & \begin{array}{cccc} 1 & x & y & x^2 \end{array} \\
\begin{array}{r} \mathcal{B}_1 \left\{ \begin{array}{c} 1 \\ x \\ y \end{array} \right. \\ \mathcal{B}_2 \left\{ \begin{array}{c} x^2 \end{array} \right. \end{array} &
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\end{array}
\boldsymbol{A}_x =
\begin{array}{cl}
\begin{array}{cccc|cc} 1 & x & y & x^2 & \overbrace{\mathbf{x^3} \quad \mathbf{x^2y}}^{\mathcal{B}_{new}} \end{array} & \\
\left[\begin{array}{cccc|cc} & & 1 & & & \\ & & & 1 & & \\ -5 & -4 & -3 & -2 & & \\ \hline & & & & 1 & \end{array}\right] &
\begin{array}{l} \left. \begin{array}{c} y \\ xy \\ y^2 \end{array} \right\} x \cdot \mathcal{B}_1 \\ \left. \mathbf{x^3} \right\} x \cdot \mathcal{B}_2 \end{array}
\end{array}
$$

$$
\begin{array}{r} \mathcal{B}_1 \left\{ \begin{array}{c} 1 \\ x \\ y \end{array} \right. \\ \mathcal{B}_2 \left\{ \begin{array}{c} x^2 \end{array} \right. \end{array}
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}
\boldsymbol{A}_y =
\left[\begin{array}{cccc|cc} & & 1 & & & \\ -5 & -4 & -3 & -2 & & \\ -5 & -4 & -3 & -2 & & \\ \hline & & & & & 1 \end{array}\right]
\begin{array}{l} \left. \begin{array}{c} y \\ xy \\ y^2 \end{array} \right\} y \cdot \mathcal{B}_1 \\ \left. \mathbf{x^2y} \right\} y \cdot \mathcal{B}_2 \end{array}
$$

Deriving new equations

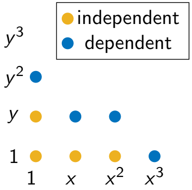
$$
\begin{array}{rcl}
xy^2 = (y^2)\cdot x = & \begin{array}{cccc} 1 & x & y & x^2 \\ \end{array} & \\
& \begin{bmatrix} -5 & -4 & -3 & -2 \end{bmatrix} \boldsymbol{A}_x & = \begin{array}{c} \begin{array}{cccccc} 1 & x & y & x^2 & x^3 & x^2y \end{array} \\ \begin{bmatrix} 15 & 7 & 9 & 2 & -2 & 0 \end{bmatrix} \end{array} \\
= (xy)\cdot y = & \begin{bmatrix} -5 & -4 & -3 & -2 \end{bmatrix} \boldsymbol{A}_y & = \begin{bmatrix} 35 & 28 & 16 & 14 & 0 & -2 \end{bmatrix}
\end{array}
$$

- New relations: $\mathcal{B}_{new} = \boldsymbol{R} \cdot \mathcal{B}$ (S-Polynomials [1])

$$
\boldsymbol{C} = \boldsymbol{A}_x(:,1{:}4)\boldsymbol{A}_y - \boldsymbol{A}_y(:,1{:}4)\boldsymbol{A}_x = \begin{array}{c} \begin{array}{cccccc} 1 & x & y & x^2 & x^3 & x^2y \end{array} \\ \begin{bmatrix} -15 & -7 & -9 & -2 & 2 & 1 \\ 20 & 21 & 7 & 12 & 2 & -2 \end{bmatrix} \end{array} \begin{array}{l} x^2\cdot y - xy\cdot x = 0 \\ xy\cdot y - y^2\cdot x = 0 \end{array}
$$

- Update underlying vector representations (null space $\boldsymbol{N}$)

$$
\begin{array}{cc} & \begin{array}{cc} \mathcal{B} & \mathcal{B}_{new} \end{array} \\ \begin{array}{c} \mathcal{B} \\ \mathcal{B}_{new} \end{array} & \begin{bmatrix} \boldsymbol{I} & \\ & \boldsymbol{I} \end{bmatrix} \end{array}
\xrightarrow{\text{Null } \boldsymbol{C}}
\begin{array}{c} \mathcal{B} \\ \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{R} \end{bmatrix} \end{array} \begin{array}{c} \mathcal{B} \\ \mathcal{B}_{new} \end{array}
\xrightarrow{\text{numerical basis}}
\begin{array}{c} \mathcal{P} \\ \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} \end{array} \begin{array}{c} \mathcal{B} \\ \mathcal{B}_{new} \end{array}
$$



