Numerical Linear Algebra Algorithms to Solve (Multivariate) Polynomial Systems

Minisymposium SIAM LA24

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STADIUS
Center for Dynamical Systems, Signal Processing, and Data Analytics

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Numerical linear algebra meets algebraic geometry

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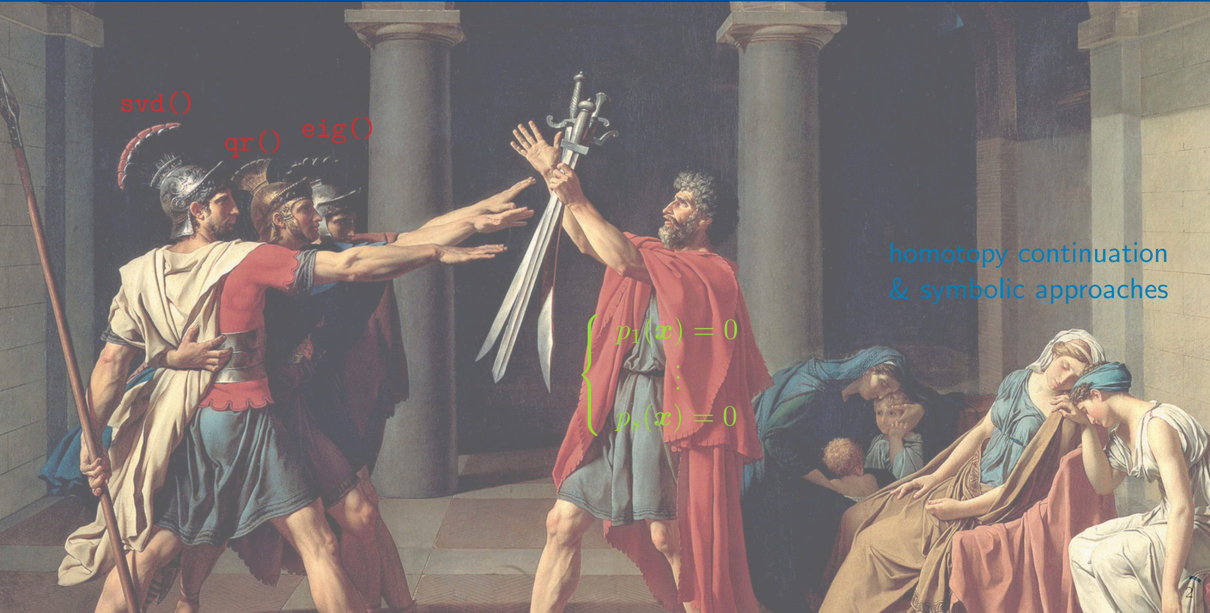

Multivariate polynomials

A **multivariate polynomial** $p(\boldsymbol{x})$, or $p(x_1, \ldots, x_n)$, in n variables is a finite linear combination of monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ from K^n with coefficients $c_{\boldsymbol{\alpha}}$ from K:

$$p(\boldsymbol{x}) = \sum_{\mathcal{A}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}},$$

where the summation runs over all the exponents in the set $\mathcal{A}$.

- K can be any field: complex numbers $\mathbb{C}$, real numbers $\mathbb{R}$, or finite numbers $\mathbb{F}_q$
- $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ indexes the monomials $\boldsymbol{x}^{\boldsymbol{\alpha}}$ and coefficients $c_{\boldsymbol{\alpha}}$
- example: $p(\boldsymbol{x}) = 3 + \sqrt{5}x_1 + (1+i)x_2 + \frac{3}{2}x_1^2x_2^8$

Multivariate polynomial systems

- Typically, multivariate polynomials appear in systems of equations:

$$\begin{cases} p_1(\boldsymbol{x}) = \sum_{\mathcal{A}^{(1)}} c_{\boldsymbol{\alpha}}^{(1)} \boldsymbol{x}^{\boldsymbol{\alpha}} = 0, \\ \vdots \\ p_s(\boldsymbol{x}) = \sum_{\mathcal{A}^{(s)}} c_{\boldsymbol{\alpha}}^{(s)} \boldsymbol{x}^{\boldsymbol{\alpha}} = 0. \end{cases}$$

- Every polynomial has a total degree: $d_i = \max(|\boldsymbol{\alpha}|)$.

