Least Squares Projection Onto the Behavior for SISO LTI Models

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Fonds Wetenschappelijk Onderzoek
Vlaanderen
Opening new horizons

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SISO LTI recurrence relation

- LTI dynamics of $(\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{y}}) \in \mathbb{R}^{2\times N}$:

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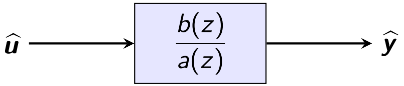

SISO LTI recurrence relation

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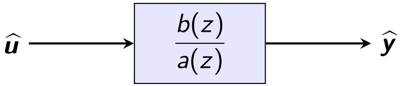
- Transfer function:

$$\widehat{H}(z) = \frac{b(z)}{a(z)} = \frac{b_0 z^n + \cdots + b_{n-1} z + b_n}{a_0 z^n + \cdots + a_{n-1} z + a_n}.$$



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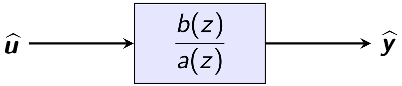
- Model params: $\boldsymbol{a} \in \mathbb{R}^{n+1}$, $\boldsymbol{b} \in \mathbb{R}^{n+1}$
  - Normalization ($a_0 = 1$)



Behavior in the null space

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where $\widehat{\boldsymbol{w}} \in \mathbb{R}^{2N}$ is the *model-compliant* data trajectory:

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Basis matrix for the null space

$$\begin{bmatrix} a_n & -b_n & a_{n-1} & -b_{n-1} & \dots & \dots & a_0 & -b_0 & & & & & \\ & & a_n & -b_n & \dots & \dots & a_1 & -b_1 & a_0 & -b_0 & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & a_n & -b_n & \dots & \dots & a_1 & -b_1 & a_0 & -b_0 \end{bmatrix} \quad \widehat{\boldsymbol{T}} \in \mathbb{R}^{2N \times (N+n)} \quad = \mathbf{0}.$$

$$\iff \widetilde{\boldsymbol{T}}^{\top} \widehat{\boldsymbol{T}} = \mathbf{0}$$

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$$
\underbrace{\begin{bmatrix}
a_n & -b_n & a_{n-1} & -b_{n-1} & \cdots & \cdots & a_0 & -b_0 & & & & \\
 & & a_n & -b_n & \cdots & \cdots & a_1 & -b_1 & a_0 & -b_0 & & \\
 & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & a_n & -b_n & \cdots & \cdots & a_1 & -b_1 & a_0 & -b_0
\end{bmatrix}}_{\widetilde{\boldsymbol{T}}^\top \in \mathbb{R}^{(N-n)\times 2N}}
\underbrace{\left[\begin{array}{ccc|cccc|ccc}
b_n & \cdots & b_1 & b_0 & & & & & & \\
a_n & \cdots & a_1 & a_0 & & & & & & \\
 & \ddots & \vdots & b_1 & b_0 & & & & & \\
 & \ddots & \vdots & a_1 & a_0 & \ddots & & & & \\
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- Model *behavior*: $\widehat{\boldsymbol{w}} \in \text{null}\left(\widetilde{\boldsymbol{T}}^{\top}\right) = \text{range}\left(\widehat{\boldsymbol{T}}\right)$
- Orthogonal decomposition of ambient space: $\mathbb{R}^{2N} = \underbrace{\text{range}\left(\widehat{\boldsymbol{T}}\right)}_{\text{behavior}} \oplus \text{range}\left(\widetilde{\boldsymbol{T}}\right)$.

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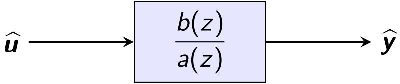
Observed data is not model-compliant

For a given model $(\boldsymbol{a}, \boldsymbol{b})$:

- In practice: e.g., measurement inaccuracies, missing data, and model mismatch



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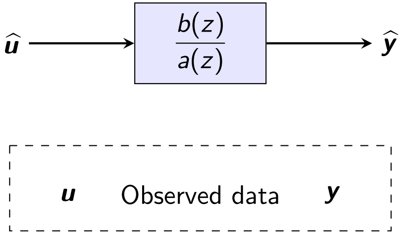

