Solving Applications from Systems Theory Via Efficient Numerical Linear Algebra Root-Finding Algorithms

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Fonds Wetenschappelijk Onderzoek Vlaanderen
Opening new horizons

Table of Contents

Polynomials and Macaulay matrix

$$\begin{cases} p_1(\boldsymbol{x}) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 = 0 \\ p_2(\boldsymbol{x}) = b_{00} + b_{10}x_1 + b_{01}x_2 + b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2 = 0 \end{cases}$$

$$\begin{array}{c} \\ p_1(\boldsymbol{x}) \\ x_1p_1(\boldsymbol{x}) \\ x_2p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ x_1p_2(\boldsymbol{x}) \\ x_2p_2(\boldsymbol{x}) \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\ \begin{bmatrix} a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} & 0 & 0 & 0 & 0 \\ 0 & a_{00} & 0 & a_{10} & a_{01} & 0 & a_{20} & a_{11} & a_{02} & 0 \\ 0 & 0 & a_{00} & 0 & a_{10} & a_{01} & 0 & a_{20} & a_{11} & a_{02} \\ b_{00} & b_{10} & b_{01} & b_{20} & b_{11} & b_{02} & 0 & 0 & 0 & 0 \\ 0 & b_{00} & 0 & b_{10} & b_{01} & 0 & b_{20} & b_{11} & b_{02} & 0 \\ 0 & 0 & b_{00} & 0 & b_{10} & b_{01} & 0 & b_{20} & b_{11} & b_{02} \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{bmatrix} = \mathbf{0}$$

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Right null space of the Macaulay matrix

$$
\begin{array}{c} \\ p_1(\boldsymbol{x}) \\ x_1 p_1(\boldsymbol{x}) \\ x_2 p_1(\boldsymbol{x}) \\ p_2(\boldsymbol{x}) \\ x_1 p_2(\boldsymbol{x}) \\ x_2 p_2(\boldsymbol{x}) \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 & x_1^3 & x_1^2x_2 & x_1x_2^2 & x_2^3 \end{array} \\
\begin{bmatrix}
a_{00} & a_{10} & a_{01} & a_{20} & a_{11} & a_{02} & 0 & 0 & 0 & 0 \\
0 & a_{00} & 0 & a_{10} & a_{01} & 0 & a_{20} & a_{11} & a_{02} & 0 \\
0 & 0 & a_{00} & 0 & a_{10} & a_{01} & 0 & a_{20} & a_{11} & a_{02} \\
b_{00} & b_{10} & b_{01} & b_{20} & b_{11} & b_{02} & 0 & 0 & 0 & 0 \\
0 & b_{00} & 0 & b_{10} & b_{01} & 0 & b_{20} & b_{11} & b_{02} & 0 \\
0 & 0 & b_{00} & 0 & b_{10} & b_{01} & 0 & b_{20} & b_{11} & b_{02}
\end{bmatrix}
\end{array}
\begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1x_2 \\ x_2^2 \\ x_1^3 \\ x_1^2x_2 \\ x_1x_2^2 \\ x_2^3 \end{bmatrix} = \mathbf{0}
$$

$$
\boldsymbol{V}_d = \left[\begin{array}{ccc}
1 & 0 & 1 \\ \hline
x_1|_{(1)} & 1 & x_1|_{(2)} \\
x_2|_{(1)} & 0 & x_2|_{(2)} \\ \hline
x_1^2|_{(1)} & 2x_1|_{(1)} & x_1^2|_{(2)} \\
x_1x_2|_{(1)} & x_2|_{(1)} & x_1x_2|_{(2)} \\
x_2^2|_{(1)} & 0 & x_2^2|_{(2)}
\end{array}\right]
$$

- Basis matrix of the right null space can be written in terms of the solutions ($d \geq d_*$).
- from the rank-nullity theorem

$$
\begin{aligned}
n_d &= q_d - r_d \\
&= \dim \mathcal{P}_d^n / \langle p_1(\boldsymbol{x}), \ldots, p_s(\boldsymbol{x}) \rangle_d
\end{aligned}
$$

- right null space $\approx$ dual vector space of the quotient space
- requires differential functionals

$$
\partial_{\boldsymbol{i}}(\cdot)|_{(j)} \triangleq \frac{1}{i_1! \ldots i_n!} \frac{\partial^{|\boldsymbol{i}|}(\cdot)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} \Bigg|_{(j)}
$$

