Exact Characterization of the Global Optima of Least Squares Realization of Autonomous LTI Models as a Multiparameter Eigenvalue Problem

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fwo Fonds Wetenschappelijk Onderzoek Vlaanderen
Opening new horizons

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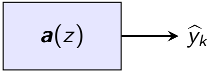
Autonomous LTI model



- LTI dynamics of model-compliant data $\widehat{\boldsymbol{y}} \in \mathbb{R}^N$:

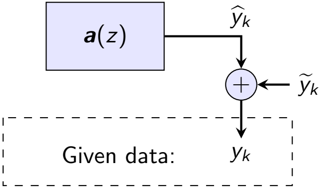
$$\widehat{y}_{k+n} + a_1 \widehat{y}_{k+n-1} + \cdots + a_n \widehat{y}_k = 0, \; \forall k{=}0,\ldots,N{-}n{-}1$$

- Kernel representation of *behavior*

$$\underbrace{\begin{bmatrix} a_n & \ldots & \ldots & a_1 & 1 & 0 & \ldots & 0 \\ 0 & a_n & \ldots & \ldots & a_1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n & \ldots & \ldots & a_1 & 1 \end{bmatrix}}_{\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}} \begin{bmatrix} \widehat{y}_0 \\ \widehat{y}_1 \\ \vdots \\ \widehat{y}_{N-1} \end{bmatrix} = \boldsymbol{0}$$

- n unknown model parameters $\boldsymbol{a} \in \mathbb{R}^n$

Autonomous LTI model



Least-squares realization:

$$\min_{\boldsymbol{a},\widehat{\boldsymbol{y}}} \quad \frac{1}{2}\|\widetilde{\boldsymbol{y}}\|_2^2 = \frac{1}{2}\|\boldsymbol{y}-\widehat{\boldsymbol{y}}\|_2^2,$$

$$\text{s.t.} \quad \boldsymbol{T}^{\boldsymbol{a}}_{N-n}\widehat{\boldsymbol{y}} = \boldsymbol{0}.$$

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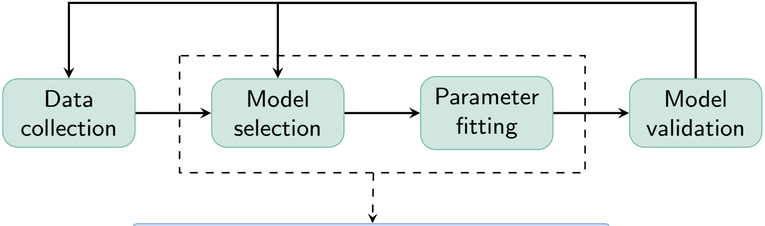
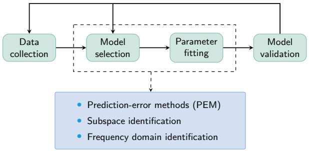
- n unknown model parameters $\boldsymbol{a} \in \mathbb{R}^n$

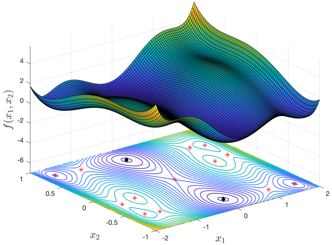
System identification for autonomous LTI models



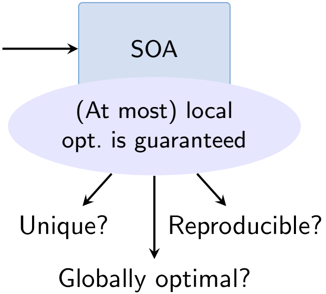
- Prediction-error methods (PEM)
- Subspace identification
- Frequency domain identification

Nonlinear optimization problem(s)