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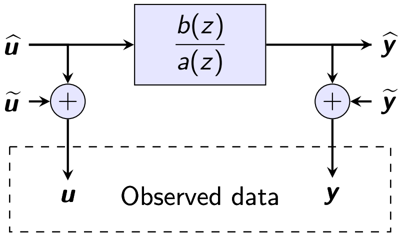

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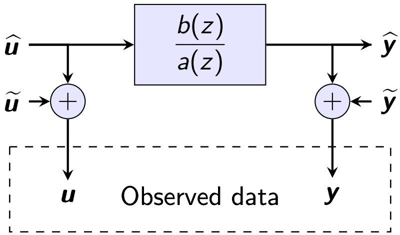

$$\begin{aligned} \min_{\widehat{\boldsymbol{u}}, \widehat{\boldsymbol{y}}} \quad & J = \left\| \begin{bmatrix} \boldsymbol{y} - \widehat{\boldsymbol{y}} \\ \boldsymbol{u} - \widehat{\boldsymbol{u}} \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} \widetilde{\boldsymbol{y}} \\ \widetilde{\boldsymbol{u}} \end{bmatrix} \right\|_2^2, \\ \text{s.t.} \quad & \widetilde{\boldsymbol{T}}^{\mathsf{T}} \widehat{\boldsymbol{w}} = \boldsymbol{0} \text{ and } a_0 = 1. \end{aligned}$$

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Orthogonality between model-compliant data and misfits

With $\boldsymbol{l} = [l_0, \ldots, l_{N-n-1}]^\mathsf{T} \in \mathbb{R}^{N-n}$ Lagrange multipliers, define

$$\mathcal{L}(\widehat{\boldsymbol{w}}, \boldsymbol{l}) = \sum_{k=0}^{N-1} \|\boldsymbol{w}_k - \widehat{\boldsymbol{w}}_k\|_2^2 + \sum_{k=n}^{N-1} l_{k-n}\left(\sum_{i=0}^{n} a_i \widehat{y}_{k-i} - \sum_{i=0}^{n} b_i \widehat{u}_{k-i}\right).$$

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First-order necessary conditions for optimality:

- $\partial \mathcal{L}(\ldots)/l_k = 0, \ \forall k \in \{0, \ldots, N-1\} \Rightarrow \widehat{\boldsymbol{w}} \in \mathrm{null}\left(\widetilde{\boldsymbol{T}}^\mathsf{T} \right)$

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- $\partial \mathcal{L}(\ldots)/\widehat{\boldsymbol{w}}_k = 0,\ \forall k \in \{0, \ldots, N-1\} \Rightarrow \widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{T}} \boldsymbol{l} \Rightarrow \widetilde{\boldsymbol{w}} \in \text{range}\left(\widetilde{\boldsymbol{T}}\right)$,

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$$\longrightarrow \mathbb{R}^{2N} = \underbrace{\operatorname{range}\left(\widehat{\boldsymbol{T}}\right)}_{\text{behavior}} \oplus \underbrace{\operatorname{range}\left(\widetilde{\boldsymbol{T}}\right)}_{\text{misfits}}.$$

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Optimal solution via orthogonal projection(s)

- The minimal norm misfits $\widetilde{\boldsymbol{w}}$: orth. projection of $\boldsymbol{w}$ onto range$\left(\widetilde{\boldsymbol{T}}\right)$,

$$\widetilde{\boldsymbol{w}} = (\widetilde{\boldsymbol{T}}^{\top})^{\dagger}\widetilde{\boldsymbol{T}}^{\top}\boldsymbol{w} = \widetilde{\boldsymbol{T}}(\widetilde{\boldsymbol{T}}^{\top}\widetilde{\boldsymbol{T}})^{-1}\widetilde{\boldsymbol{T}}^{\top}\boldsymbol{w},$$