Null space based solution approach

- Shift-trick:

$$\boldsymbol{S}_1 \boldsymbol{V}_d \boldsymbol{D}_{x_1} = \boldsymbol{S}_{x_1} \boldsymbol{V}_d$$

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- solution vectors $\boldsymbol{v}_d$ in $\boldsymbol{V}_d$ are not known in advanced:

numerical basis matrix $\boldsymbol{Z}_d$ of the right null space $(\boldsymbol{V}_d = \boldsymbol{Z}_d \boldsymbol{T})$

$$\Downarrow$$

$$(\boldsymbol{S}_1 \boldsymbol{Z}_d) \boldsymbol{T} \boldsymbol{D}_{x_1} = (\boldsymbol{S}_{x_1} \boldsymbol{Z}_d) \boldsymbol{T}$$

$$\boldsymbol{T} \boldsymbol{D}_{x_1} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{x_1} \boldsymbol{Z}_d)$$

Null space based solution approach

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$$\boldsymbol{T} \boldsymbol{D}_{x_1} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{x_1} \boldsymbol{Z}_d)$$

- it is possible to *shift with any polynomial* in the variables:

$$\boldsymbol{T} \boldsymbol{D}_{x_2} \boldsymbol{T}^{-1} = (\boldsymbol{S}_1 \boldsymbol{Z}_d)^{-1} (\boldsymbol{S}_{x_2} \boldsymbol{Z}_d),$$

which leads to the same $\boldsymbol{T}$

What is a “large enough” degree?

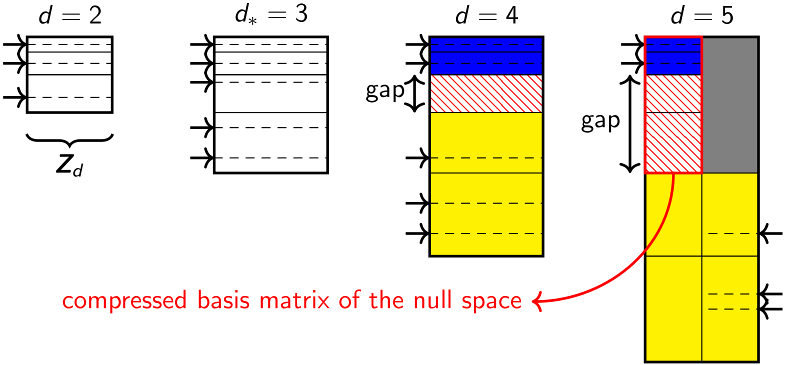

Figure: The column compression deflates the solutions at infinity from the eigenvalue problems.

MacaulayLab

For more details...

- Stetter, 2004
- Batselier et al., 2014; De Cock and De Moor, 2021; Dreesen, 2013
- Vermeersch, 2023; Vermeersch and De Moor, 2021, 2022, 2023

Software available via: `www.macaulaylab.net`

Table of Contents

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H_2-norm model reduction of SISO LTI models

Higer order model

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1}+\cdots+b_1s+b_0}{s^n+a_{n-1}s^{n-1}+\cdots+a_1s+a_0}$$

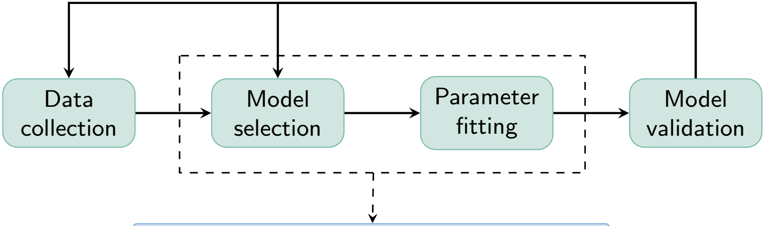
Reduced-order model $(m < n)$

$$\hat{H}(s) = \frac{\hat{b}(s)}{\hat{a}(s)} = \frac{\hat{b}_{m-1}s^{m-1}+\cdots+\hat{b}_1s+\hat{b}_0}{s^m+\hat{a}_{m-1}s^{m-1}+\cdots+\hat{a}_1s+\hat{a}_0},$$

$$\text{Min } \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega) - \hat{H}(j\omega)|^2 \, \mathrm{d}\omega$$