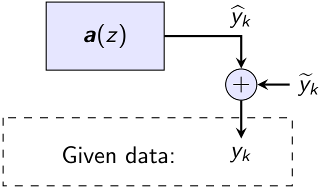




Non-convex
opt. problem

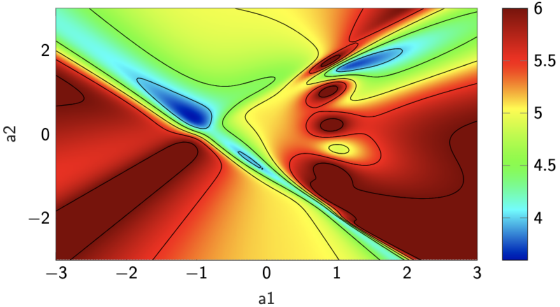


Example



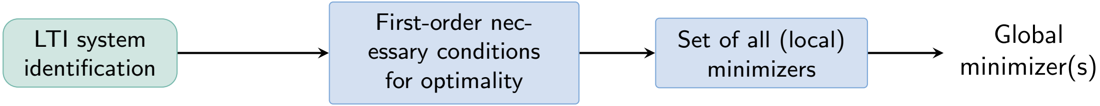
Least-squares realization:

$$\begin{aligned}\min_{\boldsymbol{a},\widehat{\boldsymbol{y}}} \quad & \frac{1}{2}\|\widetilde{\boldsymbol{y}}\|_2^2 = \frac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2, \\ \text{s.t.} \quad & \boldsymbol{T}_{N-n}^{\boldsymbol{a}}\widehat{\boldsymbol{y}} = \boldsymbol{0}.\end{aligned}$$



(Objective function $\|\widetilde{\boldsymbol{y}}\|_2^2$ for random data $\boldsymbol{y} \in \mathbb{R}^{16}$ and $n = 2$)

Global optimality



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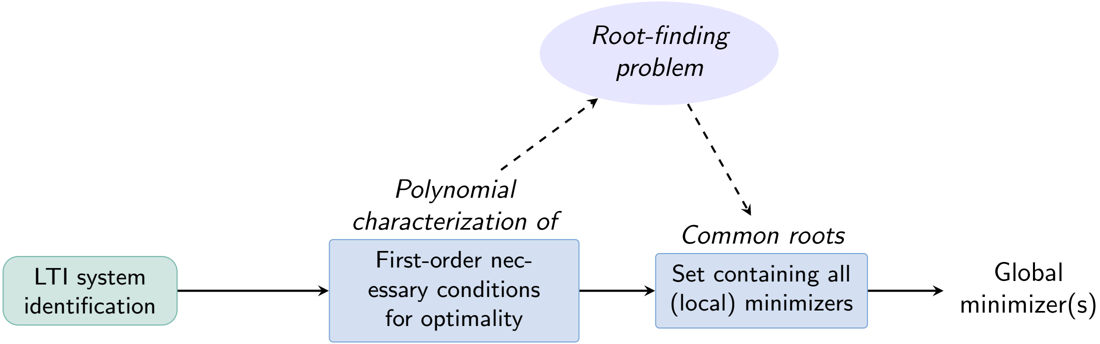

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Least-squares realization

$$\min_{\boldsymbol{a},\widehat{\boldsymbol{y}}} \quad \frac{1}{2}\|\widetilde{\boldsymbol{y}}\|_2^2 = \frac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2,$$

$$\text{s.t.} \quad \boldsymbol{T}^{\boldsymbol{a}}_{N-n}\widehat{\boldsymbol{y}} = \boldsymbol{0}.$$

$$\downarrow$$

$$\mathcal{L}(\boldsymbol{a}, \widehat{\boldsymbol{y}}, \boldsymbol{l}) = \tfrac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2 + \boldsymbol{l}^{\mathsf{T}}\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\widehat{\boldsymbol{y}},$$

$$\downarrow$$

$$\begin{aligned}
\partial\mathcal{L}/\partial\widehat{\boldsymbol{y}} &= \widehat{\boldsymbol{y}} - \boldsymbol{y} + \left(\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\right)^{\mathsf{T}}\boldsymbol{l} = \boldsymbol{0},\\
\partial\mathcal{L}/\partial\boldsymbol{a} &= \left(\widehat{\boldsymbol{Y}}_{N-n}\right)^{\mathsf{T}}\boldsymbol{l} - \boldsymbol{e}\lambda = \boldsymbol{0},\\
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Theorem (De Moor, 2020)

The minimal norm misfit $\widetilde{\boldsymbol{y}} = \boldsymbol{y} - \widehat{\boldsymbol{y}} \in \mathbb{R}^N$ corresponds to the orth. projection of $\boldsymbol{y}$ onto $\operatorname{row}\left(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\right)$,

$$\widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}}\left(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}}\right)^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\boldsymbol{y}.$$

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$$\downarrow$$

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Theorem (Lagauw, Vanpoucke, et al., 2024)

If $\boldsymbol{a}$ *is a (local) minimizer, then* $\exists\, \boldsymbol{g} \in \mathbb{R}^{N-2n}$ *s.t.,*

$$\widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathrm{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}})^{\mathrm{T}}\boldsymbol{g},$$

where $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}} \in \mathbb{R}^{(N-2n)\times(N-n)}$ *is a banded Toeplitz matrix defined similarly to the matrix* $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}} \in \mathbb{R}^{(N-n)\times N}$.