Updating maps numerically

- Construct new maps $\boldsymbol{A}_x^{\mathcal{P}}, \boldsymbol{A}_y^{\mathcal{P}}$ until they commute [3]: **linear system**

$$\boldsymbol{N}_{new} = \begin{array}{c} \\ \mathcal{B} \\ x \cdot \mathcal{B} \\ y \cdot \mathcal{B} \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{B} & \mathcal{B}_{new} \end{array} \\ \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ & \boldsymbol{A}_x \\ & \boldsymbol{A}_y \end{bmatrix} \end{array} \begin{array}{c} \mathcal{P} \\ \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \end{bmatrix} \end{array} = \begin{array}{c} \mathcal{P} \\ \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{A}_x \boldsymbol{T} \\ \boldsymbol{A}_y \boldsymbol{T} \end{bmatrix} \end{array} \begin{array}{c} \\ \mathcal{B} \\ x \cdot \mathcal{B} \\ y \cdot \mathcal{B} \end{array}$$

- Problem: $\boldsymbol{N}_{new}(x \cdot \mathcal{B}) = \boldsymbol{A}_x \boldsymbol{T}$ is known, $\boldsymbol{N}_{new}(x \cdot \mathcal{B}_{new})$ is not!
  - Row $\boldsymbol{T}_1$: subspace polynomials with known images
  - Null $\boldsymbol{T}_1$ ($\boldsymbol{K}$ = basis): subspace for polynomials without known images

$$\begin{array}{c} \\ \mathcal{B} \\ \mathcal{K} \end{array} \begin{array}{c} \mathcal{P} \\ \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{K}^\mathrm{T} \end{bmatrix} \end{array} \boldsymbol{A}_x^{\mathcal{P}} = \begin{array}{c} \begin{array}{ccc} \mathcal{P} & x \cdot \mathcal{K} & y \cdot \mathcal{K} \end{array} \\ \begin{bmatrix} \boldsymbol{A}_x \boldsymbol{T} & & \\ & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \end{array} \begin{array}{c} \\ x \cdot \mathcal{B} \\ x \cdot \mathcal{K} \end{array}$$

Numerical experiments

- We compare the proposed algorithm with an SVD-based implementation (1) to MacaulayLab (2) [5]
  - n Polynomials of degree d in n variables with random coefficients
  - Seems to converge if rank decisions are correct

n	d	runtime (1) (s)	runtime (2) (s)	avg residual (1)	avg residual (2)
2	3	0.0014	0.0030	4.15×10^{-13}	4.16×10^{-14}
2	10	0.020	0.054	5.37×10^{-13}	2.42×10^{-14}
2	13	0.06	0.13	4.33×10^{-6}	2.06×10^{-10}
3	3	0.0053	0.017	3.96×10^{-13}	4.40×10^{-14}
3	5	0.042	0.16	3.51×10^{-11}	1.24×10^{-12}
3	8	1.63	2.42	4.75×10^{-5}	5.55×10^{-8}
4	3	0.035	0.21	1.01×10^{-11}	4.29×10^{-13}
4	5	4.66	10.34	2.07×10^{-6}	6.83×10^{-11}
5	3	0.70	4.55	3.91×10^{-13}	1.05×10^{-14}
6	3	23.58	498.72	3.96×10^{-11}	8.03×10^{-13}

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Conclusion and future work

- **Conclusion**:
  - Different view on symbolic algorithms
    - Determining independent monomials with row echelon form $\rightarrow$ Linear system for multiplication maps in intermediate steps
    - S-polynomials $\rightarrow$ Non-commutativity of multiplication maps
    - Adding a select set of additional dimensions
  - Adaptable to SVD-based, basis-agnostic implementation

- **Future work**:
  - More stable computation to obtain new equations through $\boldsymbol{C}$?
  - Stability of obtaining multiplication maps in each iteration?
  - More compression by fully exploiting polynomial structure?

References

[1] D. A. Cox, J. B. Little, and D. O'Shea. *Ideals, Varieties and Algorithms*. Third Edition. Springer-Verlag, 2007.

[2] Philippe Dreesen. "Back to the Roots: Polynomial System Solving Using Linear Algebra". PhD thesis, KU Leuven, 2013.

[3] B Mourrain. "A new criterion for normal form algorithms". In: *Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. AAECC 1999* (1999), pp. 430–443.

[4] Simon Telen, Bernard Mourrain, and Marc Van Barel. "Solving Polynomial Systems via Truncated Normal Forms". In: *SIAM Journal on Matrix Analysis and Applications* 39 (3 2018), pp. 1421–1447.

[5] Christof Vermeersch and Bart De Moor. "Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems". In: *Linear Algebra and its Applications* 654 (2022), pp. 177–209.