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- or equivalently, the optimal model-compliant data $\widehat{\boldsymbol{w}}$: orth. projection of $\boldsymbol{w}$ onto $\text{range}\left(\widehat{\boldsymbol{T}}\right)$,

$$\begin{aligned}\widehat{\boldsymbol{w}} &= \widehat{\boldsymbol{T}}(\widehat{\boldsymbol{T}}^{\top}\widehat{\boldsymbol{T}})^{-1}\widehat{\boldsymbol{T}}^{\top}\boldsymbol{w},\\ &= (\boldsymbol{I} - (\widetilde{\boldsymbol{T}}^{\top})^{\dagger}\widetilde{\boldsymbol{T}}^{\top})\boldsymbol{w}.\end{aligned}$$

Optimal solution via orthogonal projection(s)

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- Generalizes results earlier results on autonomous models to the SISO case:

B. De Moor (2020). "Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem". In: *Communications in Information and Systems* 20.2, pp. 163–207

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Misfit model

The optimal misfits $\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{y}}$, are heavily structured:

$$\widehat{\boldsymbol{T}}^\top \widetilde{\boldsymbol{T}} = \mathbf{0} \iff \widehat{\boldsymbol{T}}^\top \widetilde{\boldsymbol{T}} \boldsymbol{I} = \mathbf{0} \iff \widehat{\boldsymbol{T}}^\top \widetilde{\boldsymbol{w}} = \mathbf{0}$$

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- the $(n+1)$th equation up until the Nth equation in $\widehat{\boldsymbol{T}}^\top \widetilde{\boldsymbol{w}} = \boldsymbol{0}$ show that

$$\sum_{i=0}^{n} b_{n-i} \widetilde{y}_{k-i} + \sum_{i=0}^{n} a_{n-i} \widetilde{u}_{k-i} = 0, \ \forall k \in \{n, \ldots, N-1\}.$$

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$$\sum_{i=0}^{n} b_{n-i} \widetilde{y}_{k-i} + \sum_{i=0}^{n} a_{n-i} \widetilde{u}_{k-i} = 0 \Rightarrow \widetilde{H}(z) = -\frac{a_r(z)}{b_r(z)} = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_n z^n + \cdots + b_1 z + b_0}.$$

with the 'reversed-coefficients'-polynomials: $a_r(z) = z^n a(z^{-1})$ and $b_r(z) = z^n b(z^{-1})$.

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- alternatively, express $\widehat{\boldsymbol{w}} \perp \widetilde{\boldsymbol{w}}$ in the z-domain,

$$\langle \frac{b(z)}{a(z)} \widehat{U}(z), \widetilde{Y}(z) \rangle + \langle \widehat{U}(z), \widetilde{U}(z) \rangle = \frac{1}{2\pi i} \oint_{|z|=1} \left[\widetilde{Y}(z) \frac{b(z^{-1})}{a(z^{-1})} + \widetilde{U}(z) \right] \widehat{U}(z^{-1}) \mathrm{d}z = 0$$

$$\iff \widetilde{Y}(z) = -\frac{a(z^{-1})}{b(z^{-1})} \widetilde{U}(z) = -\frac{a_r(z)}{b_r(z)} \widetilde{U}(z) = \widetilde{H}(z) \widetilde{U}(z).$$

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Isometric state-space models

- Behavioral state-space recurrence relations,

$$\begin{aligned} \widehat{\boldsymbol{x}}_{k+1} &= \boldsymbol{A}\widehat{\boldsymbol{x}}_k + \boldsymbol{B}\widehat{v}_k \\ \widehat{\boldsymbol{w}}_k &= \boldsymbol{C}\widehat{\boldsymbol{x}}_k + \boldsymbol{D}\widehat{v}_k, \end{aligned} \quad \text{for } k = 0, \ldots, N-1,$$

with $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{B} \in \mathbb{R}^{n}$, $\boldsymbol{C} \in \mathbb{R}^{2 \times n}$, $\boldsymbol{D} \in \mathbb{R}^{2}$.