System of multivariate polynomial equations

- If $\boldsymbol{a}$ is a (local) minimizer, then $\exists\, \boldsymbol{g} \in \mathbb{R}^{N-2n}$ s.t.,

$$\widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}})^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} = (\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g}.$$
$$\iff (\boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}})^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - (\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0},$$
$$\iff \boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0}.$$

- Define the algebraic variety,

$$\mathcal{V}_{\mathbb{R}} = \left\{(\boldsymbol{a}, \boldsymbol{g}) \in \mathbb{R}^{N-n} \;:\; \boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0}\right\}.$$

$\longrightarrow$ $\mathcal{V}_{\mathbb{R}}$ contains all minimizers $\boldsymbol{a}$ of the identification problem ✓

- square system of $N-n$ polynomial equations
- degree of the polynomials is at most *quartic*

Numerical example[1] I

- Find globally optimal first-order ($n = 1$) autonomous LTI realization
- Given output data $\boldsymbol{y} = [4, 3, 2, 1]^{\mathsf{T}}$ ($N = 4$)

$$\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0},$$

$$\iff \begin{cases} 0 = 4a_1 - 2a_1g_1 - a_1^2g_2 - a_1^3g_1 + 3, \\ 0 = 3a_1 - g_1 - 2a_1g_2 - 2a_1^2g_1 - a_1^3g_2 + 2, \\ 0 = 2a_1 - g_2 - a_1g_1 - 2a_1^2g_2 + 1. \end{cases}$$

[1] Toy problem from (De Moor, 2020)

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$$\Longleftrightarrow \begin{cases} 0 = 4a_1 - 2a_1g_1 - a_1^2g_2 - a_1^3g_1 + 3, \\ 0 = 3a_1 - g_1 - 2a_1g_2 - 2a_1^2g_1 - a_1^3g_2 + 2, \\ 0 = 2a_1 - g_2 - a_1g_1 - 2a_1^2g_2 + 1. \end{cases}$$

- 7 common-roots, one of which in $\mathcal{V}_\mathbb{R}$

$\lVert\tilde{\boldsymbol{y}}\rVert_2^2$	a_1	g_1	g_2
0.1486	-0.6764	-0.2525	-0.2734
/	$-0.1589 \mp 0.808j$	$1.3577 \pm 3.8194j$	$1.8359 \pm 3.3491j$
/	$0.4209 \pm 0.6233j$	$3.0425 \mp 2.9959j$	$-0.0785 \pm 1.2013j$
/	$1.3261 \pm 2.0058j$	$-0.2739 \mp 0.6279j$	$0.3793 \mp 0.3849j$

[1] Toy problem from (De Moor, 2020)

Multiparameter eigenvalue problem

$$\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0}. \qquad (1)$$

- Rewrite (1) as a *cubic n-parameter eigenvalue problem* in the parameters $a_1, \ldots, a_n$,

$$\underbrace{\begin{bmatrix}\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} & \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\end{bmatrix}}_{\mathcal{M}(\boldsymbol{a})}\begin{pmatrix}-1\\ \boldsymbol{g}\end{pmatrix} = \boldsymbol{0}.$$

- $\mathcal{M}(\boldsymbol{a}) = \sum_{\{\boldsymbol{\alpha}\}} \boldsymbol{M}_{\boldsymbol{\alpha}}\boldsymbol{a}^{\boldsymbol{\alpha}}$, is a matrix polynomial in the monomials $\boldsymbol{a}^{\boldsymbol{\alpha}} = a_1^{\alpha_1}\ldots a_n^{\alpha_n}$, with coefficient matrices $\boldsymbol{M}_{\boldsymbol{\alpha}} \in \mathbb{R}^{(N-n)\times(N-2n+1)}$.
- The values $\boldsymbol{a} \in \mathbb{C}^n$ for which $\mathcal{M}(\boldsymbol{a})$ becomes rank-deficient, such that there exists a vector $\boldsymbol{g} \in \mathbb{C}^{N-2n}$ for which these equations are satisfied, are the *affine eigentuples*[2] of this MEP.

