Back to the roots: a spectrum of what was realized

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Outline

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Eigenvalues and vectors: For matrix $A \in \mathbb{R}^{n \times n}$:

$$Ax = x\lambda \, , x \in \mathbb{C}^n, \lambda \in \mathbb{C}, x \neq 0$$

Characteristic equation - fundamental theorem of algebra:

$$p(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

Since Galois, for $n \geq 5$: no solution in radicals !

Numerical linear algebra = iterative algorithms + finite precision machines

Cayley-Hamilton:

$$A^n + \alpha_1 A^{n-1} + \ldots + \alpha_{n-1} A + \alpha_n I_n = 0$$

Eigenvalue decomposition - Jordan Canonical Form (JCF):

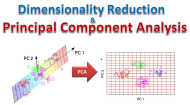
$$A = XJX^{-1}$$

Eigen-objects: Operator (object) = object × scalar

Continuous spectrum:

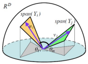
$d(e^{(\alpha \pm j\beta t)})/dt = (e^{(\alpha \pm j\beta t)})(\alpha \pm j\beta t)$, $(d\,./dt + \int .\, dt)e^{\alpha t} = e^{\alpha t}(\frac{\alpha^2+1}{\alpha})$

Discrete spectrum: e.g. standing waves



PCA

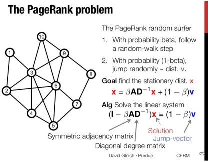
Let Y_1 and Y_2 be two orthonormal matrices of size D by m, and let $u \in \mathrm{span}(Y_1)$ and $v \in \mathrm{span}(Y_2)$ be unit vectors.



The first principal angle/canoncial corr between $\mathrm{span}(Y_1)$ and $\mathrm{span}(Y_2)$ is

$$\cos\theta_1 = \max_{u\in \mathrm{span}(Y_1)} \max_{v\in \mathrm{span}(Y_2)} u'v, \quad \text{subject to} \quad \|u\| = \|v\| = 1.$$

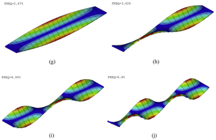
Can. Corr./Principal Angles



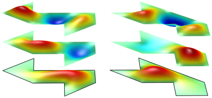
Graph spectral analysis

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{v_w^2}\frac{\partial^2 y}{dt^2}$$

Wave equation

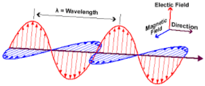


Modal shapes



Hear the shape of a drum?

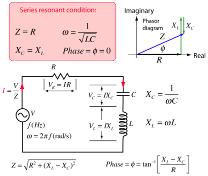
Answer: No !



Maxwell's laws

1. $\nabla \cdot \mathbf{D} = \rho_V$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
4. $\nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$

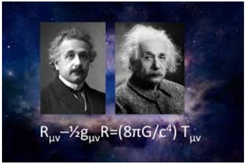
Maxwell's field equations



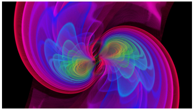
RLC circuits-resonances

$$H(t)|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$$

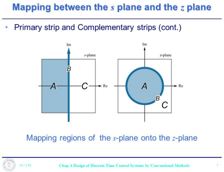
Schrödinger equation



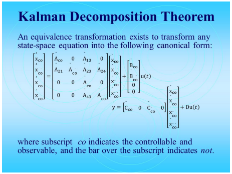
Matter curves spacetime moves matter



Gravitational waves



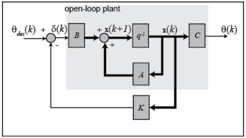
Stability


Kalman Decomposition Theorem

An equivalence transformation exists to transform any state-space equation into the following canonical form:

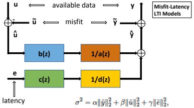
where subscript *co* indicates the controllable and observable, and the bar over the subscript indicates *not*.

Controllability/observability



Pole placement

Observers	Kalman Filter Riccati Hamil. EVP	H_∞-filter Riccati Sympl. EVP
Control	LQR Riccati Hamil. EVP	H_∞-control Riccati Sympl. EVP



LS LTI System ID = EVP !