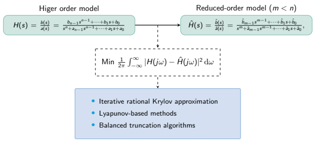
- Iterative rational Krylov approximation
- Lyapunov-based methods
- Balanced truncation algorithms

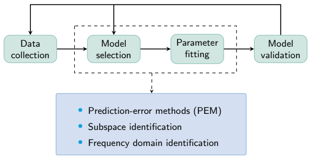
System identification for autonomous LTI models

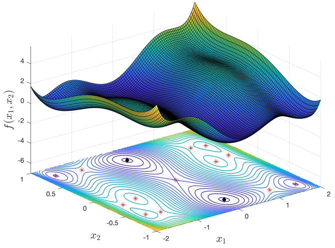


- Prediction-error methods (PEM)
- Subspace identification
- Frequency domain identification

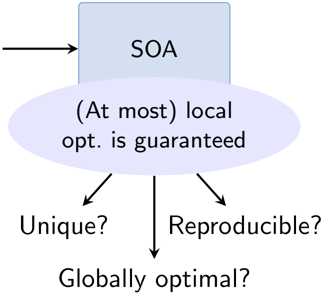
Nonlinear optimization problem(s)



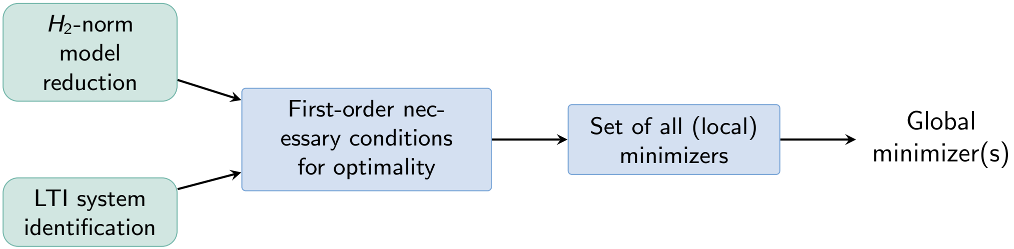




Non-convex
opt. problem



Global optimality



Global optimality

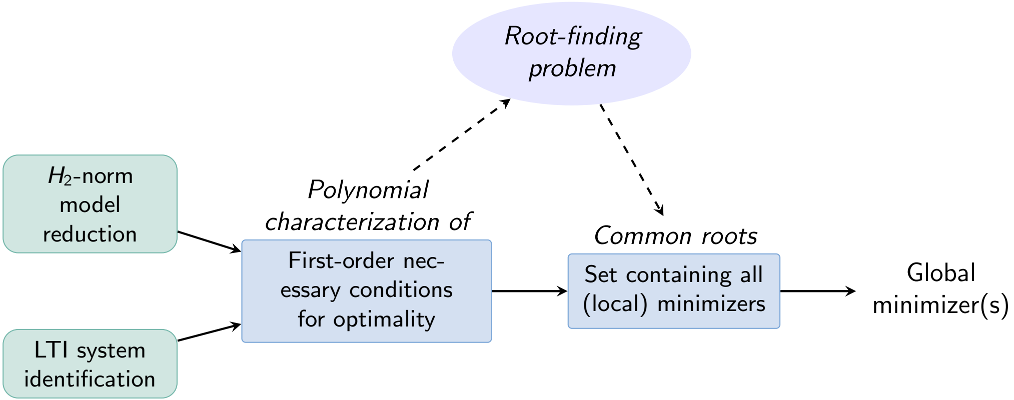


Table of Contents

Model reduction

- Class $\mathcal{M}$: minimal, stable, SISO LTI systems.

- Given $H(s) \in \mathcal{M}$ of order n,

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}, \quad (1)$$

  find $\hat{H}(s) \in \mathcal{M}$ of order $m < n$,

$$\hat{H}(s) = \frac{\hat{b}(s)}{\hat{a}(s)} = \frac{\hat{b}_{m-1}s^{m-1} + \cdots + \hat{b}_1 s + \hat{b}_0}{s^m + \hat{a}_{m-1}s^{m-1} + \cdots + \hat{a}_1 s + \hat{a}_0}, \quad (2)$$

  so that $\hat{H}(s)$ is a 'good approximation' of $H(s)$.

- $2n$ unknowns: $\hat{\boldsymbol{a}} = (\hat{a}_{m-1}, \ldots, \hat{a}_0)^{\mathsf{T}} \in \mathbb{R}^n$, $\hat{\boldsymbol{b}} = (\hat{b}_{m-1}, \ldots, \hat{b}_0)^{\mathsf{T}} \in \mathbb{R}^n$.