[2] (De Cock and De Moor, 2021)

Numerical Example I (continued)

- Given output data $\boldsymbol{y} = [4, 3, 2, 1]^\mathsf{T}$ $(N = 4)$

$$\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^\mathsf{T}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}})^\mathsf{T}\boldsymbol{g} = \boldsymbol{0},$$

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- Cubic 1-parameter eigenvalue problem:

$$\left[\underbrace{\begin{pmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}}_{\boldsymbol{M}_0} + \underbrace{\begin{pmatrix} 4 & -2 & 0 \\ 3 & 0 & -2 \\ 2 & -1 & 0 \end{pmatrix}}_{\boldsymbol{M}_1} a_1 + \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\boldsymbol{M}_2} a_1^2 + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}}_{\boldsymbol{M}_3} a_1^3\right]\begin{pmatrix} 1 \\ g_1 \\ g_2 \end{pmatrix} = \boldsymbol{0}.$$

Numerical Example II

- Find globally optimal second-order ($n = 2$) autonomous LTI realization
- Given output data[3] $\boldsymbol{y}$ ($N = 16$),

$$\boldsymbol{y} = \boldsymbol{y}_{3\text{rd}} + 0.05 * \texttt{randn(N,1)},$$

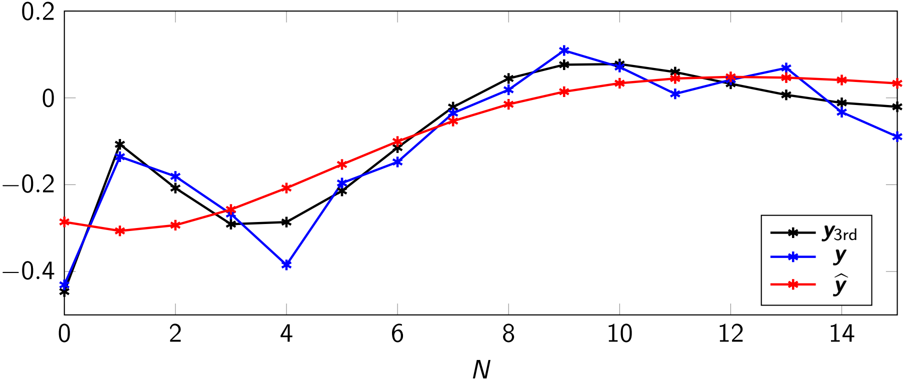
where $\boldsymbol{y}_{3\text{rd}}$ is generated by a third-order autonomous LTI model with poles $(0.2, 0.7 \pm 0.4j)$

- 739 affine common-roots, 9 of which are real-valued.

$\|\widetilde{\boldsymbol{y}}\|_2^2$	a_1	a_2	p_1	p_2
0.1327	−1.6255	0.7167	$0.8127 + 0.2369j$	$0.8127 - 0.2369j$
0.1514	−0.0752	−0.5850	0.8033	−0.7282
0.1606	−14.076	10.433	13.291	0.7849
0.5386	−0.7127	1.8381	$0.3564 + 1.3081j$	$0.3564 - 1.3081j$
⋮	⋮	⋮	⋮	⋮
0.5492	−1.3053	1.0564	$0.6527 + 0.7940j$	$0.6527 - 0.7940j$

[3] The considered instance of $\boldsymbol{y}$ has $\|\boldsymbol{y}\|_2^2 = 0.5509$.

Numerical Example II



Properties of the MEP

- One alternative MEP $\mathcal{M}_1$ described in the literature[4]
- Computational complexity of solving an MEP depends on:
  - the highest degree of the parameters
  - the number of parameters
  - the size of the coefficient matrices
- Size of the coefficient matrices the MEPs for several (N, n):

(N, n)	size($\mathcal{M}_1$)	size($\mathcal{M}_{\text{new}}$)
$(4, 1)$	7×7	3×3
$(16, 6)$	76×71	10×5
$(50, 8)$	386×379	42×35
$(200, 15)$	2975×2961	185×171

[4] (De Moor, 2020)