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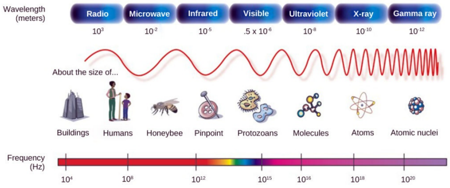
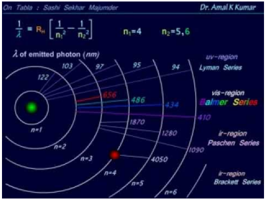
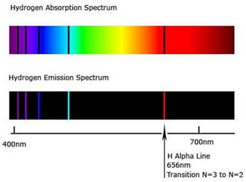
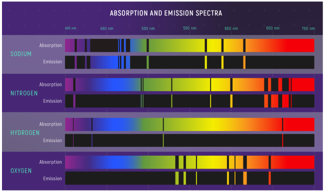
If you want to find the secrets of the universe, think in terms of energy, frequency and vibration. Nikola Tesla









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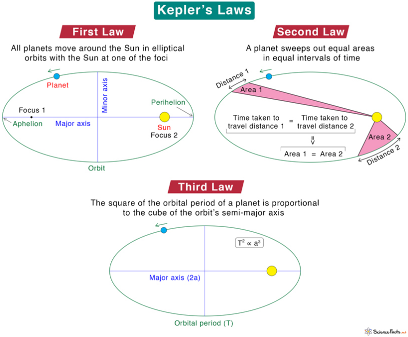
From Kepler to Newton: realization from data to (internal) state

Kepler (1571-1630)



Newton (1642-1726)

$$\mathbf{F} = m\,\mathbf{a}$$
$$F = G\,m\,M/r^2$$

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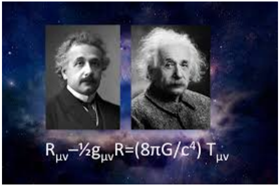
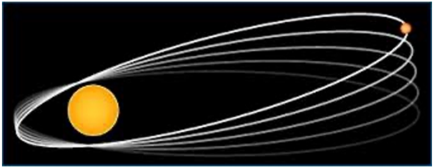
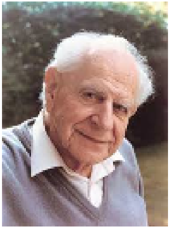
Einstein and Popper



Einstein (1879-1955)



Perihelion precession of Mercury's orbit



Popper (1902-1994)

Popper′s demarcation criterion :

A model/theory is scientific when refutable

Models forbid more than they allow

Outline

Taylor / McLaurin series expansion

$$\begin{aligned} f(z) &= f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \ldots + \frac{f^{(k)}}{k!} z^k + \ldots \\ &= \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \ldots + \gamma_k z^k + \ldots \end{aligned}$$

When rational in z^{-1} ?

$$\begin{aligned} f(z^{-1}) &= \frac{\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_{n-1} z + \beta_n}{\alpha_0 z^n + \alpha_1 z^{n-1} + \ldots + \alpha_{n-1} z + \alpha_n} \\ &= \frac{\sum_{i=0}^n \beta_i z^{n-i}}{\sum_{i=0}^n \alpha_i z^{n-i}} \\ &= \gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \ldots = \sum_{i=0}^{\infty} \gamma_i z^{-i} \end{aligned}$$

$$\Longrightarrow \sum_{i=0}^{n} \beta_i z^{n-i} = \left(\sum_{i=0}^{n} \alpha_i z^{n-i}\right)\left(\sum_{i=0}^{\infty} \gamma_i z^{-i}\right)$$

Kronecker (1823-1891)

Example: $n = 2$:

$$\beta_0 z^2 + \beta_1 z + \beta_2 = (\alpha_0 z^2 + \alpha_1 z + \alpha_2)(\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2} + \gamma_3 z^{-3} + \dots)$$

Equate likewise powers of z:

$$\begin{aligned}
z^2 : \quad \beta_0 &= \alpha_0\gamma_0 \\
z^1 : \quad \beta_1 &= \alpha_0\gamma_1 + \alpha_1\gamma_0 \\
z^0 : \quad \beta_2 &= \alpha_0\gamma_2 + \alpha_1\gamma_1 + \alpha_2\gamma_0 \\
z^{-1} : \quad 0 &= \alpha_0\gamma_3 + \alpha_1\gamma_2 + \alpha_2\gamma_1 \\
z^{-2} : \quad 0 &= \alpha_0\gamma_4 + \alpha_1\gamma_3 + \alpha_2\gamma_2 \\
\vdots \quad \vdots &\quad \vdots \\
z^{-k} : \quad 0 &= \alpha_0\gamma_{k+2} + \alpha_1\gamma_{k+1} + \alpha_2\gamma_k
\end{aligned}$$

Coefficients $\gamma_k, k \geq 2$ satisfy 3-term linear recurrence

$$\alpha_0\gamma_{k+2} + \alpha_1\gamma_{k+1} + \alpha_2\gamma_k = 0, k \geq 2,$$

with initial conditions $\gamma_0, \gamma_1, \gamma_2$ from set of linear equations.