H_2-norm model reduction of SISO LTI models

- Minimize H_2-norm of approximation error $E(s) = H(s) - \hat{H}(s)$:

$$\hat{H}(s) \in \underset{\hat{H}(s) \in \mathcal{M}}{\operatorname{argmin}} \, J^2, \tag{3}$$

where,

$$J^2 = \|E(s)\|_{H_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega) - \hat{H}(j\omega)|^2 \, \mathrm{d}\omega$$
$$= \int_0^{\infty} \left(h(t) - \hat{h}(t)\right)^2 \mathrm{d}t,$$

with $h(t)$ and $\hat{h}(t)$ the impulse responses of $H(s)$, $\hat{H}(s)$, respectively.

Interpolatory conditions for optimality

Theorem (Meier and Luenberger, 1967)

Given a stable SISO model $H(s) \in \mathcal{M}$ of order n, let $\hat{H}(s)$ of order m ($m<n$) be a stationary point of the H_2-norm model reduction problem in (3). Then for all poles p_i of $\hat{H}(s)$,

$$H(-p_i)^{(j)} = \hat{H}(-p_i)^{(j)}, \quad j = 0, \ldots, d_i,$$

where d_i is the multiplicity of the pole p_i and the superscript j denotes the jth derivative wrt. s.

Interpolatory conditions for optimality

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Theorem (Regalia, 1995)

Given a stable SISO model $H(s) \in \mathcal{M}$ of order n, let $\hat{H}(s)$ of order m with $m<n$ be a stationary point of the model reduction problem (3). Then for all $s \in \mathbb{C}$:

$$H(s) - \hat{H}(s) = \frac{b(s)}{a(s)} - \frac{\hat{b}(s)}{\hat{a}(s)} = \left[\frac{\hat{a}(-s)}{\hat{a}(s)}\right]^2 G(s),$$

with $G(s)$ the Laplace-transform of some real-valued, stable and causal signal.

Interpolatory conditions for optimality

Corollary (Lagauw, Agudelo, et al., 2023)

For any given stable SISO model $H(s)$ of order n and mth order approximant $\hat{H}(s)$ with $m<n$ as defined in (1)–(2), define the polynomial,

$$I(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2\tilde{G}(s), \tag{4}$$

where $\tilde{G}(s)$ is a polynomial parametrized in the coefficients $\boldsymbol{g} = (g_0, \ldots, g_{n-m-1})^\mathsf{T} \in \mathbb{R}^{n-m}$:

$$\tilde{G}(s) = g_{n-m-1}s^{n-m-1} + \cdots + g_1 s + g_0.$$

Then, $\hat{H}(s)$ is a stationary point of (3) if and only if,

$$\exists\, \boldsymbol{g} \quad s.t. \quad I(s) = 0,\ \forall s \in \mathbb{C}.$$

System of multivariate polynomial equations

- Let f_k be the coefficient corresponding to s^k in the polynomial,

$$l(s) = b(s)\hat{a}(s) - a(s)\hat{b}(s) - [\hat{a}(-s)]^2\,\tilde{G}(s),$$

and define the algebraic variety,

$$\mathcal{V}_{\mathbb{R}} = \left\{(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g}) \in \mathbb{R}^{m+n} \;:\; f_k(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g}) = 0, \quad \forall k = 0, \ldots, m+n-1\right\}.$$

- If $\mathcal{V}'_{\mathbb{R}} \subseteq \mathcal{V}_{\mathbb{R}}$ contains the $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g}) \in \mathcal{V}_{\mathbb{R}}$ for which $\hat{H}(s)$ is stable, then $\mathcal{V}'_{\mathbb{R}}$ contains all minimizers $\hat{H}(s)$ ✓

- square system of $m+n$ polynomial equations
- degree of the polynomials is at most *cubic*

Numerical example I

- Consider the third order model ($n = 3$) used in Agudelo et al., 2021:

$$H(s) = \frac{s^2 + 9s - 10}{s^3 + 12s^2 + 49s + 78}.$$

- For $m = 1$, the polynomial $I(s)$ from (4) is given as,

$$\begin{aligned} I(s) = {} & (\underbrace{1 - g_1 - \hat{b}_0}_{f_3})s^3 + (\underbrace{\hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9}_{f_2})s^2 \\ & + (\underbrace{9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10}_{f_1})s \\ & + (\underbrace{-g_0\hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0}_{f_0})1. \end{aligned}$$