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$$\longrightarrow \quad \text{size}(\mathcal{M}_{\text{new}}) \approx \frac{\text{size}(\mathcal{M}_1)}{(n+1)}$$

[4] (De Moor, 2020)

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H_2-norm model reduction of SISO LTI models

Minimize H_2-norm of approximation error $E(z) = H(z) - \widehat{H}(z)$:

$$\widehat{H}(z) \in \underset{\widehat{H}(z) \in \mathcal{M}}{\operatorname{argmin}} \; J^2, \tag{2}$$

where,

$$J^2 = \|E(z)\|_{H_2}^2 = \sum_{k=0}^{\infty} \left(h_k - \widehat{h}_k \right)^2,$$

with $\{h_k\}_{k=0,\ldots,\infty}$ and $\{\widehat{h}_k\}_{k=0,\ldots,\infty}$ the impulse responses of $H(z)$, $\widehat{H}(z)$, respectively, and,

$$\widehat{H}(z) = \frac{\widehat{b}(z)}{\widehat{a}(z)} = \frac{\widehat{b}_{m-1} z^{m-1} + \cdots + \widehat{b}_1 z + \widehat{b}_0}{z^m + \widehat{a}_{m-1} z^{m-1} + \cdots + \widehat{a}_1 z + \widehat{a}_0},$$

Model reduction as a limiting case ($N \to \infty$)

- Take $\boldsymbol{y}$ the impulse response of a stable mth order SISO model, then, for a given $n < m$, the LS realization problem finds the optimal $\widehat{\boldsymbol{y}}$ for which,
  - $\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2$ is minimal and,
  - $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}}\boldsymbol{a} = 0$.

- by Kronecker's Theorem[5] the rank-deficiency of the Hankel matrix $\widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}} \in \mathbb{R}^{\infty\times(n+1)}$ implies that $\widehat{\boldsymbol{y}}$ is the impulse response of an nth order SISO model.

$\longrightarrow$ for $\boldsymbol{y} \in \mathbb{R}^N$ with $N \to \infty$,

autonomous LTI LS realization $\approx$ SISO H_2-norm model reduction

[5] (Kronecker, 1881)

Interpolatory conditions for optimality

Theorem (Meier and Luenberger, 1967)

Given a stable SISO model $H(z) \in \mathcal{M}$ *of order* n*, let* $\widehat{H}(z)$ *of order* m $(m<n)$ *be a stationary point of the* H_2*-norm model reduction problem in* (2)*. Then for all poles* p_i *of* $\widehat{H}(z)$*,*

$$H(1/p_i)^{(j)} = \widehat{H}(1/p_i)^{(j)}, \quad j = 0, \ldots, d_i,$$

where d_i *is the multiplicity of the pole* p_i *and the superscript* j *denotes the* j*th derivative wrt.* z*.*

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Theorem (Meier and Luenberger, 1967)

Given a stable SISO model $H(z) \in \mathcal{M}$ of order n, let $\widehat{H}(z)$ of order m ($m<n$) be a stationary point of the H_2-norm model reduction problem in (2). Then for all poles p_i of $\widehat{H}(z)$,

$$H(1/p_i)^{(j)} = \widehat{H}(1/p_i)^{(j)}, \quad j = 0, \ldots, d_i,$$

where d_i is the multiplicity of the pole p_i and the superscript j denotes the jth derivative wrt. z.

Theorem (Regalia, 1995)

Given a stable SISO model $H(z) \in \mathcal{M}$ of order n, let $\widehat{H}(z)$ of order m with $m<n$ be a stationary point of the model reduction problem (2). Then for all $z \in \mathbb{C}$:

$$H(z) - \widehat{H}(z) = \frac{b(z)}{a(z)} - \frac{\widehat{b}(z)}{\widehat{a}(z)} = [z^m \widehat{a}(1/z)]^2 \frac{G(z)}{\widehat{a}(z)},$$

with $G(z)$ the z-transform of some real-valued, stable and causal signal.