Kronecker and Hankel

$$\begin{pmatrix} \alpha_2 & \alpha_1 & \alpha_0 & 0 & 0 & 0 & \dots \\ 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots & \ddots \\ 0 & 0 & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \dots \\ \gamma_2 & \gamma_3 & \gamma_4 & \iddots & \vdots \\ \gamma_3 & \gamma_4 & \gamma_5 & \iddots & \vdots \\ \gamma_4 & \gamma_5 & \gamma_6 & \iddots & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \end{pmatrix} = 0$$

Rational function series expansion $\Longleftrightarrow$ Hankel matrix rank deficient

Banded Toeplitz $\times$ rank deficient Hankel $= 0$
Rank Hankel $=$ degree of rational function
Recurrence relation coefficients $=$ denominator

Hankel (1839-1873)

Outline

State $x_k \in \mathbb{R}^n$, output $y_k \in \mathbb{R}^l$:

$$x_{k+1} = Ax_k \qquad X(z) = (zI_n - A)^{-1}x_0 = (I_n + Az^{-1} + Az^{-2} + \ldots)x_0$$

$$y_k = Cx_k = CA^k x_0 \qquad Y(z) = C(zI_n - A)^{-1}x_0 = Cx_0 + (CAx_0)z^{-1} + (CA^2x_0)z^{-2} + \ldots = y_0 + y_1 z^{-1} + y_2 z^{-2} + \ldots$$

Resolvent is rational:

$$(zI_n - A)^{-1} = \mathrm{adj}(A)/\det(zI_n - A)$$

Kronecker/Hankel: Factorize (SVD) to go from data to state space model

$$\begin{pmatrix} y_1 & y_2 & y_3 & y_4 & \cdots \\ y_2 & y_3 & y_4 & y_5 & \vdots \\ y_3 & y_4 & y_5 & y_6 & \vdots \\ \iddots & \iddots & \iddots & \iddots & \vdots \end{pmatrix} = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{pmatrix} \begin{pmatrix} x_0 & Ax_0 & A^2x_0 & \ldots \end{pmatrix}$$

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With characteristic equation and Cayley-Hamilton

$$p(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \ldots + \alpha_{n-1}\lambda + \alpha_n = 0$$

$$\begin{pmatrix} \alpha_n & \alpha_{n-1} & \alpha_{n-2} & \ldots & \alpha_1 & \alpha_0 & 0 & 0 & \ldots \\ 0 & \alpha_n & \alpha_{n-1} & \ddots & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \ddots \\ 0 & 0 & \alpha_n & \ddots & \ddots & \alpha_2 & \alpha_1 & \alpha_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ \vdots \end{pmatrix} = T \,.\, \Gamma = 0$$

Left null space (Γ) = banded Toeplitz T.
Right null space (T) = shift-invariant:

$$\boxed{\underline{\Gamma}\, A = \overline{\Gamma} \iff \operatorname{rank}(\, \underline{\Gamma} \;\; \overline{\Gamma}\,) = n = \operatorname{rank}(\underline{\Gamma})\ (\mathrm{PRC}) \implies A = \underline{\Gamma}^{\dagger}\, \overline{\Gamma}}$$

Enter Paul Van den Hof

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DEPARTMENT OF ELECTRICAL ENGINEERING
EINDHOVEN UNIVERSITY OF TECHNOLOGY
Group Measurement and Control

APPROXIMATE REALIZATION OF NOISY LINEAR
SYSTEMS: THE HANKEL AND PAGE MATRIX
APPROACH

by Paul Van den Hof

This report is submitted in fulfillment of the
requirements for the degree of electrical engineer
(M.Sc.) at the Eindhoven University of Technology.
The work was carried out from Jan. until Dec. 1982
in charge of Prof. dr. ir. P. Eykhoff
under supervision of dr. ir. A.A.H. Damen
and dr. ir. A.K. Hajdasinski

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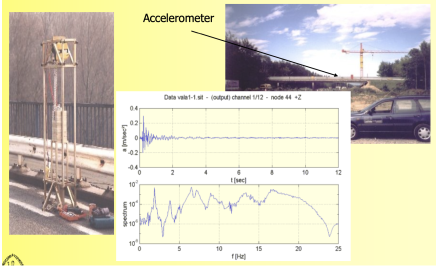
SUMMARY

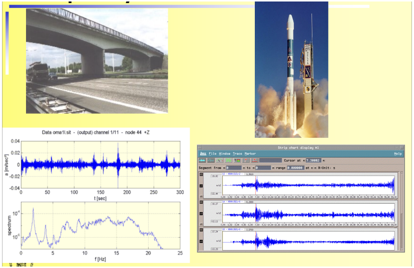
The Ho-Kalman algorithm creates a minimum realization of a linear, time invariant system, when given a sufficiently long series of deterministic Markov parameters. However if such a "truncated" series of Markov parameters has been disturbed with noise, an approximating Hankel matrix has to be constructed for applying the realization algorithm. This approximating Hankel matrix has either the improper rank, or it lacks the Hankel structure. Furthermore the Markov parameters are not processed with a constant weighting factor, which implies that the noise filtering is inadequate.
In this report an alternative matrix is introduced and investigated: the Page matrix. This matrix is much smaller than the Hankel matrix, which offers the advantage of a considerable reduction in computation. It is shown that the method using this Page matrix might be better suited for handling noisy Markov parameters. The Page matrix approach however still does not provide an optimal solution to the approximate realization problem.
The two approaches are compared theoretically and their practical performance is tested in a set of simulations.