Two limit cases

univariate polynomial

$(n = 1)$

$$p(x) = \sum_{i=0}^{d} c_i x^i$$

$$\Downarrow$$

$$\boldsymbol{C}_p = \begin{bmatrix} 0 & 0 & \cdots & 0 & -c_0/c_n \\ 1 & 0 & \cdots & 0 & -c_1/c_n \\ 0 & 1 & \cdots & 0 & -c_2/c_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1}/c_n \end{bmatrix}$$

$$\boldsymbol{C}_p \boldsymbol{x} = \lambda \boldsymbol{x}$$

linear systems

$(d_i = 1)$

$$\begin{cases} p_1(\boldsymbol{x}) = b_1 + \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ p_s(\boldsymbol{x}) = b_s + \sum_{j=1}^{n} a_{sj} x_j \end{cases}$$

$$\Downarrow$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

These are well-known problems from linear algebra!

What does solving mean?

Find all the values for $\boldsymbol{x} \in \bar{K}^n$ such that $p_1(\boldsymbol{x}) = \cdots = p_s(\boldsymbol{x}) = 0$, i.e., the variety of the polynomial system

$$\mathcal{V}(p_1, \ldots, p_s) = \left\{\boldsymbol{a} \in \bar{K}^n : p_i(\boldsymbol{a}) = 0, \forall i = 1, \ldots s\right\}$$

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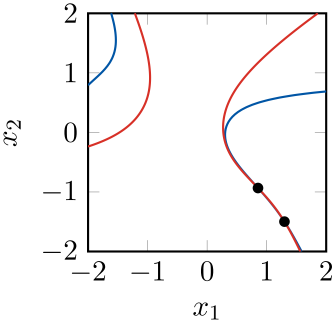
- Typically, we consider polynomial systems which are well-determined! This can be for both square and rectangular systems.



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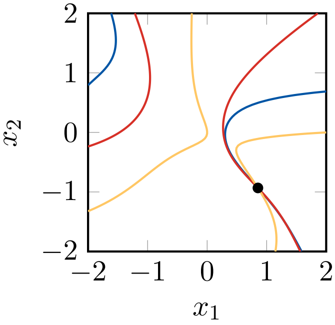
- Some solution approaches can deal with over-determined and rectangular polynomial systems.



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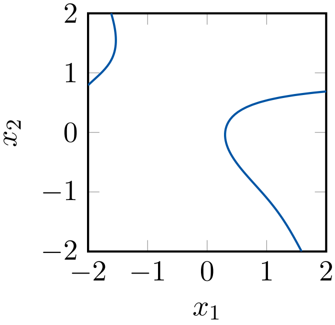
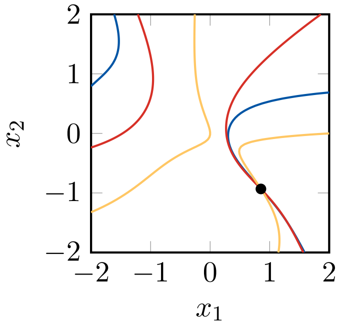
- Typically, we consider polynomial systems which are well-determined! This can be for both square and rectangular systems.
- Some solution approaches can deal with over-determined and rectangular polynomial systems.
- Under-determined polynomial systems could be solved in a certain sense, but what does it mean?



What does solving mean?

Everything depends on the ground field that you consider!