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Intermezzo: globally optimal H_2-norm SISO MOR

Theorem (Lagauw, Agudelo, et al., 2023)

Let f_k be the coefficient corresponding to s^k in the polynomial,

$$l(s) = b(s)\widehat{a}(s) - a(s)\widehat{b}(s) - [\widehat{a}(-s)]^2 \widetilde{G}(s),$$

and define the algebraic variety,

$$\mathcal{V}_{\mathbb{R}} = \left\{ (\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \boldsymbol{g}) \in \mathbb{R}^{m+n} \ : \ f_k(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \boldsymbol{g}) = 0, \quad \forall k = 0, \ldots, m+n-1 \right\}.$$

If $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ contains the $(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \boldsymbol{g}) \in \mathcal{V}_{\mathbb{R}}$ for which $\widehat{H}(s)$ is stable, then $\mathcal{V}'_{\mathbb{R}}$ contains all minimizers $\widehat{H}(s)$.

- square system of $m+n$ polynomial equations
- degree of the polynomials is at most *cubic*

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Model reduction as a limiting case ($N \to \infty$)

Theorem (Regalia, 1995)

Given a stable SISO model $H(z) \in \mathcal{M}$ of order n, let $\widehat{H}(z)$ of order m with $m<n$ be a stationary point of the model reduction problem (2). Then for all $z \in \mathbb{C}$:

$$H(z) - \widehat{H}(z) = \frac{b(z)}{a(z)} - \frac{\widehat{b}(z)}{\widehat{a}(z)} = [z^m \widehat{a}(1/z)]^2 \frac{G(z)}{\widehat{a}(z)},$$

with $G(z)$ the z-transform of some real-valued, stable and causal signal.

Theorem (Lagauw, Vanpoucke, et al., 2024)

If $\boldsymbol{a}$ is a (local) minimizer, then $\exists\, \boldsymbol{g} \in \mathbb{R}^{N-2n}$ s.t.,

$$\boldsymbol{y} - \widehat{\boldsymbol{y}} = \widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}} (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}})^{\mathsf{T}} \boldsymbol{g},$$

where $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}} \in \mathbb{R}^{(N-2n)\times(N-n)}$ is a banded Toeplitz matrix defined similarly to the matrix $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}} \in \mathbb{R}^{(N-n)\times N}$.

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Table of contents

Summary & future work

- Globally optimal LS realization of autonomous LTI models is an MEP
- LS realization of autonomous LTI models $\leftrightarrow$ SISO H_2-norm model reduction

Future work:

- Proof zero-dimensionality of the (affine) solution set
- Particular interest in the *real-valued* common-roots
- Exploit structure of MEPs in computations

S. Lagauw, L. Vanpoucke, et al. (June 2024). "Exact Characterization of the Global Optima of Least Squares Realization of Autonomous LTI Models as a Multiparameter Eigenvalue Problem". In: *Proc. of the 22nd European Control Conference (ECC)*. Stockholm, Sweden, pp. 3439–3444

S. Lagauw, O. M. Agudelo, et al. (2023). "Globally Optimal SISO H_2-Norm Model Reduction Using Walsh's Theorem". In: *IEEE Control Systems Letters* 7, pp. 1670–1675

Questions?

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References I

- Agudelo, O. M., C. Vermeersch, and B. De Moor (2021). "Globally Optimal H_2-Norm Model Reduction: A Numerical Linear Algebra Approach". In: *IFAC PapersOnLine (Special Issue: Proceedings of MTNS 2020)* 54.9, pp. 564–571.
- Collins, E.G., D.J. Phillips, and D.C. Hyland (1991). "Robust decentralized control laws for the ACES structure". In: *IEEE Control Systems Magazine* 11.3, pp. 62–70.
- De Cock, K. and B. De Moor (2021). "Multiparameter Eigenvalue Problems and Shift-Invariance". In: *IFAC-PapersOnLine (Special Issue: Proceedings of MTNS 2020)* 54.9, pp. 159–165.
- De Moor, B. (2020). "Least squares optimal realisation of autonomous LTI systems is an eigenvalue problem". In: *Communications in Information and Systems* 20.2, pp. 163–207.
- Kronecker, L. (1881). "Zur Theorie der Elimination einer Variablen aus zwei algebraischen Gleichungen". In: *Monatsberichte der Königlich Preussischen Akademie der Wissenschaften (Berlin)*. Reprinted as pp. 113–192 in Mathematische Werke, vol. 2, B.G. Teubner, Leipzig, 1897., pp. 535–600.
- Lagauw, S., O. M. Agudelo, and B. De Moor (2023). "Globally Optimal SISO H_2-Norm Model Reduction Using Walsh's Theorem". In: *IEEE Control Systems Letters* 7, pp. 1670–1675.
- Lagauw, S., L. Vanpoucke, and B. De Moor (June 2024). "Exact Characterization of the Global Optima of Least Squares Realization of Autonomous LTI Models as a Multiparameter Eigenvalue Problem". In: *Proc. of the 22nd European Control Conference (ECC)*. Stockholm, Sweden, pp. 3439–3444.
- Meier, L. and D.G. Luenberger (Oct. 1967). "Approximation of Linear Constant Systems". In: *IEEE Transactions on Automatic Control* 12.5, pp. 585–588.
- Regalia, Phillip A. (1995). *Adaptive IIR Filtering in Signal Processing and Control*. Marcel Dekker Inc., New York, NY.