Isometric state-space models

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$$\begin{bmatrix} \widehat{\boldsymbol{x}}_{k+1} \\ \widehat{\boldsymbol{w}}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \widehat{\boldsymbol{x}}_k \\ \widehat{v}_k \end{bmatrix} \quad \text{for } k = 0, \ldots, N-1, \text{ with } \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} = \boldsymbol{I}_{n+1},$$

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$$\begin{bmatrix} \widetilde{\boldsymbol{x}}_{\boldsymbol{k+1}} \\ \widetilde{\boldsymbol{w}}_{\boldsymbol{k}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \widetilde{\boldsymbol{B}} \\ \boldsymbol{C} & \widetilde{\boldsymbol{D}} \end{bmatrix} \begin{bmatrix} \widetilde{\boldsymbol{x}}_{\boldsymbol{k}} \\ \widetilde{v}_k \end{bmatrix} \quad \text{for } k = 0, \ldots, N-1, \text{ with } \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \widetilde{\boldsymbol{B}} \\ \boldsymbol{C} & \boldsymbol{D} & \widetilde{\boldsymbol{D}} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \widetilde{\boldsymbol{B}} \\ \boldsymbol{C} & \boldsymbol{D} & \widetilde{\boldsymbol{D}} \end{bmatrix}^{\mathsf{T}} = \boldsymbol{I}_{n+2}.$$

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- Isometry $\longrightarrow$ orthogonality

Ambient space

$$\widehat{\boldsymbol{H}} = \begin{bmatrix} \boldsymbol{D} & 0 & \dots & \dots & \dots & 0 \\ \boldsymbol{CB} & \boldsymbol{D} & 0 & \dots & \dots & 0 \\ \boldsymbol{CAB} & \boldsymbol{CB} & \boldsymbol{D} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & 0 \\ \boldsymbol{CA}^{N-2}\boldsymbol{B} & \boldsymbol{CA}^{N-3}\boldsymbol{B} & \dots & \dots & \boldsymbol{CB} & \boldsymbol{D} \end{bmatrix} \in \mathbb{R}^{2N \times N}, \quad \boldsymbol{\Gamma} = \begin{bmatrix} \boldsymbol{C}^\mathsf{T} & (\boldsymbol{CA})^\mathsf{T} & \dots & (\boldsymbol{CA}^{N-1})^\mathsf{T} \end{bmatrix}^\mathsf{T} \in \mathbb{R}^{2N \times n}$$

Ambient space

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- Behavior: $(N + n)$-dimensional subspace

$$\widehat{\boldsymbol{w}} = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{v}} + \boldsymbol{\Gamma}\widehat{\boldsymbol{x}}_0, \Rightarrow \widehat{\boldsymbol{w}} \in \left[\text{range}\left(\widehat{\boldsymbol{H}}\right) \oplus \text{range}(\boldsymbol{\Gamma})\right].$$

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- $(N-n)$-dimensional subspace:

$$\mathbb{R}^{2N} = \underbrace{(\text{range}(\boldsymbol{H}) \oplus \text{range}(\boldsymbol{\Gamma}))}_{\text{behavior}} \oplus \underbrace{\text{range}\left(\widetilde{\boldsymbol{H}}\boldsymbol{Z}\right)}_{\text{misfits}}.$$

(FONC $\longrightarrow \widetilde{\boldsymbol{x}}_0 = \widetilde{\boldsymbol{x}}_N = \boldsymbol{0}$)

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Summary

- LS-criterion induces orthogonal decomposition of ambient space
- Minimal misfits can be expressed in terms of the model parameters via an orth. projection
- Misfits are heavily structured
- Two alternative yet equivalent frameworks: state-space vs. input-output

Future work: (globally optimal) SISO LTI misfit modeling

Misfit modeling (SYSID!) is double optimization problem:

$$\min_{\widehat{\boldsymbol{u}},\widehat{\boldsymbol{y}},\boldsymbol{a},\boldsymbol{b}} J = \left\| \begin{bmatrix} \boldsymbol{y} - \widehat{\boldsymbol{y}} \\ \boldsymbol{u} - \widehat{\boldsymbol{u}} \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} \widetilde{\boldsymbol{y}} \\ \widetilde{\boldsymbol{u}} \end{bmatrix} \right\|_2^2,$$

$$\text{s.t.} \quad \widetilde{\boldsymbol{T}}^{\mathsf{T}} \widehat{\boldsymbol{w}} = \boldsymbol{0} \text{ and } a_0 = 1,$$

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Earlier work: globally optimal system identification

- B. De Moor (2020). "Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem". In: *Communications in Information and Systems* 20.2, pp. 163–207
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How to generalize these globally optimal approaches from autonomous to SISO models?

Questions?

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