This polynomial system has 9 solutions in $\mathbb{C}^2$, 1 solution in $\mathbb{R}^2$, and 0 solutions in $\mathbb{Q}^2$

Some examples:

- $\mathbb{C}$: homotopy continuation, particle physics
- $\mathbb{R}$: optimization, chemical reaction networks, robotics
- $\mathbb{Q}$: discriminants/resultants, Grassmannians, number theory
- $\mathbb{F}_q$: cryptography
- $\mathbb{C}\{\{t\}\}, \mathbb{Q}_p$: tropical geometry

Number of solutions?

- For univariate polynomials (i.e., $n = 1$), the **fundamental theorem of algebra** states that a degree d polynomial has d roots.
- The **theorem of Bézout** is the multivariate extension of that theorem.

For any square system (i.e., $s = n$) of multivariate polynomial equations $p_1(\boldsymbol{x}), \ldots, p_n(\boldsymbol{x})$, the number of isolated solutions in the projective space $\mathbb{P}^n$ when the solution set is zero-dimensional, i.e., the number of isolated points in the zero-dimensional variety $\mathcal{V}(p_1(\boldsymbol{x}), \ldots, p_n(\boldsymbol{x})) \subset \mathbb{P}^n$, is exactly equal to

$$m_b = d_1 \cdots d_n = \prod_{i=1}^{n} d_i,$$

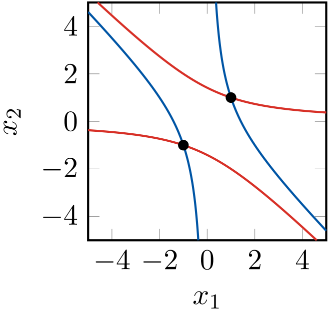
where d_i is the total degree of the polynomial $p_i(\boldsymbol{x})$.

Number of solutions?

- The theorem of Bézout counts the number of isolated solutions in the projective space:

$$m_b = m_a + m_\infty$$

- For generic systems, $m_b = m_a$, but in practice this is not the case
- There exist more refined bounds on the number of affine solutions (e.g., Kushnirenko, BKK, etc.)



Different solution approaches

When we consider an algebraically closed field, there are two main methods to solve systems of multivariate polynomial equations:

normal form methods

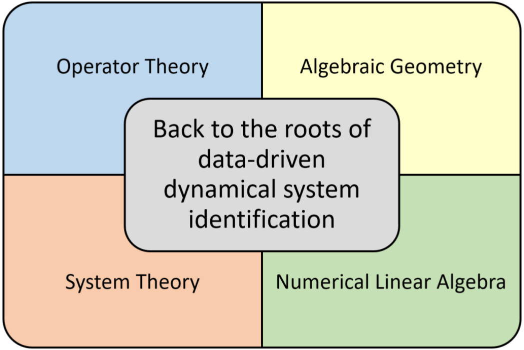
- reduce problem to a univariate problem
- (numerical) linear algebra
- any field K^n
- rectangular systems: $s \geq n$
- $m_b < \pm 10\,000$ solutions

homotopy continuation methods

- continuously deform a system with known solutions
- ordinary differential equations
- field of complex numbers $\mathbb{C}^n$
- square systems: $s = n$
- $m_b < \pm 1\,000\,000$ solutions