Numerical example (MOR) I

- Consider the third order model ($n = 3$) used in Agudelo et al., 2021:

$$H(s) = \frac{s^2 + 9s - 10}{s^3 + 12s^2 + 49s + 78}.$$

- For $m = 1$, the polynomial $l(s)$ from (**??**) is given as,

$$
\begin{aligned}
l(s) = &\underbrace{(1 - g_1 - \widehat{b}_0)}_{f_3} s^3 + \underbrace{(\widehat{a}_0 - 12\widehat{b}_0 - g_0 + 2\widehat{a}_0 g_1 + 9)}_{f_2} s^2 \\
&+ \underbrace{(9\widehat{a}_0 - 49\widehat{b}_0 + 2\widehat{a}_0 g_0 - \widehat{a}_0^2 g_1 - 10)}_{f_1} s \\
&+ \underbrace{(-g_0 \widehat{a}_0^2 - 10\widehat{a}_0 - 78\widehat{b}_0)}_{f_0} 1.
\end{aligned}
$$

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Numerical example (MOR) I

Strategy: find all $(\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \boldsymbol{g})$ for which $I(s) = 0, \ \forall s \in \mathbb{C}$"

$$\iff \begin{cases} 0 = f_3 = 1 - g_1 - \widehat{b}_0, \\ 0 = f_2 = \widehat{a}_0 - 12\widehat{b}_0 - g_0 + 2\widehat{a}_0 g_1 + 9, \\ 0 = f_1 = 9\widehat{a}_0 - 49\widehat{b}_0 + 2\widehat{a}_0 g_0 - \widehat{a}_0^2 g_1 - 10, \\ 0 = f_0 = -g_0 \widehat{a}_0^2 - 10\widehat{a}_0 - 78\widehat{b}_0. \end{cases} \tag{3}$$

- Common roots of (3) contain all stationary points $\widehat{H}(s)$ of (2):

J	$\widehat{a}_0$	$\widehat{b}_0$	g_0	g_1	stable
8.9403	$-4.1639 + 0.9026j$	$24.930 - 6.5393j$	$-106.84 - 18.287j$	$-23.930 + 6.5393j$	✗
8.9403	$-4.1639 - 0.9026j$	$24.930 + 6.5393j$	$-106.84 + 18.287j$	$-23.930 - 6.5393j$	✗
0.3982	0.2671	−0.0437	10.349	1.0437	✓
0.2784	0.6914	9.6796	1.2799	−2.0986	✓
0.5232	−16.618	1.9264	0.0576	−0.9264	✗

Numerical example (MOR) II

We search for the globally optimal 4th order reduced model ($m = 4$) of the state-space model[6] ($n = 17$).

- System of 21 polynomial equations with $d_{\text{max}} = 3$
- $\mathcal{V}_{\mathbb{R}}$ contains 290 tuples, 69 remain in $\mathcal{V}'_{\mathbb{R}}$
- Four best-performing stationary points $\widehat{H}(s)$:

	J	$p_{1,2}$	$p_{3,4}$
*	9.14×10^{-3}	$-0.032 \pm 78.54j$	$-0.111 \pm 15.43j$
l_1	9.22×10^{-3}	$-0.032 \pm 78.54j$	$-5.713 \pm 52.57j$
l_2	1.03×10^{-2}	$-0.032 \pm 78.54j$	$-0.023 \pm 3.842j$
l_3	1.09×10^{-2}	$-0.032 \pm 78.54j$	$-4.663 \pm 15.88j$

[6] The model describes the interaction between a torque activator and a torsional rate sensor for the ACES structure Collins et al., 1991.

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Numerical example (MOR) II