Outline

Application: Impulse response and stochastic realization



Impulse response



Stochastic realization

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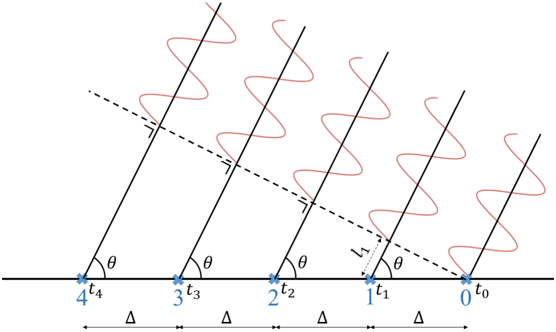
Application: Direction of Arrival: Uniform linear array, narrow band sources, far field



$$
\begin{aligned}
y_i(t) &= \sin(\omega t + \frac{\omega(i\Delta)\cos\theta}{c}) = \sin(\omega t + \varphi_i) \\
&= \sin(\omega t)\cos\varphi_i + \cos(\omega t)\sin\varphi_i = \begin{pmatrix} \cos\varphi_i & \sin\varphi_i \end{pmatrix} \begin{pmatrix} \sin\omega t \\ \cos\omega t \end{pmatrix}
\end{aligned}
$$

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Application: Shape from moments

Calculate moments of 'pdf' and show that

$$\int_{-T}^{T} p_f(t,\theta)t^k dt = \sum_{j=0}^{k} \binom{k}{j} \cos^{k-j}(\theta)\sin^j(\theta)\mu_{k-j,j}$$

$$\mu_{pq} \rightarrow \tau_k$$

$$\tau_k = \sum_{j=1}^{n} a_j z_j^k$$

→ Realization theory !

Application: Cepstrum realization

Power cepstrum = power spectrum of log of power spectrum

$$\log \Phi(z) = \sum_{-\infty}^{+\infty} c_k . z^{-k}$$

Cepstral coefficients $c_k = c_{-k}, \forall k;$

$$c_0 = 2 \log \rho$$

$$k\, c_k = \sum \alpha_i^k - \sum \beta_i^k$$

i-th cepstral coefficient = sum of i-th powers of poles and zeros

→ Realization theory !

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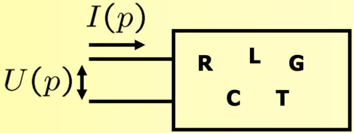
Application: Electrical circuit power spectrum by R, L, C, T, G

A transfer function $Z(p)$ is **realizable** as a passive electrical circuit

$\Leftrightarrow$ there exists an interconnection of a finite number of R's, L's, C's, T's and G's such that

$$Z(p) = \frac{U(p)}{I(p)}$$



$\Leftrightarrow Z(p)$ is positive real

$\Leftrightarrow p \in \mathbb{C}_+ \Rightarrow Z(p) \in \mathbb{C}_+$

Outline

mD shift invariant systems $(m = 2)$

$x_{k,l} \in \mathbb{R}^n$, $y_{k,l} \in \mathbb{R}$:

$$
\begin{array}{rcl}
x_{k+1,l} & = & A_1 \, x_{k,l} \\
x_{k,l+1} & = & A_2 \, x_{k,l} \qquad A_1 A_2 = A_2 A_1 \\
y_{k,l} & = & C \, x_{k,l}
\end{array}
$$

$$
Y = \left(\begin{array}{c|cc|ccc|cc}
y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} & y_{30} & \cdots \\
\hline
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} & y_{40} & \cdots \\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} & y_{31} & \cdots \\
\hline
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} & y_{50} & \cdots \\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} & y_{41} & \cdots \\
y_{02} & y_{12} & y_{13} & y_{22} & y_{13} & y_{04} & y_{32} & \cdots \\
\hline
y_{30} & y_{40} & y_{31} & y_{50} & \cdots & \cdots & \cdots & \cdots \\
y_{21} & y_{31} & y_{22} & y_{41} & \cdots & \cdots & \cdots & \cdots \\
y_{12} & y_{22} & y_{13} & y_{32} & \cdots & \cdots & \cdots & \cdots \\
y_{03} & y_{13} & y_{04} & y_{23} & \cdots & \cdots & \cdots & \cdots \\
\hline
y_{40} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right) \qquad \mathbf{rank}(Y) = n
$$

$$
= \quad \Gamma \;\; \Delta = \left(\begin{array}{c}
C \\
\hline
CA_1 \\
CA_2 \\
\hline
CA_1^2 \\
CA_1A_2 \\
CA_2^2 \\
\hline
\vdots
\end{array}\right) \left(\begin{array}{c|cc|cc} x_0 & A_1 x_0 & A_2 x_0 & A_1^2 x_0 & \cdots \end{array} \right)
$$