Numerical example I

Strategy: find all $(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}}, \boldsymbol{g})$ for which $I(s) = 0, \ \forall s \in \mathbb{C}$"

$$\iff \begin{cases} 0 = f_3 = 1 - g_1 - \hat{b}_0, \\ 0 = f_2 = \hat{a}_0 - 12\hat{b}_0 - g_0 + 2\hat{a}_0 g_1 + 9, \\ 0 = f_1 = 9\hat{a}_0 - 49\hat{b}_0 + 2\hat{a}_0 g_0 - \hat{a}_0^2 g_1 - 10, \\ 0 = f_0 = -g_0 \hat{a}_0^2 - 10\hat{a}_0 - 78\hat{b}_0. \end{cases} \tag{5}$$

- Common roots of (5) contain all stationary points $\hat{H}(s)$ of (3):

J	$\hat{a}_0$	$\hat{b}_0$	g_0	g_1	stable
8.9403	$-4.1639 + 0.9026j$	$24.930 - 6.5393j$	$-106.84 - 18.287j$	$-23.930 + 6.5393j$	✗
8.9403	$-4.1639 - 0.9026j$	$24.930 + 6.5393j$	$-106.84 + 18.287j$	$-23.930 - 6.5393j$	✗
0.3982	0.2671	-0.0437	10.349	1.0437	✓
0.2784	0.6914	9.6796	1.2799	-2.0986	✓
0.5232	-16.618	1.9264	0.0576	-0.9264	✗

Numerical example II

We search for the globally optimal 4th order reduced model ($m = 4$) of the state-space model[1] ($n = 17$) considered in Žigić et al., 1992.

- System of 21 polynomial equations with $d_{\max} = 3$
- $\mathcal{V}_{\mathbb{R}}$ contains 290 tuples, 69 remain in $\mathcal{V}'_{\mathbb{R}}$
- Four best-performing stationary points $\hat{H}(s)$:

	J	$p_{1,2}$	$p_{3,4}$
$*$	9.14×10^{-3}	$-0.032 \pm 78.54j$	$-0.111 \pm 15.43j$
l_1	9.22×10^{-3}	$-0.032 \pm 78.54j$	$-5.713 \pm 52.57j$
l_2	1.03×10^{-2}	$-0.032 \pm 78.54j$	$-0.023 \pm 3.842j$
l_3	1.09×10^{-2}	$-0.032 \pm 78.54j$	$-4.663 \pm 15.88j$

[1]The model describes the interaction between a torque activator and a torsional rate sensor for the ACES structure Collins et al., 1991.

Numerical example II

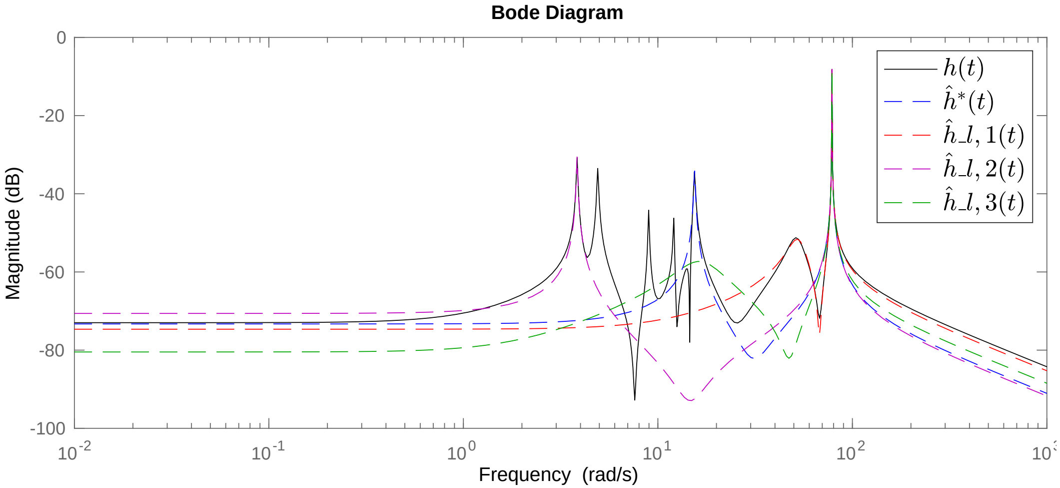
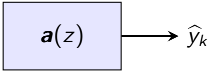