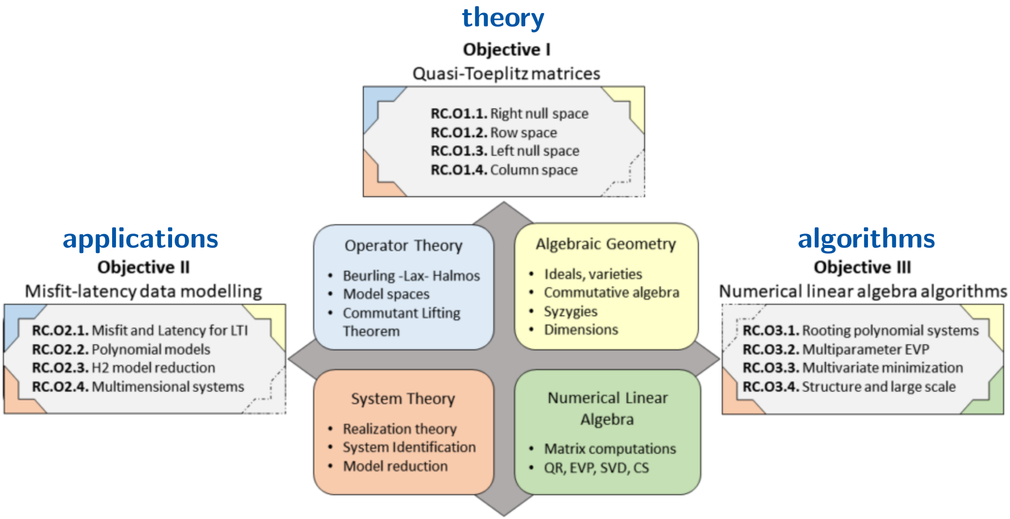
“Back to the Roots” project

- **What:** Advanced ERC grant of prof. Bart De Moor
- **Goal:** find globally optimal models for LTI systems
- **Difficulty:** large, non-convex optimization problems
- **Approach:** go “back to the roots” of mathematical modeling!



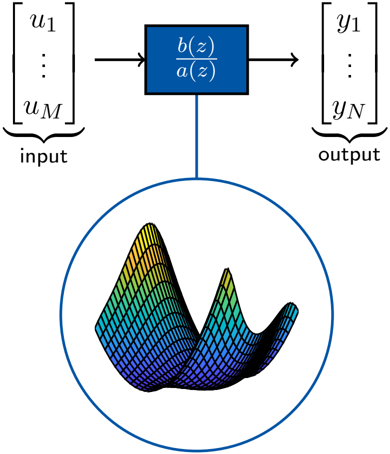
https://homes.esat.kuleuven.be/~sistawww/bdm/backtotheroots/

"Back to the Roots" project



A taste of this approach

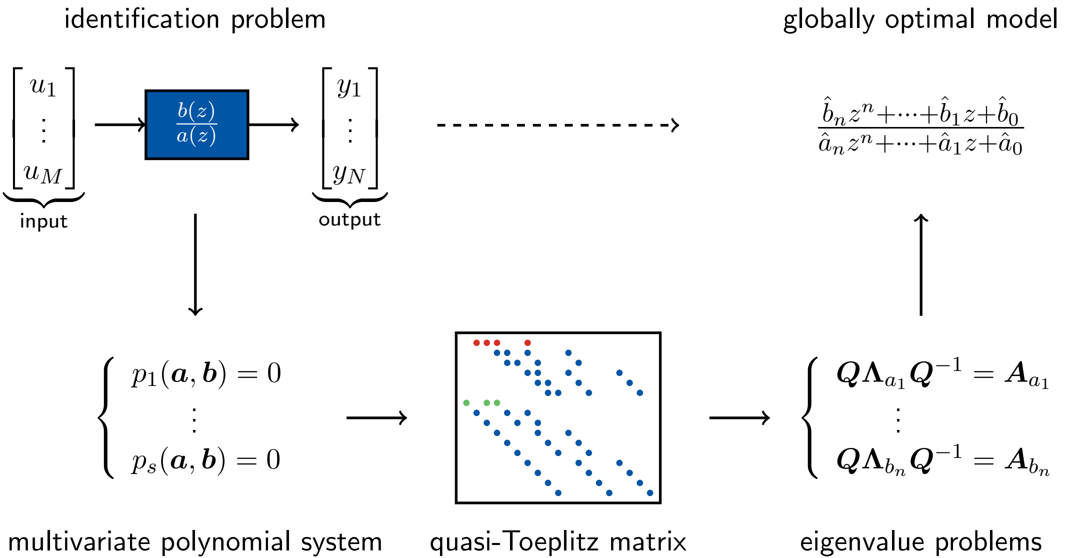
identification problem



globally optimal model

$$\frac{\hat{b}_n z^n + \cdots + \hat{b}_1 z + \hat{b}_0}{\hat{a}_n z^n + \cdots + \hat{a}_1 z + \hat{a}_0}$$