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The column space of Γ is a **multi-shift-invariant subspace**:

$$\underline{\Gamma}\ A_1 \ = S_1\Gamma = \begin{pmatrix} C \\ \hline CA_1 \\ CA_2 \\ \hline CA_1^2 \\ CA_1A_2 \\ CA_2^2 \\ \hline \vdots \\ \hline CA_1^{p-2} \\ CA_1^{p-3}A_2 \\ \vdots \\ CA_2^{p-2} \end{pmatrix} A_1 = \begin{pmatrix} CA_1 \\ CA_1^2 \\ \hline CA_1A_2 \\ CA_1^3 \\ CA_1^2A_2 \\ CA_1A_2^2 \\ \hline \vdots \\ \hline CA_1^{p-1} \\ CA_1^{p-2}A_2 \\ \vdots \\ CA_1A_2^{p-2} \end{pmatrix} \quad \text{and} \quad \underline{\Gamma}\ A_2 \ = S_2\Gamma$$

- Selector matrix S_1 selects the block rows $(2, 4, 5, 7, 8, 9, \ldots)$.
- Selector matrix S_2 selects the block rows $(3, 5, 6, 8, 9, 10, \ldots)$.
- Find A_1, A_2 by solving set of linear equations (PRC: $\text{rank}(\underline{\Gamma}) = \text{n}$)

$$\boxed{\mathsf{A}_1 = \underline{\Gamma}^\dagger S_1\Gamma \ \text{ and } \ A_2 = \underline{\Gamma}^\dagger S_2\Gamma\ .}$$

- A multi-shift invariant subspace is determined by the eigenvalues of its shifts A_1 and A_2

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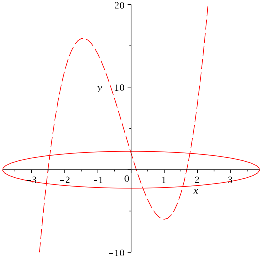
Outline

- All mD generalizations of DOA, shape-from-moments, power spectra, etc.
- Bilinear system identification
- *Rooting multivariable polynomials*
- *Multi-parameter eigenvalue problems*
- *Global optimum of prediction-error-methods*
- ...

Application: Two polynomials in two variables

- Consider

$$\begin{cases} p(x,y) &= x^2 + 3y^2 - 15 = 0 \\ q(x,y) &= y - 3x^3 - 2x^2 + 13x - 2 = 0 \end{cases}$$



- Fix a monomial order, e.g., $1 < x < y < x^2 < xy < y^2 < x^3 < x^2y < \ldots$

- Construct quasi-Toeplitz Macaulay matrix M:

$$\begin{array}{l} \\ p(x,y) \\ q(x,y) \\ x \cdot p(x,y) \\ y \cdot p(x,y) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \end{array} \\ \left[\begin{array}{cccccccccc} -15 & & & 1 & & 3 & & & & \\ -2 & 13 & 1 & -2 & & & -3 & & & \\ & -15 & & & & & 1 & & 3 & \\ & & -15 & & & & & 1 & & 3 \end{array}\right] \end{array} \begin{pmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ \vdots \\ xy^2 \\ y^3 \end{pmatrix} = 0$$

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$$\begin{cases} p(x,y) &= x^2+3y^2-15=0 \\ q(x,y) &= y-3x^3-2x^2+13x-2=0 \end{cases}$$

Continue to enlarge M:

it #	form	1	x	y	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	x^4	x^3y	x^2y^2	xy^3	y^4	x^5	x^4y	x^3y^2	x^2y^3	xy^4	y^5	→
$d=3$	p	−15			1		3																
	xp		−15					1		3													
	yp			−15					1		3												
	q	−2	13	1	−2			−3															
$d=4$	x^2p				−15							1		3									
	xyp					−15							1		3								
	y^2p						−15							1		3							
	xq		−2		13	1		−2				−3											
	yq			−2		13	1		−2				−3										
$d=5$	x^3p							−15									1		3				
	x^2yp								−15									1		3			
	xy^2p									−15									1		3		
	y^3p										−15									1		3	
	x^2q				−2			13	1				−2				−3						
	xyq					−2			13	1				−2				−3					
	y^2q						−2			13	1				−2				−3				
	↓							⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱	⋱