Table of Contents

Automous LTI model



- LTI dynamics of model-compliant data $\widehat{\boldsymbol{y}} \in \mathbb{R}^N$:

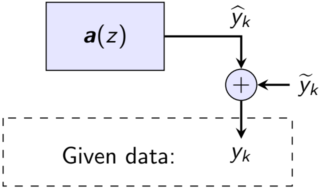
$$\widehat{y}_{k+n} + a_1 \widehat{y}_{k+n-1} + \cdots + a_n \widehat{y}_k = 0, \ \forall k{=}0, \ldots, N{-}n{-}1$$

- Kernel representation of *behavior*

$$\underbrace{\begin{bmatrix} a_n & \ldots & \ldots & a_1 & 1 & 0 & \ldots & 0 \\ 0 & a_n & \ldots & \ldots & a_1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n & \ldots & \ldots & a_1 & 1 \end{bmatrix}}_{\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}} \begin{bmatrix} \widehat{y}_0 \\ \widehat{y}_1 \\ \vdots \\ \widehat{y}_{N-1} \end{bmatrix} = \mathbf{0}$$

- n unknown model parameters $\boldsymbol{a} \in \mathbb{R}^n$

Automous LTI model



Least-squares realization:

$$\min_{\boldsymbol{a},\widehat{\boldsymbol{y}}} \quad \frac{1}{2}\|\widetilde{\boldsymbol{y}}\|_2^2 = \frac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2,$$
$$\text{s.t.} \quad \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \boldsymbol{0}.$$

- LTI dynamics of model-compliant data $\widehat{\boldsymbol{y}} \in \mathbb{R}^N$:

$$\widehat{y}_{k+n} + a_1\widehat{y}_{k+n-1} + \cdots + a_n\widehat{y}_k = 0, \ \forall k{=}0,\ldots,N{-}n{-}1$$

- Kernel representation of *behavior*

$$\underbrace{\begin{bmatrix} a_n & \ldots & \ldots & a_1 & 1 & 0 & \ldots & 0 \\ 0 & a_n & \ldots & \ldots & a_1 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n & \ldots & \ldots & a_1 & 1 \end{bmatrix}}_{\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}} \begin{bmatrix} \widehat{y}_0 \\ \widehat{y}_1 \\ \vdots \\ \widehat{y}_{N-1} \end{bmatrix} = \boldsymbol{0}$$

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$$\text{s.t.} \quad \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \boldsymbol{0}.$$

↓

$$\mathcal{L}(\boldsymbol{a}, \widehat{\boldsymbol{y}}, \boldsymbol{l}) = \tfrac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2 + \boldsymbol{l}^\top \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}},$$

↓

$$\partial\mathcal{L}/\partial\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{y}} - \boldsymbol{y} + (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^\top \boldsymbol{l} = \boldsymbol{0},$$
$$\partial\mathcal{L}/\partial\boldsymbol{a} = (\widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}})^\top \boldsymbol{l} - \boldsymbol{e}\lambda = \boldsymbol{0},$$
$$\partial\mathcal{L}/\partial\boldsymbol{l} = \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}}\boldsymbol{a} = \boldsymbol{0}$$

Least-squares realization

$$\min_{\boldsymbol{a},\widehat{\boldsymbol{y}}} \quad \frac{1}{2}\|\widetilde{\boldsymbol{y}}\|_2^2 = \frac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2,$$
$$\text{s.t.} \quad \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \boldsymbol{0}.$$

$\downarrow$

$$\mathcal{L}(\boldsymbol{a}, \widehat{\boldsymbol{y}}, \boldsymbol{l}) = \tfrac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2 + \boldsymbol{l}^{\mathsf{T}}\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}},$$

$\downarrow$

$$\begin{aligned}
\partial\mathcal{L}/\partial\widehat{\boldsymbol{y}} &= \widehat{\boldsymbol{y}} - \boldsymbol{y} + (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}}\boldsymbol{l} = \boldsymbol{0},\\
\partial\mathcal{L}/\partial\boldsymbol{a} &= (\widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}})^{\mathsf{T}}\boldsymbol{l} - \boldsymbol{e}\lambda = \boldsymbol{0},\\
\partial\mathcal{L}/\partial\boldsymbol{l} &= \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}}\boldsymbol{a} = \boldsymbol{0}
\end{aligned}$$