A taste of this approach



Minisymposium – Part I of II

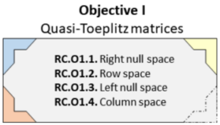
- Solving Equations Using Khovanskii Bases
  by **Barbara Betti**, Marta Panizzut, and Simon Telen
  15h45 - 16h05
- Block Krylov Methods from the Perspective of Orthogonal Matrix Polynomials
  by **Michele Rinelli**, Marc Van Barel, and Raf Vandebril
  16h10 - 16h30
- Solving Polynomial Systems Using Determinantal Formulas
  by **Matías Bender**
  16h35 - 16h55

Minisymposium – Part II of II

- Direct Numerical Computation of Polynomial Multiplication Maps
  by **Lukas Vanpoucke** and Bart De Moor
  17h10 - 16h30
- Tensor Decomposition Using Numerical (Non)Linear Algebra
  by Fulvio Gesmundo, **Leonie Kayser**, and Simon Telen
  17h35 - 17h55
- Solving Applications from Systems Theory via Efficient Numerical Linear Algebra Root-Finding Algorithms
  by **Sibren Lagauw**, Christof Vermeersch, and Bart De Moor
  18h00 - 18h20

A minisymposium filled with fascinating topics!

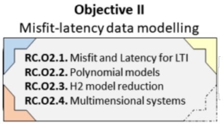
theory



Solving Equations Using Khovanskii Bases

Solving Polynomial Systems Using Determinantal Formulas

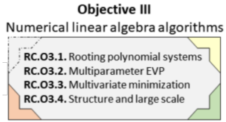
applications



Tensor Decomposition Using Numerical (Non)Linear Algebra

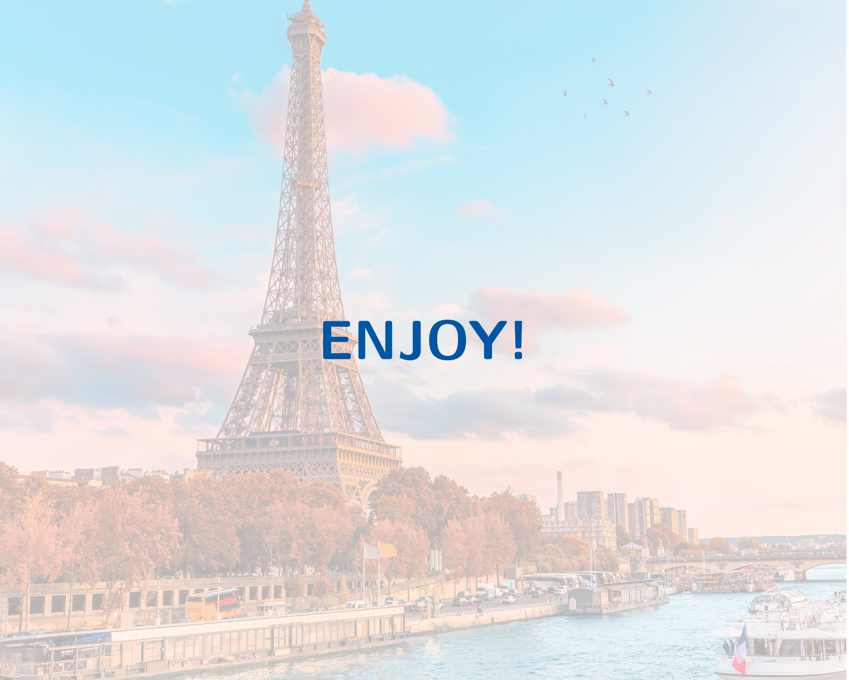
Solving Applications from Systems Theory via Efficient Numerical Linear Algebra Root-Finding Algorithms

algorithms



Block Krylov Methods from the Perspective of Orthogonal Matrix Polynomials

Direct Numerical Computation of Polynomial Multiplication Maps



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