- $\#$ rows grows faster than $\#$ cols $\Rightarrow$ overdetermined system
- If solution exists: rank deficient by construction!

nD realization in the null space

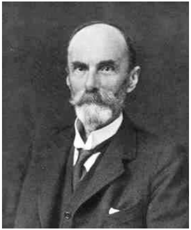
- Macaulay matrix M:

$$M = \begin{bmatrix} \times & \times & \times & \times & 0 & 0 & 0 \\ 0 & \times & \times & \times & \times & 0 & 0 \\ 0 & 0 & \times & \times & \times & \times & 0 \\ 0 & 0 & 0 & \times & \times & \times & \times \end{bmatrix}$$

- Solutions generate vectors in kernel of M:

$$MK = 0$$

- Number of solutions s follows from rank decisions 'mind-the-gap':



Francis Sowerby Macaulay

Vandermonde nullspace K built from s solutions (x_i, y_i):

$$\left[\begin{array}{c|c|c|c} 1 & 1 & \ldots & 1 \\ \hline x_1 & x_2 & \ldots & x_s \\ y_1 & y_2 & \ldots & y_s \\ \hline x_1^2 & x_2^2 & \ldots & x_s^2 \\ x_1y_1 & x_2y_2 & \ldots & x_sy_s \\ y_1^2 & y_2^2 & \ldots & y_s^2 \\ \hline x_1^3 & x_2^3 & \ldots & x_s^3 \\ x_1^2y_1 & x_2^2y_2 & \ldots & x_s^2y_s \\ x_1y_1^2 & x_2y_2^2 & \ldots & x_sy_s^2 \\ y_1^3 & y_2^3 & \ldots & y_s^3 \\ \hline x_1^4 & x_2^4 & \ldots & x_4^4 \\ x_1^3y_1 & x_2^3y_2 & \ldots & x_s^3y_s \\ x_1^2y_1^2 & x_2^2y_2^2 & \ldots & x_s^2y_s^2 \\ x_1y_1^3 & x_2y_2^3 & \ldots & x_sy_s^3 \\ y_1^4 & y_2^4 & \ldots & y_s^4 \\ \hline \vdots & \vdots & \vdots & \vdots \end{array}\right]$$

Setting up an eigenvalue problem in x

- Choose s linear independent rows in K

$$S_1 K$$

- This corresponds to finding linear dependent columns in M

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_s \\ y_1 & y_2 & \dots & y_s \\ x_1^2 & x_2^2 & \dots & x_s^2 \\ x_1 y_1 & x_2 y_2 & \dots & x_s y_s \\ y_1^2 & y_2^2 & \dots & y_s^2 \\ x_1^3 & x_2^3 & \dots & x_s^3 \\ x_1^2 y_1 & x_2^2 y_2 & \dots & x_s^2 y_s \\ x_1 y_1^2 & x_2 y_2^2 & \dots & x_s y_s^2 \\ y_1^3 & y_2^3 & \dots & y_s^3 \\ x_1^4 & x_2^4 & \dots & x_4^4 \\ x_1^3 y_1 & x_2^3 y_2 & \dots & x_s^3 y_s \\ x_1^2 y_1^2 & x_2^2 y_2^2 & \dots & x_s^2 y_s^2 \\ x_1 y_1^3 & x_2 y_2^3 & \dots & x_s y_s^3 \\ y_1^4 & y_2^4 & \dots & y_s^4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Shifting the selected rows gives (shown for 3 columns)

$$
\left[\begin{array}{c|c|c}
1 & 1 & 1 \\ \hline
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\ \hline
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
y_1^2 & y_2^2 & y_3^2 \\ \hline
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 \\
y_1^3 & y_2^3 & y_3^3 \\ \hline
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3 \\
x_1^2 y_1^2 & x_2^2 y_2^2 & x_3^2 y_3^2 \\
x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 \\
y_1^4 & y_2^4 & y_3^4 \\ \hline
\vdots & \vdots & \vdots
\end{array}\right]
\longrightarrow \text{“shift with } x\text{”} \longrightarrow
\left[\begin{array}{c|c|c}
1 & 1 & 1 \\ \hline
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\ \hline
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
y_1^2 & y_2^2 & y_3^2 \\ \hline
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1 y_1^2 & x_2 y_2^2 & x_3 y_3^2 \\
y_1^3 & y_2^3 & y_3^3 \\ \hline
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3 \\
x_1^2 y_1^2 & x_2^2 y_2^2 & x_3^2 y_3^2 \\
x_1 y_1^3 & x_2 y_2^3 & x_3 y_3^3 \\
y_1^4 & y_2^4 & y_3^4 \\ \hline
\vdots & \vdots & \vdots
\end{array}\right]
$$

simplified:

$$
\left[\begin{array}{c|c|c}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
x_1^3 & x_2^3 & x_3^3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3
\end{array}\right]
\begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix}
=
\left[\begin{array}{c|c|c}
x_1 & x_2 & x_3 \\
x_1^2 & x_2^2 & x_3^2 \\
x_1 y_1 & x_2 y_2 & x_3 y_3 \\
x_1^2 y_1 & x_2^2 y_2 & x_3^2 y_3 \\
x_1^4 & x_2^4 & x_4^4 \\
x_1^3 y_1 & x_2^3 y_2 & x_3^3 y_3
\end{array}\right]
$$