Theorem (De Moor, 2020)

The minimal norm misfit $\widetilde{\boldsymbol{y}} = \boldsymbol{y} - \widehat{\boldsymbol{y}} \in \mathbb{R}^N$ *corresponds to the orth. projection of* $\boldsymbol{y}$ *onto* $\mathrm{row}\left(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\right)$,

$$\widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathsf{T}})^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\boldsymbol{y}.$$

Least-squares realization

$$\min_{\boldsymbol{a},\widehat{\boldsymbol{y}}} \quad \frac{1}{2}\|\widetilde{\boldsymbol{y}}\|_2^2 = \frac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2,$$
$$\text{s.t.} \quad \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \boldsymbol{0}.$$

↓

$$\mathcal{L}(\boldsymbol{a}, \widehat{\boldsymbol{y}}, \boldsymbol{l}) = \tfrac{1}{2}\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\|_2^2 + \boldsymbol{l}^{\mathrm{T}}\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}},$$

↓

$$\partial\mathcal{L}/\partial\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{y}} - \boldsymbol{y} + (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathrm{T}}\boldsymbol{l} = \boldsymbol{0},$$
$$\partial\mathcal{L}/\partial\boldsymbol{a} = (\widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}})^{\mathrm{T}}\boldsymbol{l} - \boldsymbol{e}\lambda = \boldsymbol{0},$$
$$\partial\mathcal{L}/\partial\boldsymbol{l} = \boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{Y}}_{\boldsymbol{N-n}}\boldsymbol{a} = \boldsymbol{0}$$

Theorem (De Moor, 2020)

The minimal norm misfit $\widetilde{\boldsymbol{y}} = \boldsymbol{y} - \widehat{\boldsymbol{y}} \in \mathbb{R}^N$ *corresponds to the orth. projection of* $\boldsymbol{y}$ *onto* $\mathrm{row}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})$,

$$\widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathrm{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathrm{T}})^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}}\boldsymbol{y}.$$

Theorem (Lagauw, Vanpoucke, et al., 2024)

If $\boldsymbol{a}$ *is a (local) minimizer, then* $\exists\, \boldsymbol{g} \in \mathbb{R}^{N-2n}$ *s.t.,*

$$\widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}})^{\mathrm{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}})^{\mathrm{T}}\boldsymbol{g},$$

where $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-2n}} \in \mathbb{R}^{(N-2n)\times(N-n)}$ *is a banded Toeplitz matrix defined similarly to the matrix* $\boldsymbol{T}^{\boldsymbol{a}}_{\boldsymbol{N-n}} \in \mathbb{R}^{(N-n)\times N}$.

System of multivariate polynomial equations

- If $\boldsymbol{a}$ is a (local) minimizer, then $\exists\ \boldsymbol{g} \in \mathbb{R}^{N-2n}$ s.t.,

$$\begin{aligned} \widetilde{\boldsymbol{y}} = (\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}})^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} = (\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g}. \\ \iff (\boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}})^{-1}\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - (\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0}, \\ \iff \boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0}. \end{aligned}$$

- Define the algebraic variety,

$$\mathcal{V}_{\mathbb{R}} = \left\{(\boldsymbol{a}, \boldsymbol{g}) \in \mathbb{R}^{N-n}\ :\ \boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^{\mathsf{T}}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^{\mathsf{T}}\boldsymbol{g} = \boldsymbol{0}\right\}.$$

$\longrightarrow$ $\mathcal{V}_{\mathbb{R}}$ contains all minimizers $\boldsymbol{a}$ of the identification problem ✓

- square system of $N-n$ polynomial equations
- degree of the polynomials is at most *quartic*

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Numerical example III

Toy problem from De Moor, 2020

- Find globally optimal first-order ($n = 1$) autonomous LTI realization
- Given output data $\boldsymbol{y} = [4, 3, 2, 1]^\mathsf{T}$ ($N = 4$)

$$\boldsymbol{T}^{\boldsymbol{a}}_{N-n}\boldsymbol{y} - \boldsymbol{T}^{\boldsymbol{a}}_{N-n}(\boldsymbol{T}^{\boldsymbol{a}}_{N-n})^\mathsf{T}(\boldsymbol{T}^{\boldsymbol{a}}_{N-2n})^\mathsf{T}\boldsymbol{g} = \boldsymbol{0},$$

$$\iff \begin{cases} 0 = 4a_1 - 2a_1g_1 - a_1^2g_2 - a_1^3g_1 + 3, \\ 0 = 3a_1 - g_1 - 2a_1g_2 - 2a_1^2g_1 - a_1^3g_2 + 2, \\ 0 = 2a_1 - g_2 - a_1g_1 - 2a_1^2g_2 + 1. \end{cases}$$

- 7 common-roots, one of which in $\mathcal{V}_\mathbb{R}$

$\lVert\widetilde{\boldsymbol{y}}\rVert_2^2$	a_1	g_1	g_2
0.1486	-0.6764	-0.2525	-0.2734
/	$-0.1589 \mp 0.808j$	$1.3577 \pm 3.8194j$	$1.8359 \pm 3.3491j$
/	$0.4209 \pm 0.6233j$	$3.0425 \mp 2.9959j$	$-0.0785 \pm 1.2013j$
/	$1.3261 \pm 2.0058j$	$-0.2739 \mp 0.6279j$	$0.3793 \mp 0.3849j$

Numerical Example IV

- Find globally optimal first-order ($n = 2$) autonomous LTI realization
- Given output data[2] $\boldsymbol{y}$ ($N = 16$),

$$\boldsymbol{y} = \boldsymbol{y}_{3\text{rd}} + 0.05 * \texttt{randn(N,1)},$$

where $\boldsymbol{y}_{3\text{rd}}$ is generated by a third-order autonomous LTI model with poles $(0.2, 0.7 \pm 0.4j)$

- 739 affine common-roots, 9 of which are real-valued.

$\lVert\tilde{\boldsymbol{y}}\rVert_2^2$	a_1	a_2	p_1	p_2
0.1327	-1.6255	0.7167	$0.8127 + 0.2369j$	$0.8127 - 0.2369j$
0.1514	-0.0752	-0.5850	0.8033	-0.7282
0.1606	-14.076	10.433	13.291	0.7849
0.5386	-0.7127	1.8381	$0.3564 + 1.3081j$	$0.3564 - 1.3081j$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0.5492	-1.3053	1.0564	$0.6527 + 0.7940j$	$0.6527 - 0.7940j$

[2] The considered instance of $\boldsymbol{y}$ has $\lVert\boldsymbol{y}\rVert_2^2 = 0.5509$.

Numerical Example IV

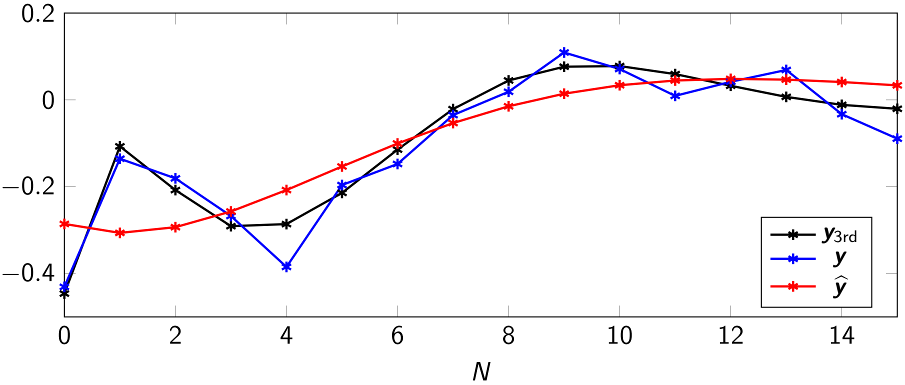

Table of Contents

Conclusions

What should we learn from these examples?

- Systems of polynomial equations from applications are *sparse*
- Systems of polynomial equations from applications are *structured*
- Particular interest in the *real-valued* common-roots
- Solving systems of polynomial equations is useful!

⟶ S. Lagauw, O. M. Agudelo, et al. (2023). "Globally Optimal SISO H_2-Norm Model Reduction Using Walsh's Theorem". In: *IEEE Control Systems Letters* 7, pp. 1670–1675

⟶ S. Lagauw, L. Vanpoucke, et al. (2024). *Exact Characterization of the Global Optima of Least Squares Realization of Autonomous LTI Models as a Multiparameter Eigenvalue Problem*. Tech. rep. Accepted for publication in the Proc. of the 22nd European Control Conference (ECC), Stockholm, Sweden. ESAT-STADIUS, KU Leuven

Questions?

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