Finding the x-roots

Let $D_x = \mathrm{diag}(x_1, x_2, \ldots, x_s)$, then

$$S_1 K D_x = S_x K,$$

where S_1 and S_x select rows from K w.r.t. shift property We have

$$S_1 K D_x = S_x K$$

Generalized Vandermonde K is not known as such, instead a null space basis Z is calculated, which is a linear transformation of K:

$$ZV = K$$

which leads to

$$(S_x Z)V = (S_1 Z)V D_x$$

Here, V is the matrix with eigenvectors, D_x contains the roots x as eigenvalues.

Setting up an eigenvalue problem in y

It is possible to shift with y as well...

We find

$$S_1 K D_y = S_y K$$

with D_y diagonal matrix of y-components of roots, leading to

$$(S_y Z)V = (S_1 Z)V D_y$$

Some interesting observations:

- same eigenvectors V!
- $(S_x Z)^{-1}(S_1 Z)$ and $(S_y Z)^{-1}(S_1 Z)$ commute
  $\Longrightarrow$ 'commutative algebra'

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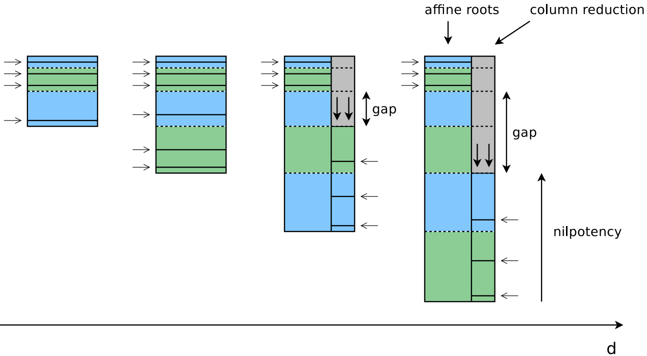
'Mind the Gap' with roots at infinity !



Application: Multiparameter Eigenvalue Problem (MEVP)

Given $A_0, \ldots, A_m \in \mathbb{R}^{p \times q}$ with $p \geq q$, find $\lambda_i \in \mathbb{C}, i = 1, \ldots, m$ and $x \neq 0 \in \mathbb{C}^q$ so that

$$(A_0 + A_1\lambda_1 + \ldots + A_m\lambda_m)\, x = 0$$

Special cases:

- Ordinary EVP: $A_0 \in \mathbb{R}^{n \times n}$, $A_1 = -I_n$, $A_i = 0, i \geq 2$
- 'Generalized' EVP: $A_0, A_1 \in \mathbb{R}^{n \times n}$, $A_i = 0, i \geq 2$

Basic idea to solve an MEVP (illustrated for $m = 2$)

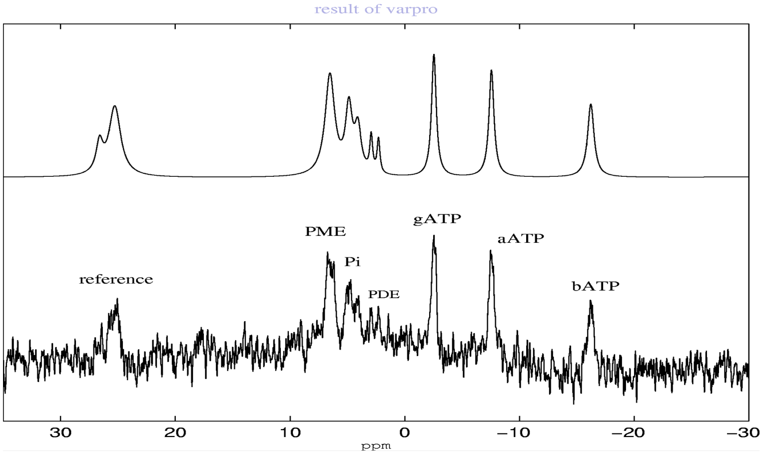
$$(A_0 + A_1\lambda_1 + A_2\lambda_2)\, x = 0$$

$$\begin{matrix} \times 1 \\ \times\lambda_1 \\ \times\lambda_2 \\ \times\lambda_1^2 \\ \vdots \end{matrix} \quad \begin{pmatrix} A_0 & A_1 & A_2 & 0 & 0 & 0 & 0 & \dots \\ 0 & A_0 & 0 & A_1 & A_2 & 0 & 0 & \dots \\ 0 & 0 & A_0 & 0 & A_1 & A_2 & 0 & \dots \\ 0 & 0 & 0 & A_0 & 0 & 0 & A_1 & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x \\ \hline x\lambda_1 \\ x\lambda_2 \\ \hline x\lambda_1^2 \\ x\lambda_1\lambda_2 \\ x\lambda_2^2 \\ \hline x\lambda_1^3 \\ \vdots \end{pmatrix} = 0$$

Block 'quasi'-Toeplitz structure + 'generalized' Vandermonde structure

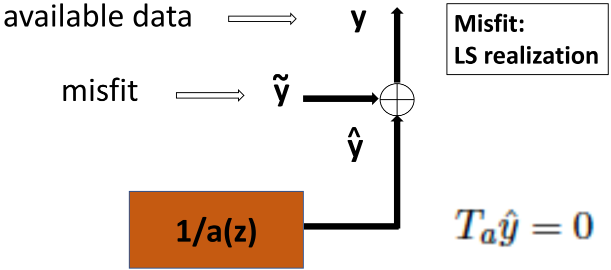
Outline

Errors using inadequate data are much less than those using no data at all.
Charles Babbage.



Data not model-compliant

Misfit case: Least squares realization (n_a)



$$T_a \hat{y} = 0$$

$$\sigma^2 = \|\tilde{y}\|_2^2$$

Misfit case: Least squares realization (ref: Kailath 80 !)

Data : $y \in \mathbb{R}^N$. **Model:** Data = model-compliant data + misfit:

$$y = \hat{y} + \tilde{y}$$

Model-compliancy (Popper: models forbid more than allow) :
Image model:

$$\hat{y} = \Gamma \, \hat{x}_0$$

Kernel model

$$\begin{aligned} \hat{Y} \, a &= T^a_{N-n} \, \hat{y} \\ &= \begin{pmatrix} \alpha_n & \alpha_{n-1} & \dots & \alpha_0 & 0 & \dots & 0 \\ 0 & \alpha_n & \alpha_{n-1} & \ddots & \alpha_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \vdots \\ \hat{y}_{N-1} \end{pmatrix} = 0 \end{aligned}$$

Least squares minimization:

$$\min \|\tilde{y}\|_2^2 \text{ subject to model} - \text{compliancy}$$

Multi-parameter EVP

$$\boxed{\begin{pmatrix} T^a_{N-n} y & T^a_{N-n} (T^a_{N-n})^T (T^a_{N-2n})^T \end{pmatrix} \begin{pmatrix} -1 \\ g \end{pmatrix} = 0}$$

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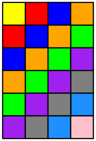
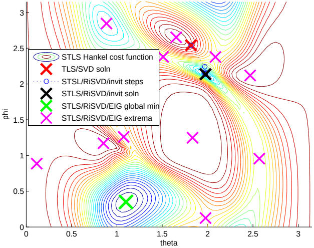
$$
\begin{aligned}
\min_{v} \quad & \tau^2 = v^T M^T D_v^{-1} M v \\
\text{s. t.} \quad & v^T v = 1.
\end{aligned}
$$





method	TLS/SVD	STLS inv. it.	STLS eig
v_1	.8003	.4922	**.8372**
v_2	-.5479	-.7757	**.3053**
v_3	.2434	.3948	**.4535**
τ^2	4.8438	3.0518	**2.3822**
global solution?	no	no	**yes**

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SUMMARY

The Ho-Kalman algorithm creates a minimum realization of a linear, time invariant system, when given a sufficiently long series of deterministic Markov parameters. However if such a "truncated" series of Markov parameters has been disturbed with noise, an approximating Hankel matrix has to be constructed for applying the realization algorithm. This approximating Hankel matrix has either the improper rank, or it lacks the Hankel structure. Furthermore the Markov parameters are not processed with a constant weighting factor, which implies that the noise filtering is inadequate.
In this report an alternative matrix is introduced and investigated: the Page matrix. This matrix is much smaller than the Hankel matrix, which offers the advantage of a considerable reduction in computation. It is shown that the method using this Page matrix might be better suited for handling noisy Markov parameters. The Page matrix approach however still does not provide an optimal solution to the approximate realization problem.
The two approaches are compared theoretically and their practical performance is tested in a set of simulations.

For the Valedictum of Paul Van den Hof:

An eigen-statement !

The optimal solution of the **least squares misfit 1D realization problem**

=

the **exact** solution of an **mD realization** problem